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TERM RANK OF 0, 1 MATRICES

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1. Introduction.

Let A be a matrix with n rows and m columns, all of whose entries are $0's$ and $1's$. Let the sum of row i of A be denoted by r_i $(i = 1, \dots, n)$, and let the sum of colum j be denoted by s_j $(j = 1, \dots, m)$. With the matrix A we associate the row sum vector

$$(1.1) \qquad R = (r_1, \dots, r_n)$$

and the column sum vector

$$(1.2) \qquad S = (s_1, \dots, s_m).$$

Let $\delta_i = (1, \dots, 1, 0, \dots, 0)$ be a vector of m components with $1's$ in the first r_i positions and $0's$ elsewhere. A matrix of row sum vector R of the form

$$(1.3) \qquad \bar{A} = \begin{pmatrix} \delta_1 \\ \vdots \\ \delta_n \end{pmatrix}$$

is called *maximal*. Throughout the discussion

$$(1.4) \qquad R' = (r_1', \dots, r_m')$$

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designates the column sum vector of $\bar{A}$. Similarly let

$$\varepsilon_i = \begin{pmatrix} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

be a vector of n components with $1's$ in the first s_i positions and $0's$ elsewhere. Then

$$A^* = (\varepsilon_1, \dots, \varepsilon_m) \tag{1.5}$$

has column sum vector S. The notation

$$S' = (s_1', \dots, s_n') \tag{1.6}$$

designates the row sum vector of A^*. Note that

$$\sum_{i=1}^{n} r_i = \sum_{i=1}^{m} r_i'$$

are conjugate partitions. Also

$$\sum_{i=1}^{m} s_i = \sum_{i=1}^{n} s_i'$$

are conjugate partitions. Moreover, the components of R' and S' always appear in descending order.

Let $U = (u_1, \dots, u_q)$ and $V = (v_1, \dots, v_q)$ be two vectors with integral components. We write

$$U \lhd V$$

or

$$V \rhd U$$

if

$$u_1 + u_2 + \dots + u_i \leq v_1 + v_2 + \dots + v_i \quad (i = 1, \dots, q-1). \tag{1.7}$$

$$u_1 + u_2 + \dots + u_q = v_1 + v_2 + \dots + v_q . \tag{1.8}$$

If, furthermore, (1.7) and (1.8) still hold when the components of U and V are reordered so that they are in nonincreasing order, we say that U is *majorized* by V, written

$$U \prec V$$

or

$$V \succ U.$$

We are now in a position to state the existence theorem for 0, 1 matrices having row sum vector R and column sum vector S [1; 3]. We give a new proof in Section 2.

EXISTENCE THEOREM. Let $R = (r_1, \ldots, r_n)$ and $S = (s_1, \ldots, s_m)$ be two vectors with nonnegative integral components. Then there exists a matrix A of size $n \times m$ with entries $0's$ and $1's$ with row sum vector R and column sum vector S if and only if

$$S' \succ R.$$

The *term rank* ρ of the 0, 1 matrix A is the order of the greatest minor of A with a non zero term in its determinant expansion. This integer is also equal to the minimal number of rows and columns that collectively contain all of the non zero elements of A [2]. Let $\mathfrak{A}$ be the class of 0, 1 matrices with row sum vector $R = (r_1, \ldots, r_m)$ and column sum vector $S = (s_1, \ldots, s_n)$. Notationally we write

$$\mathfrak{A}(R, S)$$

In [4] Ryser has found a formula for $\bar{\rho}$, the maximal term rank for matrices in $\mathfrak{A}(R, S)$. In Section 3 we derive an algorithm for finding $\tilde{\rho}$, the minimal term rank of matrices in $\mathfrak{A}(R, S)$. Unfortunately a simple formula for $\tilde{\rho}$, analogous to the formula for ρ, does not appear to be forthcoming. In Section 4 we give a method for constructing matrices of maximal term rank ρ. Sections 3 and 4 comprise the main portion of our paper.

Consider the 2×2 submatrices of A of the types

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

An *interchange* is a transformation of the elements of A that changes a minor of type A_1 into type A_2 or vice versa and leaves all other elements of A unaltered. By a theorem of Ryser [3], *if A and A^* are two elements in the class $\mathfrak{A}(R, S)$, then A is transformable into A^* by a sequence of interchanges.* We give a different proof of this interchange theorem in Section 2. Suppose now that an element $a_{uv} = 1$ of A is such that no sequence of interchanges applied to A replaces $a_{uv} = 1$ by 0. Then a_{uv} is called an *invariant* 1 of A. By the interchange theorem it is an invariant 1 of the class $\mathfrak{A}(R, S)$. In our concluding Section 5 we obtain a formula for finding which 1's of a class $\mathfrak{A}(R, S)$ are invariant 1's.

2. Existence And Interchange Theorems.

In this section we give new proofs of the existence and interchange theorems described in Section 1. We begin with the following:

LEMMA 2.1. Let $U \lhd V$. If U can be transformed into a vector with nonincreasing components by successively interchanging two adjacent components which differ by 1, then $U \prec V$.

