Rendiconti
del
Seminario Matematico
della
Università di Padova

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Rendiconti del Seminario Matematico della Università di Padova, tome 34 (1964), p. 200-203

<http://www.numdam.org/item?id=RSMUP_1964__34__200_0>

ANOTHER SAASCHÜTZIAN THEOREM FOR DOUBLE SERIES

Nota *) *di* L. CARLITZ (*a Durham*)

Put

$$S(m, n) = \sum_{r=0}^{m} \sum_{s=0}^{n} \frac{(-m)_r(-n)_s(\alpha)_{r+s}(\beta)_r(\beta')_s}{r\,!\,s\,!\,(\gamma)_{r+s}(\delta)_r(\delta')_s} .$$

The writer [1] has proved that if

$$(1) \qquad \begin{cases} \gamma = \beta + \beta', \\ \gamma + \delta = \alpha + \beta - m + 1, \\ \gamma + \delta' = \alpha + \beta' - n + 1, \end{cases}$$

then

$$(2) \qquad S = \frac{(\beta + \beta' - \alpha)_{m+n}(\beta')_m(\beta)_n}{(\beta + \beta')_{m+n}(\beta' - \alpha)_m(\beta - \alpha)_n} .$$

Moreover (2) is equaivalent to the transformation formula

$$(3) \qquad F_1(\gamma - \alpha;\ \beta', \beta;\ \gamma;\ x, y) = (1 - x)^{\beta' - \alpha}(1 - y)^{\beta - \alpha} F_1(\alpha;\ \beta, \beta';\ \gamma;\ x, y),$$

where $\gamma = \beta + \beta'$ and

$$F_1(\alpha;\ \beta, \beta';\ \gamma;\ x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{m\,!\,n\,!\,(\gamma)_{m+n}} x^m y_n .$$

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*) Pervenuto in Redazione il 31 ottobre 1963.
Indirizzo dell'A.: Depart. of Mathematics - Duke University - *Durham*, North Carolina, U.S.A.

It may be of interest to note another result somewhat similar to (2). Put

$$S(n) = \sum_{r+s\geqslant n} \frac{(-n)_{r+s}(\alpha)_r(\alpha')_s(\beta)_r(\beta')_s}{r\,!\,s\,!(\gamma)_{r+s}(\delta)_r(\delta')_s}.$$

Then

$$S(n) = \sum_{r=0}^{n} \frac{(-n)(\alpha)(\beta)_r}{r\,!\,(\gamma)_r(\delta)_r} \sum_{s=0}^{n-r} \frac{(-n+r)(\alpha')(\beta')_s}{s\,!\,(\gamma+r)_s(\delta')_s}.$$

If we assume that

$$\gamma + \delta' = \alpha' + \beta' - n + 1, \tag{4}$$

then inner sum is Saaschutzian and therefore equal to

$$\frac{(\gamma-\alpha'+r)_{n-r}(\gamma-\beta'+r)_{n-r}}{(\gamma+r)_{n-r}(\gamma-\alpha'-\beta'+r)_{n-r}} =$$

$$= \frac{(\gamma-\alpha')_n(\gamma-\beta')_n}{(\gamma)_n(\gamma-\alpha'-\beta')_n} \frac{(\gamma)_r(\gamma-\alpha'-\beta')_r}{(\gamma-\alpha')_r(\gamma-\beta')_r}.$$

It follows that

$$S_n = \frac{(\gamma-\alpha')_n(\gamma-\beta')_n}{(\gamma)_n(\gamma-\alpha'-\beta')_n} \sum_{r=0}^{n} \frac{(-n)_r(\alpha)_r(\beta)_r(\gamma-\alpha'-\beta')_r}{r\,!\,(\gamma-\alpha')_r(\gamma-\beta')_r(\delta)_r}.$$

If we assume that

$$\gamma = \beta + \beta', \tag{5}$$

the last sum becomes

$$\sum_{r=0}^{n} \frac{(-n)_r(\alpha)_r(\beta-\alpha')_r}{r\,!\,(\gamma-\alpha')_r(\delta)_r}.$$

