Rendiconti
del
Seminario Matematico
della
Università di Padova

H. M. Srivastava

**On Bessel, Jacobi and Laguerre polynomials**

*Rendiconti del Seminario Matematico della Università di Padova*, tome 35, nº 2 (1965), p. 424-432

<http://www.numdam.org/item?id=RSMUP_1965__35_2_424_0>

# ON BESSEL, JACOBI AND LAGUERRE POLYNOMIALS

*Nota di* H. M. SRIVASTAVA *) (*a Jodhpur*, India)

*Summary.* – Making use of two formulae of SRIVASTAVA, a number of Neumann type expansions involving Bessel, Jacobi and Laguerre polynomials are obtained.

**1.** Srivastava [5] has recently proved the formulae:

$$(1.1)\quad z^{\lambda}{}_pF_q\left[\begin{matrix}\alpha_1, \ldots, \alpha_p ; \\ \varrho_1, \ldots, \varrho_q ;\end{matrix}\ -a^2z^2\right]{}_PF_Q\left[\begin{matrix}\beta_1, \ldots, \beta_P ; \\ \sigma_1, \ldots, \sigma_Q ;\end{matrix}\ -b^2z^2\right] =$$
$$=\sum_{n=0}^{\infty}\frac{(\lambda+2n)\Gamma(\lambda+n)}{n!}J_{\lambda+2n}(2z)\cdot$$
$$\cdot F\left[\begin{matrix}-n, \lambda+n : \alpha_1, \ldots, \alpha_p ; & \beta_1, \ldots, \beta_P ; \\ - \quad : \varrho_1, \ldots, \varrho_q ; & \sigma_1, \ldots, \sigma_Q ;\end{matrix}\ a^2, b^2\right]$$

and

$$(1.2)\quad \left(\frac{1}{2}z\right)^{\mu+\nu}{}_pF_q\left[\begin{matrix}\alpha_1, \ldots, \alpha_p ; \\ \varrho_1, \ldots, \varrho_q ;\end{matrix}\ -a^2z^2\right]{}_PF_Q\cdot$$
$$\cdot\left[\begin{matrix}\beta_1, \ldots, \beta_P ; \\ \sigma_1, \ldots, \sigma_Q ;\end{matrix}\ -b^2z^2\right]\frac{\Gamma(\mu+\nu+1)}{\Gamma(\mu+1)\Gamma(\nu+1)} =$$

---

*) Pervenuta in redazione il 15 marzo 1965.
Indirizzo dell'A.: Department of Mathematics, The University, Jodhpur (India).

$$=\sum_{n=0}^{\infty}\frac{(\mu+\nu+2n)\Gamma(\mu+\nu+n)}{n!}J_{\mu+n}(z)J_{\nu+n}(z)\cdot$$

$$\cdot F\left|\begin{matrix}-n,\mu+1,\nu+1,\mu+\nu+n:\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_P;\\ \frac{1}{2}(\mu+\nu+1),\frac{1}{2}(\mu+\nu+2):\varrho_1,\ldots,\varrho_q;\sigma_1,\ldots,\sigma_Q;\end{matrix}\;a^2,\,b^2\right|,$$

where the notation for the double hypergeometric functions is due to Burchnall and Chaundy [3, p. 112] in preference, for the sake of brevity, to that introduced by Kampé De Fériet.

We note that when $p=P=Q-1=q-1=0$, which we shall call a 'confluent case' of the functions involved, the double series in (1.1) reduces to Appell's $F_4$ which can be further expressed as a product of two ${}_2F_1$'s [2, § 9.6], and by a suitable choice of $\lambda$ we have Bateman's well-known expansion [6, p. 370]. It is not difficult to show that (1.2) reduces to Carlitz's formula [4, p. 134] under similar restrictions on the integers $p$, $q$, $P$ and $Q$.

