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A note on definite integrals involving the derivatives of hypergeometric polynomials

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A NOTE ON DEFINITE INTEGRALS INVOLVING THE DERIVATIVES OF HYPERGEOMETRIC POLYNOMIALS

Nota (*) *di* JYOTI CHAUDHURI (*a Calcutta*)

Chatterjee [1] (1957) evaluated certain definite integrals involving Legendre polynomials and Popov [2] (1959) gave an alternative shorter proof of one of the results proved by Chatterjee and also gave a generalisation of one of the results of Chatterjee. The object of this present note is to evaluate certain definite integrals involving derivatives of Jacobi polynomials and Laguerre polynomials and the results obtained by the above two authors can be easily obtained as particular cases from the results given here.

1. - *We shall first evaluate the integral*

$$\int_{-1}^{1} D^p P_l(x)\ D^r P_m^{(\alpha,\beta)}(x) \cdot D^s P_n^{(\alpha,\beta)}(x) dx$$

where D^p stands for $\dfrac{d^p}{dx^p}$, and $P_l(x)$, $P_n^{(\alpha,\beta)}(x)$ $(\alpha > -1, \beta > -1)$ denote Legendre polynomial and Jacobi polynomial respectively.

Taking $D^r P_m^{(\alpha,\beta)}(x) D^s P_n^{(\alpha,\beta)}(x)$ as the first function and $D^p P_l(x)$ as the second function we integrate by parts and thus we obtain

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from the well-known formula

$$\int_{-1}^{1} u v^{(n)} dx = \Big| \sum_{k=1}^{n} (-1)^{k-1} u^{(k-1)} v^{(n-k)} \Big|_{-1}^{1} + (-1)^n \int_{-1}^{1} v u^{(n)} dx$$

that

$$\int_{-1}^{1} D^p P_l(x) D^r P_m^{(\alpha,\beta)}(x) D^s P_n^{(\alpha,\beta)}(x) dx$$

$$= \Big| \sum_{k=1}^{p} (-1)^{k-1} D^{k-1}\{D^r P_m^{(\alpha,\beta)}(x) D^s P_n^{(\alpha,\beta)}(x)\} D^{p-k} P_l(x) \Big|_{-1}^{1}$$

$$+ (-1)^p \int_{-1}^{1} P_l(x) \cdot D^p\{D^r P_m^{(\alpha,\beta)}(x) D^s P_n^{(\alpha,\beta)}(x)\} dx \,.$$

Now, by applying Leibnitz's theorem for the nth derivative of the product of two functions to $D^p \{D^r P_m^{(\alpha,\beta)}(x) D^s P_n^{(\alpha,\beta)}(x)\}$ and noting that the values of $D^m P_n^{(\alpha,\beta)}(x)$ at $x = 1$ and $x = -1$ as given by Carlitz (1954), can be written as

$$(1) \qquad | D^m P_n^{(\alpha,\beta)}(x) |_{x=1} = \frac{m!}{2^m} \binom{n+m+\alpha+\beta}{n+\alpha+\beta} \binom{n+\alpha}{n-m}$$

$$(2) \qquad | D^m P_n^{(\alpha,\beta)}(x) |_{x=-1} = \frac{(-1)^{n-m} m!}{2^m} \binom{n+m+\alpha+\beta}{n+\alpha+\beta} \binom{n+\beta}{n-m}$$

we get

$$\int_{-1}^{1} D^p P_l(x) D^r P_m^{(\alpha,\beta)}(x) D^s P_n^{(\alpha,\beta)}(x) dx$$

$$= 2^{1-p-r-s} \sum_{k=1}^{p} \sum_{\nu=0}^{k-1} (-1)^{k-1} \binom{k-1}{\nu} \binom{m+r+k-\nu-1+\alpha+\beta}{r+k-\nu-1} \cdot$$

$$\cdot \binom{n+s+\nu+\alpha+\beta}{s+\nu} \binom{l+p-k}{p-k} \binom{l}{p-k} \cdot$$

$$(1.1)\quad (s+\nu)!\,(r+k-\nu-1)!\,(p-k)!\left\{\binom{m+\alpha}{m-r-k+\nu+1}\cdot\right.$$

$$\left.\cdot\binom{n+\alpha}{n-s-\nu}+(-1)^{l+m+n-p-r-s}\binom{m+\beta}{m-r-k+\nu+1}\binom{n+\beta}{n-s-\nu}\right\}$$

if $l > (m-r)+(n-s)-p$, for in that case the integral vanishes,
and

$$= 2^{1-p-r-s}\sum_{k=1}^{p}\sum_{\nu=0}^{k=1}(-1)^{k-1}\binom{k-1}{\nu}\binom{m+r+k-\nu-1+\alpha+\beta}{r+k-\nu-1}\cdot$$

$$\cdot\binom{n+s+\nu+\alpha+\beta}{s+\nu}\binom{l+p-k}{p-k}\binom{l}{p-k}(s+\nu)!\,\cdot$$

