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Notes on a formula of Carlitz

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NOTES ON A FORMULA OF CARLITZ

Nota (*) *di* S. K. CHATTERJEA (*a Calcutta*)

1. - In a previous note [1]), Carlitz has proved the formula

$$(1.1)\quad \left(\frac{\text{Sin}\,\beta}{\text{Sin}\,\alpha}\right)^n P_n^{\lambda+1/2}(\cos\alpha) = \\ = \sum_{r=0}^{n} \binom{2\lambda+n}{r} \left[\frac{\text{Sin}\,(\beta-\alpha)}{\text{Sin}\,\alpha}\right]^r P_{n-r}^{\lambda+1/2}(\cos\beta)$$

where $P_n^\lambda(x)$ denotes the ultraspherical polynomial of degree n. For $\lambda = 0$, we get the formula of Rainville [2]);

$$(1.2)\quad \left(\frac{\text{Sin}\,\beta}{\text{Sin}\,\alpha}\right)^n P_n(\cos\alpha) = \sum_{r=0}^{n} \binom{n}{r} \left[\frac{\text{Sin}\,(\beta-\alpha)}{\text{Sin}\,\alpha}\right]^r P_{n-r}(\cos\beta)$$

where $P_n(x)$ denotes the Legendre polynomial of degree n.

Following the method of Carlitz, one can obtain

$$(1.3)\quad (\cot\alpha\ \text{Sin}\,\beta)^n \Phi_n(\tan\alpha) = \\ = \sum_{r=0}^{n} \binom{n}{r} \left[\frac{\text{Sin}\,(\beta-\alpha)}{\text{Sin}\,\alpha}\right]^r \cos^{n-r}\beta\,\Phi_{n-r}(\tan\beta)$$

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1) CARLITZ, L., Bull. Cal. Math. Soc., 51 (1959), pp. 132-133.

2) RAINVILLE, E. D., Bull. Amer. Math. Soc., 51 (1945), pp. 268-271.

and

(1.4) $(\cot\alpha \operatorname{Sin}\beta)^n L_n(\tan\alpha) =$

$$= \sum_{r=0}^{n} \binom{n}{r} \left[\frac{\operatorname{Sin}(\beta-\alpha)}{\operatorname{Sin}\alpha}\right]^r \cos^{n-r}\beta L_{n-r}(\tan\beta)$$

$$\alpha \neq (2n+1)\pi/2, \quad \beta \neq (2n+1)\pi/2; \quad n = 0, \pm 1, \pm 2, \ldots$$

where $\Phi_n(x)$ denotes the polynomial of degree n, encountered by Karle (2, p. 269) in the study of the contribution to electron scattering of a freely rotating group within molecules comprising a jet of gas, and $L_n(x)$ denotes the Laguerre polynomial of degree n and of order zero.

The generating function

$$e^t I_0(tx) = \sum_{n=0}^{\infty} \Phi_n(x) \frac{t^n}{n!} \tag{1.5}$$

for the polynomials $\Phi_n(x)$ is obtained by Rainville (2, p 269). From (1.5) we have

$$e^{t\cos\beta} I_0(tx\cos\beta) = \sum_{n=0}^{\infty} \Phi_n(x) \frac{t^n}{n!} \cos^n\beta \tag{1.6}$$

$\therefore$

$$\sum_{r=0}^{\infty} \frac{t^r}{r!} \left[\frac{\operatorname{Sin}(\beta-\alpha)}{\operatorname{Sin}\alpha}\right]^r \sum_{n=0}^{\infty} \Phi_n(x) \frac{t^n}{n!} \cos^n\beta = \tag{1.7}$$
$$= e^{t \operatorname{Sin}\beta \cos\alpha/\operatorname{Sin}\alpha} I_0(tx\cos\beta)$$

Writing $\beta = \alpha$ and $t = t \operatorname{Sin}\beta/\operatorname{Sin}\alpha$ in (1.6) we obtain

$$\sum_{n=0}^{\infty} \Phi_n(x) \left(\frac{\operatorname{Sin}\beta}{\operatorname{Sin}\alpha}\right)^n \frac{t^n}{n!} \cos^n\alpha = e^{t\operatorname{Sin}\beta\cos\alpha/\operatorname{Sin}\alpha} I_0(tx\cot\alpha \operatorname{Sin}\beta) \tag{1.8}$$

