S. K. CHATTERJEA

On a generalization of Laguerre polynomials

<http://www.numdam.org/item?id=RSMUP_1964__34__180_0>

ON A GENERALIZATION OF LAGUERRE POLYNOMIALS

Nota *) *di* S. K. CHATTERJEA (*a Calcutta*)

1. – In a recent paper [1], the writer has defined the polynomials $T_{kn}^{(\alpha)}(x)$ by the Rodrigues' formula

$$(1.1) \qquad T_{kn}^{(\alpha)}(x) = \frac{1}{n!} \, x^{-\alpha} e^{x^k} D^n (x^{\alpha + n} e^{-x^k}) \, ,$$

where k is a natural number. The polynomials $T_{kn}^{(\alpha)}(x)$ are of exactly degree kn $(n = 0, 1, 2, ...)$. They satisfy the operational formula

$$(1.2) \qquad \prod_{j=1}^{n} (xD - kx^k + \alpha + j) = n! \sum_{r=0}^{n} \frac{x^r}{r!} T_{k(n-r)}^{(\alpha + r)}(x) D^r \, .$$

The following are the consequences of the operational formula (1.2):

$$(1.3) \qquad n T_{kn}^{(\alpha)}(x) = (xD - kx^k + \alpha + n) T_{k(n-1)}^{(\alpha)}(x)$$

$$(1.4) \qquad \binom{m+n}{m} T_{k(m+n)}^{(\alpha)}(x) = \sum_{r=0}^{\min(m,n)} \frac{x^r}{r!} T_{k(m-p)}^{(\alpha + n + r)}(x) D^r T_{kn}^{(\alpha)}(x) \, .$$

The polynomials $T_{kn}^{(\alpha)}(x)$ are generated by the function

$$(1.5) \qquad (1 - t)^{-\alpha - 1} \exp [x^k u(t)] = \sum_{n=0}^{\infty} T_{kn}^{(\alpha)}(x) t^n$$

where
$$u(t) = 1 - (1-t)^{-k}.$$

In the same paper the writer has also proved the following properties

$$\sum_{r=0}^{k+1} (-1)^r \binom{k+1}{r} (n+1-r) T^{(\alpha)}_{k(n+1-r)}(x)$$

$$(1.6) \qquad = (\alpha + 1) \sum_{r=0}^{k} (-1)^r \binom{k}{r} T^{(\alpha)}_{k(n-r)}(\dot{x}) - kx^k T^{(\alpha)}_{kn}(x)$$

$$(1.7) \quad \sum_{r=0}^{k} (-1)^r \binom{k}{r} D T^{(\alpha)}_{k(n-r)}(x) = kx^{k-1} \sum_{r=1}^{k} (-1)^r \binom{k}{r} T^{(\alpha)}_{k(n-r)}(x)$$

$$(1.8) \qquad T'^{(\alpha)}_{kn}(x) = \sum_{r=0}^{n} \frac{(\alpha - \beta)_r}{r!} T^{(\beta)}_{k(n-r)}(x)$$

This work of the writer generalizes some properties of the Laguerre polynomials $L^{(\alpha)}_n(x)$. Indeed, when $k = 1$, $T^{(\alpha)}_{kn}(x) \equiv \equiv L^{(\alpha)}_n(x)$. A similar generalization viz., $T^{(0)}_{kn}(x)$, has been previously studied by Palas [2]. The purpose of this paper is to discuss a more general class of Laguerre polynomials.

2. DEFINITION: We first make the definition

$$(2.1) \qquad T^{(\alpha)}_{kn}(x, p) = \frac{1}{n!} x^{-\alpha} e^{px^k} D^n(x^{\alpha+n} e^{-px^k})$$

where k is a natural number.

We now show that the polynomial $T^{(\alpha)}_{kn}(x, p)$ is of exactly degree kn $(n = 0, 1, 2, ...)$. In this connection we know the result [3], for which I must thank Prof. H. W. Gould:

$$(2.2) \qquad D^s_x(z) = \sum_{k=0}^{s} \frac{(-1)^k}{k!} D^k_z f(z) \sum_{j=0}^{k} (-1)^j \binom{k}{j} z^{k-j} D^s_x z^j$$

