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David Sachs

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RECIPROCITY IN MATROID LATTICES

di DAVID SACHS (*a Urbana, Ill.*) *)

A considerable number of papers have been written on matroid lattices (also known as exchange or geometric lattices) because these lattices appear in such diverse areas as the foundations of geometry, graph theory, and field theory. Nevertheless, a method for describing these lattices in terms of more easily handled structures has not yet been discovered, and much remains to be known about them. If L is a matroid lattice of finite length, then its dual is a matroid lattice $\Leftrightarrow L$ is modular. In this paper we shall define and study a matroid lattice connected with a given matroid lattice of finite length by a method which is closely related to « turning a lattice upside down ». This process is connected with certain other problems studied by the author and others, and we shall mention these questions as we proceed.

For the convenience of the reader we include some definitions and results without proof. (See [1], [2], [5]). A *matroid* lattice of finite length is a relatively complemented semi-modular lattice of finite length. If L is a lattice with operations $+$, $\cdot$, then we write $(a, b)M \Leftrightarrow (c + a)b = c + ab$ for every $c \leqslant b$. If

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this relation is symmetric, we say the lattice is *semi-modular*, and if it is universal, then we say the lattice is *modular*. A lattice of finite length, is semi-modular $\Leftrightarrow a \succ ab \Rightarrow a + b \succ b$ (where $a \succ b$ (a covers b) means $a > b$ and there is no x with $a > x > b$). A *point* (*atom*) is an element which covers 0. A matroid lattice of finite length is a semi-modular lattice in which every element is a join of points. A lattice L is *left-complemented* if for every $a, b \in L$, there exists $b' \leqslant b$ with $a + b' = a + b$, $ab' = 0$, and $(b', a)M$. Left-complemented lattices are semimodular, and lattices of finite length are left-complemented $\Leftrightarrow$ they are matroid. Since maximal chains between elements in a semi-modular lattice have the same number of elements, one can define a dimension function $D(x)$ which we normalize by setting $D(0) = 0$. In semi-modular lattices of finite length, $D(x) + D(y) \geqslant D(x + y) + D(xy)$, and equality holds $\Leftrightarrow (x, y)M$. Finally, a *hyperplane* (*coatom*) is an element which is covered by 1.

Let L be a matroid lattice of finite length with operations $+$ and $\cdot$ denoting join and meet respectively. Let M be a subset of L having the following properties:

(1) M contains 0, 1 and all the points of L,

(2) $a \in M,\ b \in M \Rightarrow ab \in M$,

(3) $a \in M,\ b \in M \Rightarrow (a, b)M$,

(4) $x \in M,\ z \in L$ with $z \geqslant x$ implies the existence of $y \in M$ with the property that $z + y = 1$, $zy = x$.

We define $\bar{M}$ to be the dual of the poset M. $\bar{M}$ is said to be a *reciprocal* of L, and L is said to be an *inverse* of $\bar{M}$.

THEOREM 1. – *$\bar{M}$ is a matroid lattice of the same length as that of L, and meets in M and L agree. If $\cup$ is the join operation in M, then $(a, b)\bar{M}$ in the lattice $\bar{M} \Leftrightarrow a + b = a \cup b$. Finally, M generates L.*

Proof. – M generates L because M contains the points of L. M is meet closed by definition so that it must be a lattice, and meets in M and L must agree. We observe that if $a, b \in M$, then $a \cup b \geqslant a + b$. If $a, b \in M$, then there exists $b' \in M$ such that $(a + b)b' = b$, $(a + b) + b' = 1$. Thus $ab = ab'$, $a + b' = 1$ so that $a \cup b' = 1$. If $c \in M$ with $c \geqslant a$, then $c(b' \cup a) = c$, and also $c(b' + a) = cb' + a$ or $c = cb' + a$. Thus $c \leqslant cb' \cup a$, and since the reverse inequality is obvious, $c = cb' \cup a$. This equation implies that $\overline{M}$ is a left-complemented lattice and therefore a matroid lattice. Now condition (4) implies that if h is a hyperplane in M, then h must be a hyperplane in L. Since M is a matroid lattice, we can find a finite sequence of hyperplanes (h_i) such that $h_1 \succ h_1h_2 \succ ... \succ h_1h_2 ... h_n = 0$ where covering is in the sense of $\overline{M}$. But then $h_1 \succ h_1h_2 \succ ... \succ h_1h_2 ... h_n = 0$ in the sense of L because of the modularity of the elements. Thus $\overline{M}$and L have the same length.