Again using $x = x \cot\alpha \tan\beta$ in (1.7) we get

$$\sum_{r=0}^{\infty} \frac{t^r}{r!} \left[\frac{\operatorname{Sin}(\beta-\alpha)}{\operatorname{Sin}\alpha}\right]^r \sum_{n=0}^{\infty} \Phi_n(x\cot\alpha\tan\beta) \frac{t^n}{n!} \cos^n\beta = \tag{1.9}$$
$$= e^{t\operatorname{Sin}\beta\cos\alpha/\operatorname{Sin}\alpha} I_0(tx\cot\alpha \operatorname{Sin}\beta)$$

Comparing (1.8) and (1.9) we have on equating coefficients,

$$(1.10)\quad (\cot \alpha \operatorname{Sin} \beta)^n \Phi_n(x) = \\ = \sum_{r=0}^{n} \binom{n}{r} \left[\frac{\operatorname{Sin}(\beta-\alpha)}{\operatorname{Sin}\alpha}\right]^r \cos^{n-r}\beta \Phi_{n-r}(x \cot \alpha \tan \beta)$$

Thus we get finally

$$(\cot \alpha \operatorname{Sin} \beta)^n \Phi_n(\tan \alpha) = \\ = \sum_{r=0}^{n} \binom{n}{r} \left[\frac{\operatorname{Sin}(\beta-\alpha)}{\operatorname{Sin}\alpha}\right]^r \cos^{n-r}\beta \Phi_{n-r}(\tan \beta)$$

$$\alpha \neq (2n+1)\pi/2\,,\quad \beta \neq (2n+1)\pi/2\,;\quad n=0,\ \pm 1,\ \pm 2, \ldots$$

Secondly, starting with the generating function [3])

$$e^t J_0[2(tx)^{1/2}] = \sum_{r=0}^{\infty} L_n(x) \frac{t^n}{n!}$$

we can prove, in precisely the same way, that

$$(\cot \alpha \operatorname{Sin} \beta)^n L_n(\tan \alpha) = \\ = \sum_{r=0}^{n} \binom{n}{r} \left[\frac{\operatorname{Sin}(\beta-\alpha)}{\operatorname{Sin}\alpha}\right]^r \cos^{n-r}\beta\, L_{n-r}(\tan \beta)$$

$$\alpha \neq (2n+1)\pi/2\,,\quad \beta \neq (2n+1)\pi/2\,;\quad n=0,\ \pm 1,\ \pm 2, \ldots$$

2. - Special cases of the formulae just proved:

Putting $x = i \tan \alpha$ in (1.10) and observing that (2, p 269)

$$\Phi_n(x) = (1-x^2)^{n/2} P_n[(1-x^2)^{-1/2}]\,,$$

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[3]) RAINVILLE, E. D., Bull. Amer. Math. Soc., 51 (1945), pp. 266-267.

we get

$$\left(\frac{\text{Sin }\beta}{\text{Sin }\alpha}\right)^n P_n(\cos\alpha) = \sum_{r=0}^{n}\binom{n}{r}\left[\frac{\text{Sin }(\beta-\alpha)}{\text{Sin }\alpha}\right]^r P_{n-r}(\cos\beta),$$

$$\because \Phi_n(i\tan\alpha) = \sec^n\alpha P_n(\cos\alpha)\,.$$

which in (1.2).

Next using $\beta = 2\alpha$ we derive from (1.3)

$$(2.1)\quad (2\cos^2\alpha)^n\Phi_n(\tan\alpha) = \sum_{r=0}^{n}\binom{n}{r}\cos^{n-r}2\alpha\ \Phi_{n-r}(\tan 2\alpha)$$

Further using $\cos 2\alpha = x$, (2.1) can be put in the form:

$$(2.2)\quad (1+x)^n\Phi_n\left(\sqrt{\frac{1-x}{1+x}}\right) = \sum_{r=0}^{n}\binom{n}{r}x^{n-r}\Phi_{n-r}(\sqrt{1-x^2}/x)$$

In like manner, we get from (1.4)

$$(2.3)\quad (1+x)^n L_n\left(\sqrt{\frac{1-x}{1+x}}\right) = \sum_{r=0}^{n}\binom{n}{r}x^{n-r}L_{n-r}(\sqrt{1-x^2}/x)$$