Thus we obtain from (2.1)

$$T'^{(\alpha)}_{kn}(x, p) =$$

$$= \frac{1}{n!} x^{-\alpha} e^{px^k} \sum_{s=0}^{n} \binom{n}{s} (D^{n-s} x^{\alpha+n})(D^s e^{-px^k}) =$$

$$= \frac{1}{n!} e^{pz^k} \sum_{s=0}^{n} \binom{n}{s} \binom{\alpha+n}{n-s} (n-s)! \, x^s (D^s e^{-pz^k})$$

$$= \sum_{s=0}^{n} \binom{\alpha+n}{n-s} \sum_{i=0}^{s} \frac{p^i}{i!} x^{ki} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \binom{kj}{s}$$

$$= \sum_{i=0}^{n} \frac{p^i}{i!} x^{ki} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \sum_{s=i}^{n} \binom{\alpha+n}{n-s} \binom{kj}{s}$$

Now we know that

$$\sum_{j=0}^{i} (-1)^j \binom{i}{j} \sum_{s=0}^{i-1} \binom{\alpha+n}{n-s} \binom{kj}{s} = 0 \, .$$

Thus we finally obtain

$$(2.3) \qquad T_{kn}^{(\alpha)}(x, p) = \sum_{i=0}^{n} \frac{p^i}{i!} x^{ki} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \binom{\alpha+n+kj}{n} \, ,$$

which is the explicit formula for $T_{kn}^{(\alpha)}(x, p)$.

In particular, when $k = 1$, and $p = 1$, we derive

$$(2.4) \qquad T_n^{(\alpha)}(x, 1) = \sum_{i=0}^{n} \frac{x^i}{i!} \sum_{j=0}^{i} (-1)^i \binom{i}{j} \binom{\alpha+n+j}{n} =$$

$$= \sum_{i=0}^{n} \frac{x^i}{i!} \cdot (-1)^i \binom{\alpha+n}{n-i}$$

which is the explicit formula for the general Laguerre polynomials $L_n^{(\alpha)}(x)$. Thus $T_n^{(\alpha)}(x, 1) \equiv L_n^{(\alpha)}(x)$.

3. – OPERATIONAL FORMULAE: Recently we [4] have derived the general operational formula

$$(3.1) \qquad x^{-\alpha} D^n(x^{kn+\alpha} Y) = \prod_{j=1}^{n} \{x^{k-1}(z + \alpha + kj)\} Y \, ,$$

$$(k = 1, 2, 3, \dots),$$

where $z \equiv x D$ and Y is any sufficiently differentiable function of x. The operators on the right of (3.1) commute only when $k = 1$.

Thus we derive

$$(3.2) \quad x^{-\alpha}e^{px^k}D^n(x^{\alpha+n}e^{-px^k}Y) = \prod_{j=1}^{n}(xD - pkx^k + \alpha + j)Y$$

Again we observe

$$D^n(x^{\alpha+n}e^{-px^k}Y) = \sum_{r=0}^{n}\binom{n}{r}D^{n-r}(x^{\alpha+n}e^{-px^k})D^rY =$$

$$= n!\,x^{\alpha}e^{-px^k}\sum_{r=0}^{n}\frac{x^r}{r!}T_{k(n-r)}^{(\alpha+r)}(x,p)D^rY,$$

whence we obtain

$$(3.3) \quad \frac{1}{n!}x^{-\alpha}e^{px^k}D^n(x^{\alpha+n}e^{-px^k}Y) = \sum_{r=0}^{n}\frac{x^r}{r!}T_{k(n-r)}^{(\alpha+r)}(x,p)D^rY$$

It therefore follows from (3.2) and (3.3) that

$$(3.4) \quad \prod_{j=1}^{n}(xD - pkx^k + \alpha + j)Y = n!\sum_{r=0}^{n}\frac{x^r}{r!}T_{k(n-r)}^{(\alpha+r)}(x,p)D^rY$$

If we set $Y = 1$, we derive from (3.4)

$$(3.5) \quad n!\,T_{kn}^{(\alpha)}(x,p) = \prod_{j=1}^{n}(xD - pkx^k + \alpha + j)\cdot 1$$

Further if $k = 1$, and $p = 1$, we obtain from (3.4)

$$(3.6) \quad \prod_{j=1}^{n}(xD - x + \alpha + j)Y = n!\sum_{r=0}^{n}\frac{x^r}{r!}T_{n-r}^{(\alpha+r)}(x,1)D^rY;$$

which may be compared with the operational formula for the general Laguerre polynomials, derived by Carlitz [5].