If $a + b = a \cup b$ and $c \geqslant b$ with $c \in M$, then $c(a \cup b) = c(a + b) = ca + b$, and so $c(a \cup b) \leqslant ca \cup b$. This implies that $(a, b)M$ in $\overline{M}$. Now suppose that $a \cup b > a + b$. Using the element b' of the preceding paragraph, we see that $b'a \cup b = b$. But $b'(a \cup b) > b$, for if not, then $a \cup b = (a \cup b)(b' + (a + b)) = (a \cup b)b' + (a + b)$ (since $(b', a \cup b)M$ in L) $= a + b$. Hence $(a, b)M'$ in $\overline{M}$.

Corollary. – In (4), we can make the further assumption that $(z, y)M$.

Proof. – Let m be a maximal element with the properties $m \in M$, $zm = x$, $(z, m)M$. If $m + z \neq 1$, then since $\overline{M}$ is a matroid lattice of the same length as L, there exists $n \in M$ with $n \succ m$ and $m + z \not\geqslant n$. But since $(n, m + z)M$, if $t \leqslant z$, then $(t + n)z = (t + n)(m + z)z = (t + m)z = t + mz$. Thus $(n, z)M$ and we have a contradiction since $zn = (m + z)zn = mz = x$.

Remark. – We needed the fact that M contained all the points of L only to show that M generates L. In the subsequent material we shall often make statements in terms of M instead of $\overline{M}$ because the order relation and the meet operation in M and L are identical (relative to elements of M, of course).

THEOREM 2. – *Condition* (4) *can be replaced by*

(4′) *M is a matroid lattice of the same length as that of L.*

Proof. – Since $\overline{M}$ and L have the same length, a hyperplane in M is a hyperplane in L. Let $c \in M$ and $b \in L$ with $b > c$. There exists a minimal element m with the property that $b + m = 1$,

$m \geqslant c$, $m \in M$. If $mb > c$, there must exist a hyperplane $h \in M$ with $h \geqslant c$, $h \ngeqslant mb$ because $\overline{M}$ is a matroid lattice with the same length as that of L. Thus $b + mh = b + mh + mb = b + m(h + mb) = b + m$ (since $h + mb = 1$) $= 1$. Thus $mh = m$ which is impossible since $h \ngeqslant mb$. It follows that condition (4) is satisfied. The converse follows from Theorem 1.

Does every matroid lattice of finite length have a reciprocal? In the finite case we can give a negative answer without even giving an explicit counter-example. We first observe that if L is not a modular lattice, then neither is $\overline{M}$. For by Theorem 1, M is modular $\Leftrightarrow$ M is join closed. But since M contains the set of points, M must then be equal to L. Now if L is finite and not modular, then $\overline{M}$ is not modular and must contain fewer elements then L, but it must be of the same length. If $\overline{M}$ has a reciprocal, we can repeat the above process. If this process never terminates, then we obtain a strictly decreasing sequence of finite lattices, each of the same length. This is clearly impossible. Since for each number $n \geqslant 4$, finite non-modular matroid lattices of length n exist, we have the fact that matroid lattices of any length $\geqslant 4$ exist which have no reciprocals.

The above argument cannot be applied to infinite lattices, so we shall give an explicit example in this case. Let B be the lattice of subsets of an infinite set. The sets with n or fewer elements form a matroid lattice L with the natural partial ordering if we adjoin a unit element 1. Now two n-element sets are modular precisely when their intersection contains $n - 1$ elements. Thus the set $S = [1, 2, 3, \dots, n]$ $(n \geqslant 3)$ is modular with the other n-element sets $\Leftrightarrow$ it differs from them by one element. Suppose that S lies in the reciprocal of L. Let a and b be distinct elements not in S. Now $[a]$ and $[b]$ must be the intersection of n-element subsets of L which are pairwise modular. If T and W contain a and b respectively, then it is impossible that S, T, and W be pairwise modular unless $S \cap T = S \cap W = T \cap W =$ a set with $n = 1$ elements. Thus $W = (S \cap T) \cup [b]$. It is then impossible that $[b]$ be the intersection of n-element subsets which are modular with both S and T. Since S is typical, L has no reciprocal.