Now the hypergeometric series on the right of (1.1) is equal to:

$$\sum_{r=0}^{n}\frac{(-n)_r(\lambda+n)_r(\beta_1)_r\ldots(\beta)_r}{r!\,(\sigma_1)_r\ldots(\sigma_Q)_r}b^{2r}\cdot$$

$$\cdot\,{}_{p+2}F_q\left[\begin{matrix}-n+r,\lambda+n+r,\alpha_1,\ldots,\alpha_p;\\ \varrho_1,\ldots,\varrho_q;\end{matrix}\;a^2\right]$$

and this reduces to a ${}_{p+2}F_q$ when $b=0$. Therefore, if in (1.1) we set $b=0$ and change the notation slightly we get:

$$x^{\lambda}_{p}F_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p;\\ \varrho_1,\ldots,\varrho_q;\end{matrix}\;-x^2u^2\right]=\sum_{n=0}^{\infty}\frac{(\lambda+2n)\Gamma(\lambda+n)}{n!}J_{\lambda+2n}(2x)\cdot \tag{1.3}$$

$$\cdot\,{}_{p+2}F_q\left[\begin{matrix}-n,\lambda+n,\alpha_1,\ldots,\alpha_p;\\ \varrho_1,\ldots,\varrho_q;\end{matrix}\;u^2\right].$$

Similarly, (1.2) yields:

$$\left(\frac{1}{2}x\right)^{\mu+\nu}{}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p;\\ \varrho_1,\ldots,\varrho_q;\end{matrix}\;-x^2u^2\right]= \tag{1.4}$$

$$=\frac{\Gamma(\mu+1)(\Gamma\nu+1)}{(\mu+\nu)}\sum_{n=0}^{\infty}\frac{(\mu+\nu+2n)(\mu+\nu)_n}{n!}J_{\mu+n}(x)J_{\nu+n}(x)\cdot$$

$$\cdot\,{}_{p+4}F_{q+2}\left|\begin{matrix}-n,\mu+1,\nu+1,\mu+\nu+n,\alpha_1,\ldots,\alpha_p;\\ \frac{1}{2}(\mu+\nu+1),\frac{1}{2}(\mu+\nu+2),\varrho_1,\ldots,\varrho_q;\end{matrix}\;u^2\right|.$$

**2.** From the definition of the Bessel polynomials:

$$Y_n^{(\alpha)}(u) = {}_2F_0\left(-n,\ n+\alpha+1;\ -;\ \frac{1}{2}u\right)$$

we have:

$$Y_n^{(\alpha)}(u)Y_n^{(\alpha)}(-u) =$$

$$= \sum_{k=0}^{n} \frac{(-n)_k(\alpha+n+1)_k}{k!}\left(\frac{1}{2}u\right)^k {}_3F_2\left[\begin{matrix}-k,\ -n,\ \alpha+n+1;\\ 1+n-k,\ -\alpha-n-k\end{matrix}\right],$$

and on summing the well-poised terminating ${}_3F_2$ by Dixon's theorem [2, § 3.1] we see that:

$$Y_n^{(\alpha)}(u)Y_n^{(\alpha)}(-u) =$$

$$= {}_4F_1\left[\begin{matrix}-n,\ \alpha+n+1,\ \frac{1}{2}\alpha+\frac{1}{2},\ \frac{1}{2}\alpha+1;\\ \alpha+1;\end{matrix}\ u^2\right].$$