In a recent paper [6], Gould and Hopper have generalized the Hermite polynomials by the definition

$$(3.7) \quad H_n^k(x,\alpha,p) = (-1)^n x^{-\alpha}e^{px^k}D^n(x^{\alpha}e^{-px^k})$$

We remark that $x^n H_n^k(x, \alpha, p)$ yeilds a generalized class of polynomials of exactly degree kn ($n = 0, 1, 2, \ldots$), provided k is a natural number. Consequently if we write

$$x^n H_n^k(x, \alpha, p) = H_{kn}^{(\alpha)}(x, p)$$

then

(3.8) $$H_{kn}^{(\alpha)}(x, p) = (-1)^n x^{n-\alpha} e^{pz^k} D^n(x^\alpha e^{-pz^k})$$

Thus the polynomials $H_{kn}^{(\alpha)}(x, p)$ are related to our polynomials by

(3.9) $$H_{kn}^{(\alpha)}(x, p) = (-1)^n n! \, T_{kn}^{(\alpha-n)}(x, p).$$

Now returning to the operational formula (3.5) we obtain

(3.10) $$(-1)^n H_{kn}^{(\alpha)}(x, p) = \prod_{j=1}^{n} (xD - pkx^k + \alpha - n + j) \cdot 1$$

More generally we have from (3.4)

(3.11) $$\prod_{j=1}^{n} (xD - pkx^k + \alpha - n + j) Y =$$

$$= \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} x^r H_{k(n-r)}^{(\alpha)}(x, p) D^r Y$$

This operational formula viz., (3.11) seems to be of particular interest. Indeed, using $k = 2$, $p = 1$, and $\alpha = 0$, we have

$$\prod_{j=1}^{n} (xD - 2x^2 - n + j) Y =$$

$$= \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} x^r H_{2(n-r)}^{(0)}(x, 1) D^r Y$$

Now noticing that

$$H_{2(n-r)}^{(0)}(x, 1) = x^{n-r} H_{n-r}(x),$$

where $H_n(x)$ denotes the ordinary Hermite polynomials defined by

$$H_n(x) = (-1)^n e^{x^2} D^n e^{-x^2},$$

we obtain

$$\prod_{j=1}^{n} (xD - 2x^2 - n + j) Y = x^n \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} H_{n-r}(x) D^r Y$$

Now we note that

(3.12) $$x^{-n} \prod_{j=1}^{n} (xD - 2x^2 - n + j) \equiv (D - 2x)^n.$$

For, (3.12) is evidently true for $n = 1$. Next assume that (3.12) is true for $n = m$. Then we have

$$x^{-(m+1)} \prod_{j=1}^{m+1} (xD - 2x^2 - m - 1 + j) =$$

$$= x^{-(m+1)}(xD - 2x^2 - m) \prod_{j=1}^{m} (xD - 2x^2 - m + j) =$$

$$= x^{-(m+1)}(xD - 2x^2 - m)x^m(D - 2x)^m =$$

$$= x^{-(m+1)} \cdot x^m(xD - 2x^2)(D - 2x)^m =$$

$$= (D - 2x)^{m+1}.$$

Hence by induction (3.12) is true for all positive integers n. Thus we finally derive

(3.13) $$(D - 2x)^n = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} H_{n-r}(x) D^r,$$

a formula which Burchnall [7] derived some years ago.

4. – SOME APPLICATIONS OF THE OPERATIONAL FORMULA: From (3.5) we note that

(4.1) $$n T_{kn}^{(\alpha)}(x, p) = (xD - pkx^k + \alpha + n) T_{k(n-1)}^{(\alpha)}(x, p).$$

In particular, when $k = 1$, and $p = 1$, we derive

(4.2) $$n T_{n}^{(\alpha)}(x, 1) = (xD - x + \alpha + n) T_{n-1}^{(\alpha)}(x, 1)$$

which is well-known for the Laguerre polynomials $L_n^{(\alpha)}(x)$.

Again in terms of the polynomials of Gould and Hopper, (4.1) stands thus

(4.3) $H^{(\alpha)}_{kn}(x, p) + (xD - pkx^k + \alpha)H^{(\alpha-1)}_{k(n-1)}(x, p) = 0.$