We shall see later that not every matroid lattice of finite length has an inverse. It is also not true, in general, that the reciprocal characterizes the inverse (despite the fact that it generates it), but later, we shall add other conditions that are sufficient to guarantee that this does happen.

EXAMPLES. – Let Γ be a complemented modular lattice of finite length and let S be a set of hyperplanes in Γ with the property that 0 is the meet of S. The set M of elements of Γ which are meets of hyperplanes in S is a lattice under the induced ordering because it is a meet closed subsystem of Γ. Its dual $\overline{M}$ is a matroid lattice because if $m \in M$ and h is a hyperplane in M, then h is a hyperplane in Γ, and mh is covered by m because that is what occurs in Γ. Now Γ and $\overline{M}$ have the same length so that points in M are points in Γ. Let T be the set of points in M and let L be the set of elements in Γ which are joins (in the sense of Γ) of the elements in T. L is clearly a matroid lattice which has the same length as that of Γ and M. If $m \in M$, then m is the join of the points it contains (in the sense of M) since $\overline{M}$ is a matroid lattice. The same set W of points has a join in Γ which is $\leqslant m$. However, there exists a maximal chain of elements in M between 0 and m, and this chain is maximal in Γ because M and Γ have the same length. It follows that m is the join of W in the sense of Γ. Thus every element in M lies in L.

It is evident that M satisfies condition (1). If $a \in M$ and $b \in M$, then ab (in Γ) lies in M by the very definition of M. But obviously ab must then be the greatest lower bound of a and b in L. Thus condition (2) is satisfied. As we have just seen, $a \in M$ and $b \in M$ have the same meet in L as they do in Γ. As L is join closed, a and b also have the same join in L as they do in Γ. As the dimension of an element in L is the same as it is in Γ, $D(a) + D(b) = D(a + b) + D(ab)$. Thus $(a, b)M$ in L, and condition (3) is satisfied. We have already seen that condition (4′) is satisfied, so it follows that $\overline{M}$ is a reciprocal of L.

Another example of a lattice with a reciprocal is the lattice of partitions on a finite set. The set of singular partitions (at most one subset is a non-singleton) form a reciprocal of the lattice.

(See [5]). Later we shall see that this example is really a special case of the previous one.

If Γ is a matroid lattice of not necessarily finite length, and L is a join-closed system of Γ formed by taking a set of points of Γ and all their joins, then Γ is a matroid lattice. We shall call L *representable* if Γ is modular. Our $\overline{M}$ and L lattices of the first example are representable. It is easily seen that if L is representable, then it can be represented by a lattice Γ in which the unit elements coincide. In particular, a representable lattice of finite length can be represented by a lattice of the same length.

Suppose now that we have a sequence of lattices $L_1 \to L_2 \to \to \ldots \to L_k \to \ldots$ where L_n is a reciprocal of L_{n+1} for each $n = 1, 2, 3, \ldots$. Evidently $L_1 \subseteq L_2 \subseteq L_3 \ldots$ (as sets). Now the union $\mathcal{O}$ (*odd limit*) of L_1, L_3, L_3, ... has a partial ordering on it induced in the natural way, and the same is true for the *even limit* $\mathcal{E}$. Suppose that b and c lie in L_n. Then they lie in L_{n+1} and by definition they form a modular pair in L_{n+1}. Consider now bc and $b + c$ in L_{n+1}. The element bc lies in L_n and the element $b + c$ lies in $L_n \Leftrightarrow (b, c)M$ in L_n. The same reasoning applied to L_{n+1} and L_{n+2} shows that the join of b and c in L_{n+2} is bc and the meet is $b + c$ because $(b, c)M$ in L_{n+1}. By induction we see that b and c have the same join and meet in L_{n+4}, L_{n+6}, ... as well as in L_{n+3}, L_{n+5}, ... so that they are modular pairs in both $\mathcal{O}$ and $\mathcal{E}$. Thus $\mathcal{O}$ and $\mathcal{E}$ are dually isomorphic complemented modular lattices. It is to be observed that length of $\mathcal{E}$ is the same as the length of L_1.