From (1.3) we therefore have:

$$x^\lambda\, {}_2F_1\left(\frac{1}{2}\lambda,\ \frac{1}{2}\lambda+\frac{1}{2};\ \lambda;\ -x^2u^2\right) = \tag{2.1}$$

$$= \sum_{n=0}^{\infty} \frac{(\lambda+2n)\Gamma(\lambda+n)}{n!} J_{\lambda+2n}(2x) Y_n^{(\lambda-1)}(u) Y_n^{(\lambda-1)}(u),$$

and (1.4) gives:

$$\frac{\left(\frac{1}{2}x\right)^{\mu+\nu}}{\Gamma(\mu+1)\Gamma(\nu+1)}\ \cdot \tag{2.2}$$

$$\cdot\, {}_4F_3\left[\begin{matrix}\frac{1}{2}(\mu+\nu),\ \frac{1}{2}(\mu+\nu+1),\ \frac{1}{2}(\mu+\nu+1),\ \frac{1}{2}(\mu+\nu+2);\\ \mu+1,\ \nu+1,\ \mu+\nu;\end{matrix}\ -x^2u^2\right] =$$

$$= \sum_{n=0}^{\infty} \frac{(\mu+\nu+2n)(\mu+\nu)_n}{n!\,(\mu+\nu)} Y_n^{(\mu+\nu-1)}(u) Y_n^{(\mu+\nu-1)}(-u) J_{\mu+n}(x) J_{\nu+n}(x).$$

Al-Salam and Carlitz [1, p. 157] proved the last formula in a different way.

Next we consider the Jacobi polynomials:

$$P_n^{(\alpha,\beta)}(z) = \binom{n+\alpha}{\alpha} {}_2F_1\left(-n,\ n+\alpha+\beta+1;\ \alpha+1;\ \frac{1-z}{2}\right),$$

$$P_n^{(\alpha,\beta)}(-z) = (-)^n P_n^{(\beta,\alpha)}(z);$$

and we find that:

$$P_n^{(\alpha,\beta)}(z) P_n^{(\alpha,\beta)}(-z) = (-)^n \binom{n+\alpha}{\alpha}\binom{n+\beta}{\beta} \cdot$$

$$\cdot\ {}_4F_3\left[\begin{matrix} -n,\ \alpha+\beta+n+1,\ \frac{1}{2}(\alpha+\beta+1),\ \frac{1}{2}(\alpha+\beta+2); \\ \alpha+1,\ \beta+1,\ \alpha+\beta+1; \end{matrix}\ (1-z^2)\right].$$

A special case of (1.3) is:

$$\text{(2.3)}\qquad \sum_{n=0}^{\infty} n!\,\frac{(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}{(\alpha+1)_n(\beta+1)_n}\cdot$$

$$\cdot\ J_{\alpha+\beta+2n+1}(2x) P_n^{(\alpha,\beta)}(z) P_n^{(\beta,\alpha)}(z) = x^{\alpha+\beta+1}\cdot$$

$$\cdot\ {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta+1),\ \frac{1}{2}(\alpha+\beta+2); \\ \alpha+1,\ \beta+1, \alpha+\beta+1; \end{matrix}\ x^2(z^2-1)\right],$$

and from (1.4) we similarly have:

$$\text{(2.4)}\qquad \left(\frac{1}{2}x\right)^{\mu+\nu}\cdot$$

$$\cdot\ {}_4F_5\left|\begin{matrix} \frac{1}{2}(\mu+\nu), \frac{1}{2}(\mu+\nu+1), \frac{1}{2}(\mu+\nu+1), \frac{1}{2}(\mu+\nu+2); \\ \mu+\frac{1}{2},\ \nu+\frac{1}{2},\ \mu+1,\ \nu+1,\ \mu+\nu\quad; \end{matrix}\ x^2(z^2-1)\right| =$$

$$= \frac{\Gamma(\mu+1)\Gamma(\nu+1)}{(\mu+\nu)} \sum_{n=0}^{\infty} n!\,\frac{(\mu+\nu+2n)(\mu+\nu)_n}{\left(\mu+\frac{1}{2}\right)_n\left(\nu+\frac{1}{2}\right)_n} J_{\mu+n}(x) J_{\nu+n}(x)\cdot$$

$$\cdot\ P_n^{\left(\mu-\frac{1}{2},\ \nu-\frac{1}{2}\right)}(z) P_n^{\left(\nu-\frac{1}{2},\ \mu-\frac{1}{2}\right)}(z).$$