If in addition we take

$$\gamma + \delta = \alpha + \beta - n + 1, \tag{6}$$

the sum reduces to

$$\frac{(\gamma-\alpha-\alpha')_n(\gamma-\beta)_n}{(\gamma-\alpha')_n(\gamma-\alpha-\beta)_n},$$

so that finally

$$(7) \qquad S(n) = \frac{(\beta)_n(\beta')_n(\gamma - \alpha - \alpha')_n}{(\gamma)_n(\beta - \alpha')_n(\beta' - \alpha)_n},$$

provided

$$(8) \qquad \begin{cases} \gamma = \beta + \beta' \\ \gamma + \delta = \alpha + \beta - n + 1, \\ \gamma + \delta' = \alpha' + \beta' - n + 1 . \end{cases}$$

If $\gamma = \beta + \beta'$ and λ is arbitrary, it follows from (7) that

$$\sum_{n=0}^{\infty} \frac{(\beta)_n(\beta')_n(\gamma - \alpha - \alpha')_n}{n\,!\,(\gamma)_n(\lambda)_n} x^n = \sum_{n=0}^{\infty} \frac{(\beta - \alpha')_n(\beta' - \alpha)_n}{n\,!\,(\lambda)_n} x^n,$$

$$\sum_{r+s\geqslant n} \frac{(-n)_{r+s}(\alpha)_r(\alpha')_s(\beta)_r(\beta')_s}{r\,!\,s\,!\,(\gamma)_{r+s}(\alpha - \beta' - n + 1)_r(\alpha' - \beta - n + 1)_s}.$$

Since

$$(\alpha - \beta' - n + 1)_r = (-1)^r \frac{(\beta' - \alpha)_n}{(\beta' - \alpha)_{n-r}},$$

the right member reduces to

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{x^n} \sum_{r+s\leqslant n} (-1)^{r+s} \frac{(\alpha)_r(\alpha')_s(\beta)_r(\beta')_s}{(n - r - s)\,!\,r\,!\,s\,!\,(\gamma)_{r+s}} \cdot$$

$$\cdot (\beta' - \alpha)_{n-r}(\beta - \alpha')_{n-s} =$$

$$= \sum_{r,s=0}^{\infty} \frac{(\alpha)_r(\alpha')_s(\beta)_r(\beta')_s(\beta - \alpha')_r(\beta' - \alpha)_s}{r\,!\,s\,!\,(\gamma)_{r+s}(\lambda)_{r+s}} x^{r+s} \cdot$$

$$\cdot \sum_{n=0}^{\infty} \frac{(\beta - \alpha' + r)_n(\beta' - \alpha + s)_n}{n\,!\,(\lambda + r + s)_n} x^n .$$

We have therefore

$$(9) \qquad {}_3F_2 \begin{bmatrix} \beta, \beta', \gamma - \alpha - \alpha';\ x \\ \gamma, \lambda \end{bmatrix} =$$

$$= \sum_{rs=0}^{\infty} \frac{(\alpha)_r(\alpha')_s(\beta)_r(\beta')_s(\beta - \alpha')_r(\beta' - \alpha)_s}{r\,!\,s\,!\,(\gamma)_{r+s}(\lambda)_{r+s}} x^{r+s} \cdot$$

$$\cdot F[\beta - \alpha' + r, \beta' - \alpha + s;\ \lambda + r + s;\ x],$$

where $\gamma = \beta + \beta'$ and λ is arbitrary. Alternatively this can be written in the form

(10) $${}_3F_2\left[\frac{\beta, \beta', \gamma - \alpha - \alpha' ;\ x}{\gamma, \lambda}\right] =$$

$$= \sum_{n=0}^{\infty} \frac{(\beta - \alpha')_n(\beta' - \alpha)_n}{n\,!(\lambda)_n} \,.$$

$$\cdot \sum_{r,\,s=0}^{\infty} \frac{(\alpha)_r(\alpha')_s(\beta)_r(\beta')_s(\beta - \alpha' + n)_r(\beta' - \alpha + n)_s}{r\,!\,s\,!\,(\gamma)_{r+s}(\lambda + n)_{r+s}} x^{r+s} \,.$$

REFERENCE

[1] Carlitz L.: *A Saalschützian theorem for double series*. Journal ot the London Mathematical Society, Vol. 38 (1963), 415-418.