Now let us look at the relationship among L_1, L_2, $\mathcal{O}$, and $\mathcal{E}$. The poset L_1 was meet closed in the lattice L_2, and the analysis in the previous paragraph shows that it is meet closed in any of the even L_k. Hence it is meet closed in $\mathcal{E}$. Furthermore, the points of L_1 are hyperplanes in $\mathcal{E}$ and the hyperplanes of L_1 (points of L_2) are points of $\mathcal{E}$. Thus L_1 and L_2 are lattices of the type we considered in our first example. Evidently the process in the first example can be repeated indefinitely to give a sequence of reciprocals.

Thus a necessary and sufficient condition that a lattice and its reciprocal be of the type considered in the first example is that a *sequence of reciprocals exist. The existence of a sequence of reciprocals is also necessary and sufficient for the representability of a lattice of finite length.*

The notion of such a sequence of reciprocals can be used to prove that there exists a matroid lattice of finite length which is not the reciprocal of another lattice L. If $L_1 \to L_2 \to L_3 \to \ldots$ is a sequence of lattices with L_n a reciprocal of L_{n+1}, then as was noted previously, L_1 is a representable lattice. If we can show the existence of a lattice L of finite length which cannot be represented then the sequence must ultimately terminate with a lattice which is not the reciprocal of any lattice.

Let C be the direct union of infinitely many projective geometries k_i, each of length n $(n \geqslant 6)$, over fields of different characteristics. If we consider the poset L of elements of dimension $\leqslant n$ and adjoin 1, we obtain a matroid lattice of finite length which contains intervals $[0, l_i]$ isomorphic to the various projective geometries k_i with which we started. If L can be represented by P, a complemented modular lattice of finite length, then P is a direct union of finitely many projective geometries G_j (some of which may be degenerate). In such a representation meets are preserved for a given pair $(a, b) \Leftrightarrow (a, b)M$ in L. This is because $(a, b)M$ in $L \Leftrightarrow D(a) + D(b) = D(a + b) + D(ab)$. It follows that each of the projective geometries k_i with which we started will be sublattices of P since they are modular, and their points will be points in P. Since every line in a k_i contains at least 3 points, each k_i must be a sublattice of some G_j. By the pigeon hole principle, some projective geometry G_j of P contains at least two of the k_i. It is then easily seen that planes of the two k_i can be viewed as subplanes of a plane in the projective geometry G_j. But this is impossible as the characteristic of the coördinatizing division ring of G_j is determined by any subplane.

A simple modification of this example can be used to show the existence of a finite matroid lattice which cannot be represented by a finite projective geometry. (Use two finite geometries of different characteristics.) Lazarson [3] first produced an example of a finite matroid lattice which cannot be represented by a finite projective geometry by making some vector space calculations. His example is, however, easily represented by a direct union of two finite geometries.

We now consider a situation where the reciprocal satisfies the following stronger property:

(4″) $x \in M$, z, $w \in L$ with $w > z > x$ implies the existence of $y \in M$ such that $z + y = w$, $zy = x$.

It is easily seen that (4″) says that the M elements contained within w form a reciprocal of $[0, w]$ if we adjoin w to them. We then obtain the following result analogous to the corollary to Theorem 2.

Corollary. – In (4″) we can make the further assumption that $(y, z)M$.

L. R. Wilcox [6] considered the dual of property (4″), but made the assumption that L was a complemented modular lattice (no finiteness restrictions). We can obtain a result about $\overline{M}$ which he obtained in his situation.

Definition 1. – A semi-modular lattice T is *full* if when $a + b$ does not cover b, there exists x with $a + b > x > b$ such that $(a, x)M$.

Remark. – The conclusion always holds if $(a, b)M$.

Theorem 3. – *If* (4″) *is satisfied, then* $\overline{M}$ *is full.*

Proof. – Suppose that a, $b \in M$ and that b does not cover ab. Since $(a, b)M$ in L, $a + b$ does not cover a, By (4″) there exists $y \in M$ with $a + b > y > a$. If we define $x = by$ then $x + a = yb + a = y(b + a) = y$. Thus $(a, x)M$ in $\overline{M}$.