The polynomials $\Phi_n(x)$ are defined by (2, p 269)

$$\Phi_n(x) = \frac{1}{\pi}\int_0^\pi (1 + x\cos\theta)^n d\theta\,.$$

$$(2.4)\quad \therefore (1+x)^n\Phi_n\left(\sqrt{\frac{1-x}{1+x}}\right) =$$

$$= \sum_{k=0}^{n}\binom{n}{k}x^k\cdot\frac{1}{\pi}\int_0^\pi\left(1+\frac{\sqrt{1-x^2}}{x}\cos\theta\right)^k d\theta\cdot$$

$$= \frac{1}{\pi}\int_0^\pi\left\{\sum_{k=0}^{n}\binom{n}{k}(x+\sqrt{1-x^2}\cos\theta)^k\right\}d\theta\cdot$$

$$= \frac{1}{\pi}\int_0^\pi[(1+x)+\sqrt{1-x^2}\cos\theta]^k d\theta\,,$$

which is analogous to the result [4]) for the Legendre polynomials.

3. - Some Definite Integrals:

when $\beta = 2\alpha$ and $\cos 2\alpha = x$, (1.1) can be put in the form:

$$(3.1)\quad 2^{n/2}(1+x)^{n/2}P_n^{\lambda+1/2}\left(\sqrt{\frac{1+x}{2}}\right) = \sum_{k=0}^{n}\binom{2\lambda+n}{n-k}P_k^{\lambda+1/2}(x)$$

From (3.1) and from the orthogonal property of ultraspherical polynomials we easily obtain

$$(3.2)\quad \int_{-1}^{1}(1+x)^{n/2}(1-x^2)^{\lambda}P_n^{\lambda+1/2}\left(\sqrt{\frac{1+x}{2}}\right)P_r^{\lambda+1/2}(x)dx$$

$$= 0, \quad r > n$$

$$= 2^{-(n+4\lambda)/2}\binom{2\lambda+n}{n-r}\pi\cdot\frac{\Gamma(2\lambda+r+1)}{\left(\lambda+r+\frac{1}{2}\right)\Gamma(r+1)\left[\Gamma\left(\lambda+\frac{1}{2}\right)\right]^2},$$

$$0 \leqslant r \leqslant n \ (\lambda > -1)$$

For $\lambda = 0$, we obtain

$$(3.3)\quad \int_{-1}^{1}(1+x)^{n/2}P_n\left(\sqrt{\frac{1+x}{2}}\right)P_r(x)dx$$

$$= 0, \quad r > n$$

$$= 2^{-(n-2)/2}\binom{n}{r}\Big/(2r+1), \quad 0 \leqslant r \leqslant n.$$

which was established by Bhonsle (4, p 9).

[4]) BHONSLE, B. R., Ganita., 8 (1957), pp. 9-16.

4. - Recently Rainville (3, p 267) has obtained the following expression for the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ as a series of products of Laguerre polynomials of orders α and β and of different arguments:

$$(4.1)\quad P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} (-)^k \frac{\Gamma(n+\beta+1)\Gamma(n+\alpha+1)}{\Gamma(k+\beta+1)\Gamma(n-k+\alpha+1)} \cdot L_k^{(\beta)}\left(\frac{1+x}{2}\right) L_{n-k}^{(\alpha)}\left(\frac{1-x}{2}\right)$$

We at once obtain from (4.1)

$$(4.2)\quad P_n^{\lambda}(\cos 2\theta) \equiv P_n^{(\lambda-1/2,\lambda-1/2)}(\cos 2\theta)\cdot g_n$$

$$= \sum_{k=0}^{n} (-)^k \frac{\left[\Gamma\left(n+\lambda+\frac{1}{2}\right)\right]^2 \cdot g_n}{\Gamma\left(k+\lambda+\frac{1}{2}\right)\Gamma\left(n-k+\lambda+\frac{1}{2}\right)} \cdot L_k^{(\lambda-1/2)}(\cos^2\theta) L_{n-k}^{(\lambda-1/2)}(\mathrm{Sin}^2\theta)$$

$$\text{where } g_n = \frac{(2\lambda)_n}{(\lambda+1/2)_n}; \qquad (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}.$$

Again from (1.1) we obtain

$$(4.3)\quad (2\cos\alpha)^n P_n^{\lambda}(\cos\alpha) = \sum_{k=0}^{n} \binom{2\lambda+n-1}{r} P_{n-r}^{\lambda}(\cos 2\alpha)$$

$$\therefore\ P_n^{\lambda}(\cos 2\theta) = (2\cos 2\theta)^{-n} \sum_{k=0}^{n} \binom{2\lambda+n-1}{k} P_{n-k}^{\lambda}(\cos 4\theta).$$