Next we consider

$(m + n)! \, T^{(\alpha)}_{k(m+n)}(x, p) =$

$$= \prod_{j=1}^{m} (xD - pkx^k + \alpha + n + j) \prod_{i=1}^{n} (xD - pkx^k + \alpha + i) \cdot 1$$

$$= n! \prod_{j=1}^{m} (xD - pkx^k + \alpha + n + j) \cdot T^{(\alpha)}_{kn}(x, p)$$

$$= m! \, n! \sum_{r=0}^{m} \frac{x^r}{r!} T^{(\alpha+n+r)}_{k(m-r)}(x, p) D^r T^{(\alpha)}_{kn}(x, p);$$

which implies that

(4.4) $\displaystyle \binom{m + n}{m} T^{(\alpha)}_{k(m+n)}(x, p) =$

$$= \sum_{r=0}^{\min(m,n)} \frac{x^r}{r!} T^{(\alpha+n+r)}_{k(m-r)}(x, p) D^r T^{(\alpha)}_{kn}(x, p).$$

The formula (4.4) readily yields the corresponding formula for the polynomials of Gould and Hopper:

(4.5) $H^{(\alpha+m+n)}_{k(m+n)}(x, p)$

$$= \sum_{r=0}^{\min(m,n)} (-1)^r \binom{m}{r} x^r H^{(\alpha+m+n)}_{k(m-r)}(x, p) D^r H^{(\alpha+n)}_{kn}(x, p).$$

5. – GENERATING FUNCTION: We shall now show that the polynomials $T^{(\alpha)}_{kn}(x, p)$ are generated by

(5.1) $\displaystyle g(x, t) = (1 - t)^{-\alpha-1} \exp [px^k u(t)] = \sum_{n=0}^{\infty} T^{(\alpha)}_{kn}(x, p)t^n,$

where

$$u(t) = 1 - (1 - t)^{-k}.$$

From the definition (2.1) we observe

$$(5.2) \qquad T_{kn}^{(\alpha)}(x,\, p) = e^{px^k} \sum_{r=0}^{\infty} \frac{(-p)^r}{r!} \binom{kr + \alpha + n}{n} x^{kr}.$$

It may be noted that (2.3) is a consequence of (5.2).

Now we notice that

$$(5.3) \qquad T_{kn}^{(\alpha)}(x,\, p) = \frac{1}{n!} \left[\frac{\partial^n}{\partial t^n} g(x,\, 0) \right]$$

Also

$$(5.4) \quad \left[\frac{\partial^n}{\partial t^n} \{(1 - t)^{-\alpha-1} \exp\,(px^k u(t))\} \right]_{t=0}$$

$$= e^{px^k} \left[\frac{\partial^n}{\partial t^n} \left\{ (1 - t)^{-\alpha-1} \exp\left(-p \left(\frac{x}{1-t} \right)^k \right) \right\} \right]_{t=0}$$

$$= n!\, e^{px^k} \sum_{r=0}^{\infty} \frac{(-p)^r}{r!} \binom{kr + \alpha + n}{n} x^{kr}$$

Thus a comparison of (5.3) and (5.4) with (5.2) confirms (5.1).

Now from the generating function (5.1) we easily derive the following multiplication formula:

$$(5.5) \qquad T_{kn}^{(\alpha)}(x m^{1/k},\, p) = T_{kn}^{(\alpha)}(x,\, mp)\,,$$

which, in terms of the polynomials of Gould and Hopper, shapes into

$$(5.6) \qquad H_{kn}^{(\alpha)}(x m^{1/k},\, p) = H_{kn}^{(\alpha)}(x,\, mp)\,,$$

which may well be compared with (3.9) of [6, p. 54].

It is also interesting to note from (5.5) that

$$(5.7) \qquad T_{n}^{(\alpha)}(x,\, m) = L_{n}^{(\alpha)}(mr)\,.$$

Again we observe

$$(1 - t)^{-\alpha-1} \exp\,[px^k\{1 - (1 - t)^{-k}\}]$$

$$= (1 - t)^{-(\alpha-\beta)}(1 - t)^{-\beta-1} \exp\,[px^k\{1 - (1 - t)^{-k}\}]$$

whence we obtain

$$\sum_{n=0}^{\infty} T_{kn}^{(\alpha)}(x, p)t^n = (1 - t)^{-(\alpha-\beta)} \sum_{n=0}^{\infty} T_{kn}^{(\beta)}(x, p)t^n .$$

Now comparing the coefficients of t^n on both sides we get

$$(5.8) \qquad T_{kn}^{(\alpha)}(x, p) = \sum_{r=0}^{n} \frac{(\alpha - \beta)_r}{r!} T_{k(n-r)}^{(\beta)}(x, p) ,$$

where α and β are arbitrary real numbers.