This property is not satisfied by the reciprocal of the partition lattices cited in our second example. As an example where (4″) is satisfied, we consider the following: Γ is a projective geometry of finite length, L is $\Gamma - [p]$ where p is a point is Γ, and M is 1 and the set of elements in L which do not contain p. M is precisely the set of elements in L which are modular with every element in L, and, of course, it is an affine geometry. (see [5;

p. 327] for the details.) Property (4″) imposes rather strong conditions on L as we shall see in the following lemma and theorem.

LEMMA 1. – *Let $h \in M$ with $hk \neq 0$. Then $(h, k)M$.*

Proof. – We assume that $k > hk > 0$ for otherwise there is nothing to prove. By (4″), there exists $l \in M$ such that $hk + l = k$, $hkl = 0$, $(hk, l)M$. Now $D(h) + D(k) = D(h) + D(hk) + D(l)$. But $h + l = h + hk + l = h + k$, $hl = hkl = 0$. Therefore $D(h) + D(l) = D(h + l) = D(h + k)$. It follows that $D(h) + D(k) = D(hk) + D(h + k)$. Hence $(h, k)M$.

THEOREM 4. – *If $hk > p$, where p is a point, then $(h, k)M$.*

Proof. – We assume $k > hk$. By (4″) there exists $l \in M$ such that $(l, hk)M$, $l + hk = k$, $hkl = p$. By Lemma 1 $(h, l)M$. Now $h + l = h + k$, $hl = p$. Thus $D(h + k) + D(p) = D(h) + D(l)$. Since $D(l) + D(hk) = D(k) + D(p)$, $D(h + k) - D(h) = D(k) - D(hk)$. Therefore $(h, k)M$.

From Lemma 1 we deduce that an interval [p, 1] is a *modulated* lattice (see [5]). Whether or not it is actually a modular lattice is unknown.

We now look into the problem of when a lattice is uniquely determined by its reciprocal. The following notion plays a central role here.

DEFINITION 2. – Let $\overline{M}$ be a matroid lattice of finite length. A set H of elements in M (the dual of $\overline{M}$) is said to be a *quasi-dual ideal* of $\overline{M} \Leftrightarrow$

$$x \in H, \quad y \leqslant x (\text{in } M) \Rightarrow y \in H \tag{5}$$

$$x, y \in H, \quad (x, y)M \quad \text{in} \quad \overline{M} \Rightarrow x \cup y \in H\,. \tag{6}$$

A *maximal* quasi-dual ideal is a proper one contained in no other. The quasi-dual ideal generated by A will be denoted by $\{A\}$.

If now M is a reciprocal of L, then each element z in L determines a quasi-dual ideal of $\overline{M}$, namely the set of elements in M that it contains. For condition (5) is immediately satisfied. If $z \in L$ has the property $z \geqslant x$, $y \in M$ and $(x, y)M$ in $\overline{M}$, then $z \geqslant x \cup y$ because $x \cup y = x + y$. Thus (6) is satisfied. However, one cannot expect all quasi-dual ideals of $\overline{M}$ to correspond to elements in L. If the hyperplanes of L are uniquely determined, then L is uniquely determined because L is the completion by cuts of its hyperplanes and points. Our first result guaranteeing the uniqueness of L is given completely in terms of $\overline{M}$.

THEOREM 5. – *If $\overline{M}$ is a lattice in which each maximal quasi-dual ideal B not a dual ideal has a pair of elements which generate it, then $\overline{M}$ has at most one inverse L.*

Proof. – Evidently maximal dual ideals in $\overline{M}$ correspond to hyperplanes in M and conversely. Suppose that $\overline{M}$ is the reciprocal of L and that h is a hyperplane in L not in M. The set S of elements in $M \leqslant h$ is a quasi-dual ideal of $\overline{M}$. If it is not maximal, then we can extend it to a maximal one S' which cannot be a dual ideal. By hypothesis $S' = \{a, b\}$. Now $a \cup b = 1$ for otherwise S' could not be a maximal quasi-dual ideal which was not a dual ideal. Furthermore, $a + b$ must be a hyperplane h', again because of the maximality of S'. The quasi-dual ideal determined by h' evidently contains a and b and thus S' since $S' = \{a, b\}$. Hence by the maximality of S', this quasi-dual ideal is equal to S'. But this implies that every element in S in contained in h', so that $h \leqslant h'$. Since both are hyperplanes, $h = h'$ and $S = S'$.