An implication of (2.3) is the elegant formula:

$$(2.5)\quad xJ_\mu(xt)J_\nu(xt)=\left(\frac{1}{2}t\right)^{\mu+\nu}\sum_{n=0}^{\infty} n\,!\frac{(\mu+\nu+2n+1)\Gamma(\mu+\nu+n+1)}{\Gamma(\mu+n+1)\Gamma(\nu+n+1)}\cdot$$
$$\cdot\, J_{\mu+\nu+2n+1}(2x)\,P_n^{(\mu,\nu)}(z)\,P_n^{(\nu,\mu)}(z)\;,$$

where $t=\sqrt{(1-z^2)}$. It is not difficult to give a direct proof of (2.5).

Expansions involving product of two terminating ${}_1F_1$ 's can also be derived from (1.3) and (1.4). Thus we have:

$$(2.6)\quad x^\lambda\,{}_0F_3\left[-\;;\quad \lambda,\ \frac{1}{2}\lambda,\ \frac{1}{2}(\lambda+1);\quad -\frac{1}{4}x^2z^2\right]=$$
$$=\sum_{n=0}^{\infty}\frac{(\lambda+2n)\Gamma(\lambda+n)}{n\,!}J_{\lambda+2n}(2x){}_1F_1(-n;\ \lambda;\ z){}_1F_1(-n;\ \lambda;\ -z)\;,$$

$$(2.7)\quad \left(\frac{1}{2}x\right)^{\mu+\nu}{}_1F_4\left[\begin{array}{c} \frac{1}{2}(\mu+\nu+2)\;; \\ \mu+\nu,\ \frac{1}{2}(\mu+\nu),\ \mu+1,\ \nu+1; \end{array}\ -\frac{1}{4}x^2z^2\right]=$$
$$=\frac{\Gamma(\mu+1)\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)}\sum_{n=0}^{\infty}\frac{(\mu+\nu+2n)\Gamma(\mu+\nu+n)}{n\,!}J_{\mu+n}(x)J_{\nu+n}(x)\cdot$$
$$\cdot\,{}_1F_1(-n;\ \mu+\nu;\ z){}_1F_1(-n;\ \mu+\nu;\ -z)\;;$$

and since the Laguerre polynomial:

$$L_n^{(\alpha)}(z)=\binom{n+\alpha}{\alpha}{}_1F_1(-n;\ \alpha+1;\ z)\;,$$

from (2.6) and (2.7) we obtain:

$$(2.8)\quad \sum_{n=0}^{\infty}\frac{n\,!}{(\lambda)_n}(\lambda+2n)J_{\lambda+2n}(2x)L_n^{(\lambda-1)}(z)L_n^{(\lambda-1)}(-z)=$$
$$=x^\lambda\Gamma(\lambda){}_0F_3\left[-\;;\quad \lambda,\ \frac{1}{2}\lambda,\ \frac{1}{2}(\lambda+1);\quad -\frac{1}{4}x^2z^2\right]$$

and:

$$(2.9)\quad x^{\mu+\nu}{}_1F_4\left[\begin{array}{c} \frac{1}{2}(\mu+\nu+2)\;; \\ \mu+\nu,\ \frac{1}{2}(\mu+\nu),\ \mu+1,\ \nu+1; \end{array}\ -x^2z^2\right]=$$

$$= \frac{\Gamma(\mu+1)\Gamma(\nu+1)}{(\mu+\nu)} \sum_{n=0}^{\infty} n!\, \frac{(\mu+\nu+2n)}{(\mu+\nu)_n} J_{\mu+n}(2x) J_{\nu+n}(2x) \cdot$$

$$\cdot L_n^{(\mu+\nu-1)}(z) L_n^{(\mu+\nu-1)}(-z)$$

respectively.