Next we notice that

$$\sum_{n=0}^{\infty} T_{kn}^{(\alpha+\beta+1)}(x, p+q)t^n$$

$$= (1 - t)^{-\alpha-1} e^{pzk\{1-(1-t)^{-k}\}} \cdot (1 - t)^{-\beta-1} e^{qzk\{1-(1-t)^{-k}\}}$$

$$= \sum_{m=0}^{\infty} T_{km}^{(\alpha)}(x, p)t^m \cdot \sum_{n=0}^{\infty} T_{kn}^{(\beta)}(x, q)t^n$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} T_{km}^{(\alpha)}(x, p) T_{k(n-m)}^{(\beta)}(x, q)t^n .$$

Thus we obtain the following ' doubly-additive ' addition formula

$$(5.9) \qquad T_{kn}^{(\alpha+\beta+1)}(x, p+q) = \sum_{m=0}^{n} T_{km}^{(\alpha)}(x, p) T_{k(n-m)}^{(\beta)}(x, q) .$$

In particular, when $p = q = 1$, and $k = 1$, we derive

$$(5.10) \qquad T_n^{(\alpha+\beta+1)}(x, 2) = \sum_{m=0}^{n} L_m^{(\alpha)}(x) L_{n-m}^{(\beta)}(x) .$$

It follows therefore from (5.7) and (5.10) that

$$(5.11) \qquad L_n^{(\alpha+\beta+1)}(2x) = \sum_{m=0}^{n} L_m^{(\alpha)}(x) L_{n-m}^{(\beta)}(x) ,$$

which is implied by the well-known formula of the Laguerre polynomials

$$(5.12) \qquad L_n^{(\alpha+\beta+1)}(x + y) = \sum_{m=0}^{n} L_m^{(\alpha)}(x) L_{n-m}^{(\beta)}(y) \, .$$

Again returning to (5.1) we obtain

$$(1 - t)^{k+1} \frac{\partial g(x, t)}{\partial t} = [(\alpha + 1)(1 - t)^k - pkx^k] g(x, t)$$

whence we notice

$$(1 - t)^{k+1} \sum_{n=1}^{\infty} n t^{n-1} T_{kn}^{(\alpha)}(x, p) = [(\alpha + 1)(1 - t)^k - pkx^k] \cdot \sum_{n=0}^{\infty} T_{kn}^{(\alpha)}(x, p) t^n \, .$$

Performing the indicated multiplication on both sides and comparing coefficients of t^n on both sides, we derive

$$(5.13) \qquad \sum_{r=0}^{k+1} (- 1)^r \binom{k + 1}{r} (n + 1 - r) T_{k(n+1-r)}^{(\alpha)}(x, p) \, .$$

$$= (\alpha + 1) \sum_{r=0}^{k} (- 1)^r \binom{k}{r} T_{k(n-r)}^{(\alpha)}(x, p) - pkx^k T_{kn}^{(\alpha)}(x, p) \, .$$

Lastly we observe

$$(1 - t)^k \frac{\partial g(x, t)}{\partial x} = pkx^{k-1} \{(1 - t)^k - 1\} g(x, t) \, ,$$

whence we obtain in like manner

$$(5.14) \qquad \sum_{r=0}^{k} (- 1)^r \binom{k}{r} D T_{(n-r)}^{(\alpha)}(x, p)$$

$$= pkx^{k-1} \sum_{r=1}^{k} (- 1)^r \binom{k}{r} T_{(n-r)}^{(\alpha)}(x, p) \, .$$

13

REFERENCES

[1] S. K. CHATTERJEA: *A generalization of Laguerre polynomials*, Communicated to the Canad. Jour. Math.

[2] FRANCK J. PALAS: *A Rodrigues' formula*, Amer. Math. Monthly, Vol. 66 (1959), pp. 402-404.

[3] I. J. SCHWATT: *An introduction to the operations with series*, Philadelphia, 1924.

[4] S. K. CHATTERJEA: *Operational formulae for certain classical polynomials*, I, to appear in the Quart. Jour. Math. (Oxford).

[5] L. CARLITZ: *A note on Laguerre polynomials*, Michigan Math. Jour. Vol. 7 (1960), pp. 219-223.

[6] H. W. GOULD, and A. T. HOPPER: *Operational formulas connected with two generalizations of Hermite polynomials*, Duke Math. Jour. Vol. 29 (1962), pp. 51-64.

[7] J. L. BURCHNALL: *A note on the polynomials of Hermite*, Quart. Jour. Math. (Oxford), Vol. 12 (1941), pp. 9-11.