Conversely, if S is a maximal quasi-dual ideal but not a dual ideal in $\overline{M}$, then $S = \{a, b\}$. Now $a + b \neq 1$, for then S would be $\overline{M}$. If $a + b$ is not a hyperplane, then S cannot be generated by a and b since S is maximal. Thus S is exactly the quasi-dual ideal determined by $a + b$. Therefore there is a one-to-one correspondence between the maximal quasi-dual ideals of $\overline{M}$ and the hyperplanes of any L of which it is the reciprocal. In summary, L is isomorphic to the completion by cuts of the partially ordered system of maximal quasi-dual ideals of $\overline{M}$ and points of M.

Q.E.D.

The reciprocals discussed for the partition lattices satisfy the hypothesis of the above theorem (see [5; p. 339]), and there are other examples as well. But in many cases it is difficult to determine the structure of the maximal quasi-dual ideals. In many of the important examples a lattice arises from another lattice as a reciprocal. Examples of these are reciprocals satisfying property (4″). Our next aim is to show that these lattices characterize their inverses.

LEMMA 2. – *If x, b, $c \in M$ and $x + bc = b + c$ (in L), then x belongs to the quasi-dual ideal $\{b, c\}$.*

Proof. – Now xb and xc belong to $\{b, c\}$. But

$$\begin{aligned} xb + xc &= x(b + xc) \\ &= x(b + bc + xc) \\ &= x(b + c(b + xc)) \\ &= x(b + c(b + c)) = x(b + c) = x(x + bc) = x\,. \end{aligned}$$

Thus $(xb, xc)M$ in $\overline{M}$. Hence x belongs to $\{b, c\}$.

THEOREM 6. – *If h is a hyperplane in L (a lattice of length $\geqslant 5$) and $h = b + c$, $b \neq c$, with b, $c \in M$, then the quasi-dual ideal in $\overline{M}$ determined by h is $\{b, c\}$*.*

Proof. – Without loss of generality we can assume that $bc \neq 0$. For suppose that $bc = 0$. It is impossible that both b and c are points because of the length of L. If b is not a point, then h does not cover c, and therefore there exists $c' \in M$ with $c < c' < h$. Thus $bc' \neq 0$, $b + c' = h$, and since $bc' + c = c'(b + c) = c'$, $c' \in \{b, c\}$. But obviously $\{b, c'\}$ contains c since $c' > c$. Thus $\{b, c\} = \{b, c'\}$.

Now let $x \in M$ with $h \geqslant x$. We first suppose that $x \not\geqslant bc$. Since $x + bc \geqslant x$ and $x \not\geqslant bc$, $x + bc > x$. Now $b + c \geqslant x + bc$. If $b + c = x + bc$, then by Lemma 2, $x \in \{b, c\}$. Thus suppose that $b + c > x + bc$. There exists $y \in M$ such that $y(x + bc) = x$, $y + x + bc = b + c$. Now $b + c = y + x + bc = y + bc$ since $y \geqslant x$. Therefore by Lemma 2, $y \in \{b, c\}$. Thus $x \in \{b, c\}$ since $y \geqslant x$.

Suppose however that $x \geqslant bc$. Now $b + c > bc > 0$. Therefore there exists $y \in M$ such that $y + bc = b + c$, $ybc = 0$.

*) L. R. WILCOX [6] has proved a stronger result assuming that L is modular. A stronger result can be proved in this context, but it is unnecessary for our purposes.

By Lemma 2, $y \in \{b, c\}$. Thus $xy \in \{b, c\}$. But $xy + bc = x(y + bc)$ (since $x \geqslant bc$) $= x(b + c) = x$. Hence $x \in \{b, c\}$.

THEOREM 7. – *Suppose that $\overline{M}$ is a reciprocal of L_1 and satisfies* (4″), *and that $\overline{M}$ is a reciprocal of L_2 and satisfies merely* (4). *Then L_1 and L_2 are isomorphic.* (*We assume that $\overline{M}$ has length* $\geqslant 5$.)