We remark in passing that formulae involving product of two Gegenbauer polynomials or two Legendre polynomials are particular cases of (2.3), (2.4) and (2.5), and that from (2.1) and (2.2) we can deduce expansions involving product of two polynomials of the type:

$$y_n(z) = Y_n^{(0)}(z) .$$

**3.** Another confluent case of (1.3) is:

$$(3.1) \qquad x^{\nu+1} J_\mu(2ax) = a^\mu \sum_{n=0}^{\infty} \frac{(\mu+\nu+2n+1)\Gamma(\mu+\nu+n+1)}{n!} \cdot$$

$$\cdot J_{\mu+\nu+2n+1}(2x)\, {}_2F_1(-n, \mu+\nu+n+1; \mu+1; a^2) ,$$

and replacing the ${}_2F_1$ by Jacobi polynomial we have:

$$(3.2) \qquad x^{\nu+1} J_\mu(2ax) = a^\mu \sum_{n=0}^{\infty} n!\, \frac{(\mu+\nu+2n+1)\Gamma(\mu+\nu+n+1)}{(\mu+1)_n} \cdot$$

$$\cdot J_{\mu+\nu+2n+1}(2x) P_n^{(\mu,\nu)}(1-2a^2) .$$

In a similar way (1.4) gives [1, p. 153]:

$$(3.3) \qquad (ax)^{\mu+\nu-\lambda} J_\lambda(2ax) = (2a)^{\mu+\nu} \frac{\Gamma(\mu+1)\Gamma(\nu+1)}{\Gamma(\lambda+1)\Gamma(\mu+\nu+1)} \cdot$$

$$\cdot \sum_{n=0}^{\infty} \frac{(\mu+\nu+2n)\Gamma(\mu+\nu+n)}{n!} \cdot J_{\mu+n}(x) J_{\nu+n}(x) \cdot$$

$$\cdot {}_4F_3 \left[ \begin{matrix} -n, \mu+1, \nu+1, \mu+\nu+n \ ; \\ \lambda+1, \frac{1}{2}(\mu+\nu+1), \frac{1}{2}(\mu+\nu+2) ; \end{matrix} \ a^2 \right] .$$

In particular, (3.2) yields the Jacobi expansion [6, p. 22]:

$$\sin x = 2\pi \sum_{n=0}^{\infty} \frac{J_{2n+1}(x)}{\Gamma\left(\frac{1}{2}-n\right)\Gamma\left(\frac{1}{2}+n\right)}, \tag{3.4}$$

and a necessary consequence of (3.3) is the formula [4, p. 136]:

$$\frac{\sin 2ax}{2\pi a} = \sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right) J^2_{n+\frac{1}{2}}(x) P_n(1-2a^2), \tag{3.5}$$

which is obviously a special case of the Gegenbauer addition theorem [6, p. 362].

When $p = q = 2$, $u^2 = \frac{1}{2}a$, (1.4) yields:

$$\left(\frac{1}{2}x\right)^{\mu+\nu} {}_2F_2\left[\begin{matrix}\frac{1}{2}(\mu+\nu+1),\ \frac{1}{2}(\mu+\nu+2); \\ \mu+1,\ \nu+1 \quad ;\end{matrix}\ -\frac{1}{2}ax^2\right] = \tag{3.6}$$

$$= \frac{\Gamma(\mu+1)\Gamma(\nu+1)}{(\mu+\nu)} \sum_{n=0}^{\infty} \frac{(\mu+\nu+2n)(\mu+\nu)_n}{n!} \cdot$$

$$\cdot\ Y_n^{(\mu+\nu-1)}(a) J_{\mu+n}(x) J_{\nu+n}(x),$$

a formula that has been proved in [1] in a different way.