$$\cdot(r+k-\nu-1)!\,(p-k)!\left\{\binom{m+\alpha}{m-r-k+\nu+1}\binom{n+\alpha}{n-s-\nu}+\right.$$

$$\left.+(-1)^{l+m+n-p-r-s}\binom{m+\beta}{m-r-k+\nu+1}\binom{n+\beta}{n-s-\nu}\right\}$$

$$(1.2)\quad +\frac{(-1)^{p}2^{l-m-n+1}\{\Gamma(l+1)\}^{2}\Gamma(2m+\alpha+\beta+1)\Gamma(2n+\alpha+\beta+1)\Gamma(p+1)}{\Gamma(2l+2)\Gamma(m+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}$$

$$\cdot\sum_{\nu=0}^{p}\frac{1}{\Gamma(\nu+1)\Gamma(p-\nu+1)\Gamma(m-r-p+\nu+1)\Gamma(n-s-\nu+1)}$$

if

$$l=(m-r)+(n-s)-p\,.$$

If in (1.1) we put $\alpha=\beta=0$, we get the result given by Popov (1959). And if we put $\alpha=\beta=0$ & $n=s=1$ so that $\nu=0$ and then changing p to s & l to n, (1.1) will reduce to the result (4.1) of S. K. Chatterjee (1957).

2. - Next we shall evaluate

$$I=\int_{-1}^{1}\prod_{r=1}^{k}D^{m_r}P_{n_r}^{(\alpha,\beta)}(x)\cdot DP_n(x)\,dx$$

On integrating by parts we get

$$I = \left| \prod_{r=1}^{k} D^{m_r} P_{n_r}^{(\alpha,\beta)}(x) \cdot P_n(x) \right|_{-1}^{1} - \int_{-1}^{1} D^{m_1+1} P_{n_1}^{(\alpha,\beta)}(x) \cdot \prod_{r=2}^{k} D^{m_r} P_{n_r}^{(\alpha,\beta)}(x) \cdot P_n(x) dx$$

$$- \int_{-1}^{1} D^{m_1} P_{n_1}^{(\alpha,\beta)}(x) \cdot D^{m_2+1} P_{n_2}^{(\alpha,\beta)}(x) \cdot \prod_{r=3}^{k} D^{m_r} P_{n_r}^{(\alpha,\beta)}(x) \cdot P_n(x) dx$$

$$- \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$- \int_{-1}^{1} \prod_{r=1}^{k-1} D^{m_r} P_{n_r}^{(\alpha,\beta)}(x) \cdot D^{m_k+1} P_{n_k}^{(\alpha,\beta)}(x) \cdot P_n(x) \cdot dx \,. \tag{2.1}$$

If $n > n_1 + n_2 + \ldots + n_k - m_1 - m_2 - \ldots - m_k - 1$, then all the integrals vanish. Then by applying (1) & (2), we get from (2.1)

$$I = \prod_{r=1}^{k} \frac{\Gamma(n_r + m_r + \alpha + \beta + 1)}{2^{m_r} \Gamma(n_r + \alpha + \beta + 1) \Gamma(n_r - m_r + 1)} \cdot \left\{ \prod_{r=1}^{k} \frac{\Gamma(n_r + \alpha + 1)}{\Gamma(m_r + \alpha + 1)} - (-1)^n \prod_{r=1}^{k} \frac{(-1)^{n_r - m_r} \Gamma(n_r + \beta + 1)}{\Gamma(m_r + \beta + 1)} \right\} \tag{2.2}$$

where

$$n > n_1 + n_2 + \ldots + n_k - m_1 - m_2 - \ldots - m_k - 1 \,.$$

The result (5.3) of Chatterjee's paper becomes a particular case of this and is obtained by putting $\alpha = \beta = 0$, $m_r = 1$ $(r = 1, 2 \ldots k)$ and $n_k = 1$ (so that $DP_1(x) = 1$).

If $n = n_1 + n_2 + \ldots + n_k - m_1 - m_2 - \ldots - m_k - 1$ the part of the integrand of each integral, leaving $P_n(x)$ aside, is a polynomial of degree $n_1 + n_2 + \ldots + n_k - m_1 - m_2 - \ldots - m_k - 1$, i · e · , n, and hence by the orthogonal property of $P_n(x)$ we can evaluate them. So, in this case we have

$$(2.3)\qquad \int_{-1}^{1}\prod_{r=1}^{k} D^{m_r}P_{n_r}^{(\alpha,\beta)}(x)\cdot DP_n(x)dx =$$

$$=\prod_{r=1}^{k}\frac{\Gamma(n_r+m_r+\alpha+\beta+1)}{2^{m_r}\Gamma(n_r+\alpha+\beta+1)\Gamma(n_r-m_r+1)}\left[\frac{\Gamma(n_r+\alpha+1)}{\Gamma(m_r+\alpha+1)}+\right.$$