Proof. – We shall first show that the hyperplanes in L_1 not in M are determined by maximal quasi-dual ideals in $\overline{M}$ with two generators, and conversely. Any hyperplane h in L_1 is the join of two elements b and c in M. By Theorem 6, the quasi-dual ideal determined by h is $\{b, c\}$. Let p be a point in L_1 not contained in h. There exists $d \in M$ with $h \succ d$. Now $d + p$ is a hyperplane which determines a quasi-dual ideal in $\overline{M}$. Again by Theorem 6, this is simply $\{d, p\}$. Now there exists $f \in M, f \neq d$ with $d + p \succ f$. Thus f lies in $\{d, p\} \subseteq \{b, c, p\}$. (Note that $h \not\succ f$ since $h(d + p) = d$). If we apply (4″), we deduce the existence of $d' \in M$ with $h \succ d'$ and $d' \ngeqslant df$. Thus $d' + f = d' + f + df = h + f = 1$. Hence $b, c, \{p\}$ is the entire lattice $\overline{M}$. Thus $\{b, c\}$ is maximal. Conversely, if $\{b, c\}$ is maximal, then $b + c$ must be a hyperplane.

Now let h_1 be a hyperplane in L_1. The assertion is that L_2 has a hyperplane h_2 which contains precisely the same elements of M as does h_1 (in particular, the same set of points). If $h_1 \in M$, this is obvious. If $h_1 = b + c$ where $b, c \in M$, then the join of b and c in L_2 must also be a hyperplane, say h_2. This follows from dimensionality consideration. Now h_2 determines a quasi-dual ideal in $\overline{M}$ which must contain b and c. Since $\{b, c\}$ is maximal, h_2 determines precisely $\{b, c\}$ (as does h_1).

Conversely, suppose that h_2 is a hyperplane in L_2 not in M. Then $h_2 = p_1 + p_2 + \dots + p_n$ where the set P of points is independent. Since L_1 and L_2 must have the same length (that of $\overline{M}$), there exists a hyperplane h_1 in L_1 which contains all the p_i. Now $h_1 = b + c$ where $b, c \in M$. As was shown in the previous paragraph, there exists $h_1' \in L_2$ which determines $\{b, c\}$. But then h_1' must contain all the p_i because $P \subseteq \{b, c\}$. Thus $h_1' = h_2$. Hence h_2 and h_1 contain precisely the same set of points in M, namely those in $\{b, c\}$. Thus there is an isomorphism between the poset of hyperplanes and points of L_1 and the corresponding poset in L_2. Thus L_1 and L_2 are isomorphic.

Corollary. – Let L_1 be a matroid lattice with reciprocal $\overline{M}$ in which every hyperplane in L_1 (not in M) is determined by a maximal quasi-dual ideal in $\overline{M}$ with two generators, and conversely. Then if $\overline{M}$ is also the reciprocal of L_2, then L_1 and L_2 are isomorphic.

Proof. – Follows from the proof of the theorem.

Remark. – The partition lattices and their reciprocals of singular partitions satisfy the hypothesis of the corollary but not that of the theorem. We mention in passing that if we require (4″) to hold merely for w a hyperplane (assuming (4), of course), then the hypothesis of the corollary of Theorem 7 will be satisfied. In fact, the proofs for Theorems 6 and 7 would go through without a change.

R. Rado [4] has shown that a partition lattice of finite length can be represented by a projective space over any division ring. This result can be deduced from the point of view of reciprocals. The reciprocal of singular partitions is easily seen to be a Boolean algebra with the hyperplanes deleted. Such lattices can readily be represented in any n-dimensional projective space by considering an independent set of points and the lattice generated by them in $n + 1$-dimensional space. If one projects the lattice onto a hyperplane not containing any of the points using a center not contained in the lattice generated by the points, one gets a representation. By dualizing and taking meets, we must obtain the partition lattices because of the uniqueness theorem. As a reciprocal, a partition lattice cannot uniquely determine its inverse because over a field of characteristic 2 the diagonals of a plane quadralateral meet in a common point, while for other characteristics they do not. Continuation of the process of taking inverses leads to many different lattices having more than one inverse because the partition lattices can be represented by projective spaces over a division ring of any characteristic. The bond-lattices of linear graphs [4; p. 312] can also be represented by projective spaces over any division ring, and these also lead to lattices which have more than one inverse.

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