Also, for $p = q - 1 = 2$, (1.3) leads to:

$$x^{\mu+\nu} {}_2F_3\left[\begin{matrix}\frac{1}{2}(\mu+\nu+1),\ \frac{1}{2}(\mu+\nu+2); \\ \mu+1,\ \nu+1,\ \mu+\nu+1 \quad ;\end{matrix}\ -a^2x^2\right] = \tag{3.7}$$

$$= \sum_{n=0}^{\infty} \frac{(\mu+\nu+2n)\Gamma(\mu+\nu+n)}{n!} J_{\mu+\nu+2n}(2x) \cdot$$

$$\cdot\ {}_4F_3\left[\begin{matrix}-n,\ \mu+\nu+n,\ \frac{1}{2}(\mu+\nu+1),\ \frac{1}{2}(\mu+\nu+2); \\ \mu+1,\ \nu+1,\ \mu+\nu+1 \quad ;\end{matrix}\ a^2\right],$$

and similarly, from (1.4) we get:

$$(3.8) \qquad \left(\frac{1}{2}x\right)^{\mu+\nu} \cdot$$

$$\cdot {}_2F_3\left[\begin{matrix}\frac{1}{2}(\mu+\nu+1),\ \frac{1}{2}(\mu+\nu+2); \\ \mu+1,\ \nu+1,\ \mu+\nu+1 \quad ;\end{matrix} \quad -a^2x^2\right] =$$

$$= \frac{\Gamma(\mu+1)\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)} \sum_{n=0}^{\infty} \frac{(\mu+\nu+2n)\Gamma(\mu+\nu+n)}{n!} \cdot$$

$$\cdot J_{\mu+n}(x)J_{\nu+n}(x) \cdot {}_2F_1(-n,\ \mu+\nu+n; \quad \mu+\nu+1; \quad a^2) .$$

From (3.7) we have:

$$(3.9) \qquad J_\mu(ax)J_\nu(ax) = \frac{\left(\frac{1}{2}a\right)^{\mu+\nu}}{\Gamma(\mu+1)\Gamma(\nu+1)} \cdot$$

$$\cdot \sum_{n=0}^{\infty} \frac{(\mu+\nu+2n)\Gamma(\mu+\nu+n)}{n!} \cdot J_{\mu+\nu+2n}(2x) \cdot$$

$$\cdot {}_4F_3\left[\begin{matrix}-n,\ \mu+\nu+n,\ \frac{1}{2}(\mu+\nu+1),\ \frac{1}{2}(\mu+\nu+2); \\ \mu+1,\ \nu+1,\ \mu+\nu+1 \quad ;\end{matrix} \quad a^2\right]$$

and (3.8) readily gives:

$$(3.10) \qquad J_\mu(ax)J_\nu(ax) = a^{\mu+\nu} \cdot$$

$$\cdot \sum_{n=0}^{\infty} \frac{(\mu+\nu+2n)(\mu+\nu)_n}{n!\ (\mu+\nu)} J_{\mu+n}(x)J_{\nu+n}(x) \cdot$$

$$\cdot {}_2F_1(-n,\ \mu+\nu+n; \quad \mu+\nu+1; \quad a^2) ,$$

a formula proved by Carlitz [4].

## REFERENCES

[1] AL-SALAM W. A., CARLITZ L.: *Some expansions in products of Bessel functions*, Portugaliae Mathematica, vol. 22 (1963), pp. 153-160.

[2] BAILEY W. N.: *Generalised hypergeometric series*, Cambridge, 1935.

[3] BURCHNALL J. L., CHAUNDY T. W.: *Expansions of Appell's double hypergeometric functions II*, Quarterly Journal of Mathematics (Oxford), vol. 12 (1941), pp. 112-128.

[4] CARLITZ L.: *Some expansions in products of Bessel functions*, Quarterly Journal of Mathematics (Oxford) (2), vol. 13 (1962), pp. 134-136.

[5] SRIVASTAVA H. M.: *Some expansions in products of hypergeometric functions, to appear.*

[6] WATSON G. N.: *Theory of Bessel functions*, second edition, Cambridge and New York, 1944.