$$\left.+\frac{\Gamma(n_r+\beta+1)}{\Gamma(m_r+\beta+1)}\right]-\frac{2^{n+1}(n!)^2(n+1)}{\Gamma(2n+2)}\prod_{r=1}^{k}\frac{\Gamma(2n_r+\alpha+\beta+1)}{2^{n_r}\Gamma(n_r+\alpha+\beta+1)\Gamma(n_r-m_r+1}$$

where

$$n = n_1 + n_2 + \ldots + n_k - m_1 - m_2 - \ldots - m_k - 1\ .$$

Putting $\alpha = \beta = 0$ we get the corresponding result in $P_n(x)$. Similarly, if $L_n^{(\alpha)}(x)$ denotes Laguerre polynomial, we get

$$(2.4)\qquad \int_{0}^{\infty} e^{-x}\prod_{r=1}^{k} D^{m_r}L_{n_r}^{(\alpha_r)}(x)\cdot DL_n(x)dx$$

$$= \left| e^{-x}\prod_{r=1}^{k} D^{m_r}L_{n_r}^{(\alpha_r)}(x)\cdot L_n(x)\right|_0^\infty + \int_0^\infty e^{-x}\prod_{r=1}^{k} D^{m_r}L_{n_r}^{(\alpha_r)}(x)\cdot L_n(x)\cdot dx$$

$$-\int_0^\infty e^{-x}D^{m_1+1}L_{n_1}^{(\alpha_1)}(x)\cdot\prod_{r=2}^{k} D^{m_r}L_{n_r}^{(\alpha_r)}(x)\cdot L_n(x)dx$$

$$-\int_0^\infty e^{-x}D^{m_1}L_{n_1}^{(\alpha_1)}(x)\cdot D^{m_2+1}L_{n_2}^{(\alpha_2)}(x)\cdot\prod_{r=3}^{k} D^{m_r}L_{n_r}^{(\alpha_r)}(x)\cdot L_n(x)dx$$

$$-\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$-\int_0^\infty e^{-x}\prod_{r=1}^{k-1} D^{m_r}L_{n_r}^{(\alpha_r)}(x)\cdot D^{m_k+1}L_{n_k}^{(\alpha_k)}(x)\cdot L_n(x)\cdot dx\ .$$

Since $\underset{x\to\infty}{Lt}\ e^{-x}x^n = 0$, it follows that $\underset{x\to\infty}{Lt}\ [e^{-x}\prod_{r=1}^{k} D^{m_r}L_{n_r}^{(\alpha_r)}(x)\cdot$ $\cdot L_n(x)] = 0$ for $\prod_{r=1}^{k} D^{m_r}L_{n_r}^{(\alpha_r)}(x)\cdot L_n(x)$ is a polynomial of degree $n + n_1 + n_2 + \ldots + n_k - m_1 - m_2 - \ldots - m_k$. Also $D^m L_n^{(\alpha)}(0) =$

$= (-1)^m \binom{n+\alpha}{n-m}$. Hence the first term of the right hand side of (2.4) vanishes at infinity.

Now, we shall consider two cases:

Case 1: $n > n_1 + n_2 + ... + n_k - m_1 - m_2 - ... - m_k$.

Case 2: $n = n_1 + n_2 + ... + n_k - m_1 - m_2 - ... - m_k$.

For *case* 1, all the integrals on the right hand side of (2.4) are zero. Hence

$$(2.5)\quad \int_0^\infty e^{-x} \prod_{r=1}^{k} D^{m_r} L_{n_r}^{(\alpha_r)}(x) \cdot DL_n(x)dx = -\prod_{r=1}^{k} \frac{(-1)^{m_r}\Gamma(n_r+\alpha_r+1)}{\Gamma(n_r-m_r+1)\Gamma(m_r+\alpha_r+1)}$$

where

$$n > n_1 + n_2 + ... + n_k - m_1 - m_2 - ... - m_k .$$

For, *case* 2, i.e., when $n = n_1 + n_2 + ... + n_k - m_1 - m_2 - - ... - m_k$, the first integral on the right hand side of (2.4) is non-zero and all the other integrals vanish. Now, evaluating the first integral by the orthogonal property of Laguerre polynomial, we find

$$(2.6)\quad \int_0^\infty e^{-x} \prod_{r=1}^{k} D^{m_r} L_{n_r}^{(\alpha_r)}(x) \cdot DL_n(x)dx = (-1)^{m_1+m_2+...+m_k} \cdot$$

$$\cdot \left[\prod_{r=1}^{k} \frac{1}{\Gamma(n_r - m_r + 1)}\right] \left[\Gamma(n+1) - \prod_{r=1}^{k} \frac{\Gamma(n_r + \alpha_r + 1)}{\Gamma(m_r + \alpha_r + 1)}\right]$$

where

$$n = n_1 + n_2 + ... + n_k - m_1 - m_2 - ... - m_k .$$

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[3] Carlitz L.: Bulletin of the Calcutta Mathematical Society, vol. 46, 1954, p. 94.