PROOF: We may suppose V has nonincreasing components for this does not upset the hypothesis $U \lhd V$. Let $U = (u_1, \dots, u_q)$ and $V = (v_1, \dots, v_q)$ $(v_1 \geq v_2 \geq \dots \geq v_q)$. Suppose that $u_j = u_{j-1} + 1$. We assert that if we interchange these, the new vector $\bar{U}$ will satisfy $\bar{U} \lhd V$. For if not,

$$u_1 + u_2 + \dots + u_{j-2} \leq v_1 + \dots + v_{j-2}, \tag{2.1}$$

$$u_1 + u_2 + \dots + u_{j-1} = v_1 + \dots + v_{j-1}, \tag{2.2}$$

$$u_1 + u_2 + \dots + u_j \leq v_1 + \dots + v_j. \tag{2.3}$$

Then (2.1) and (2.2) imply

$$u_j > u_{j-1} \geq v_{j-1}. \tag{2.4}$$

But (2.2) and (2.3) imply

$$u_j \leq v_j. \tag{2.5}$$

This contradicts the assertion V is nonincreasing, and Lemma 2.1 follows.

EXISTENCE THEOREM. Let $R = (r_1, \dots, r_n)$ and $S = (s_1, \dots, s_m)$ $r_1 \geq \dots \geq r_n$. The proof is by induction on m. For $m = 1$ the there exists a matrix A of sixe $n \times m$ with entries $0's$ and $1's$ with row sum vector R and column sum vector S if and only if

$$S' \succ R.$$

PROOF: For the necessity see [3]. We may suppose $r_1 \geq \dots \geq r_n$. The proof is by induction on m. For $m = 1$ the theorem is clear. Suppose the theorem is true if S has $m - 1$ components. Let $s_m = t$. Define

$$R_1 = (r_1 - 1, \dots, r_t - 1, r_{t+1}, \dots, r_n),$$

$$S_1 = (s_1, \dots, s_{m-1}).$$

Now the number of positive components of R is $\geq$ the number of positive components of S'. For otherwise we could not have $R \prec S'$. Also the number of positive components of S' equals the largest s_i. This implies that R_1 has nonnegative components. Now $S_1' - R_1 = S' - R$ so that

$$S_1' \rhd R_1. \tag{2.6}$$

Since R_1 is transformable into a vector with nonincreasing components by successively interchanging two adjacent elements which differ by 1, by Lemma 2.1, $S_1' \succ R_1$, and the class $\mathfrak{A}(R_1, S_1)$ is nonempty by induction. Adjoining the column vector

$$\begin{pmatrix} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

with t $1's$ in the initial positions to an element of $\mathfrak{A}(R_1, S_1)$ gives an element of $\mathfrak{A}(R, S)$.

INTERCHANGE THEOREM. Let A and A^* be two $n \times m$ matrices in the class $\mathfrak{A}(R, S)$. Then A^* is transformable into A by a finite number of interchanges.

PROOF: Let $r_1 \geq ... \geq r_n$, $s_1 \geq ... \geq s_m$. By a finite sequence of interchanges, $1's$ may be shifted to the left in the first row until they occupy the first r_1 positions. Now applying the same argument to the matrix with sum vector $R_1 = (r_2, ..., r_n)$ and column sum vector $S_1 = (s_1 - 1, ..., s_{r_1} - 1, s_{r_1+1}, ..., s_m)$, we can put $1's$ in the r_2 columns where S_1 has the largest components. Continuing in this manner we see that there are two sequences of interchanges, one taking A into a matrix Z and the other taking A^* into Z. Suppose that the intermediate matrices taking A into Z are $A_1, ..., A_q$. Then since there is an interchange taking Z into A_q and one taking A_q into A_{q-1}, etc., there is a sequence of interchanges taking Z into A_1. Hence there is a sequence of interchanges taking Z into A and a sequence taking A^* into A.

3. An Algorithm for $\tilde{\rho}$.

Let $\mathfrak{A} = \mathfrak{A}(R, S)$ be the class of 0,1 matrices with row sum vector $R = (r_1, ..., r_n)$ and column sum vectors $S = (s_1, ..., s_m)$. Let A be in $\mathfrak{A}$ and let

$$A = \begin{pmatrix} A_i^u \\ A_{n-i}^l \end{pmatrix} \tag{3.1}$$

where A_i^u denotes the upper i rows of A and A_{n-i}^l denotes the lower $n-i$ rows of A. Let the column sum vector of A_i^u be denoted by S_i^u, the row sum vector of A_i^u by R_i^u, and similarly S_{n-i}^l and R_{n-i}^l will denote respectively the column sum vector and the row sum vector of A_{n-i}^l.

LEMMA 3.1. Let A be an element of $\mathfrak{A}(R, S)$ and let

$$A = \begin{pmatrix} A_i^u \\ A_{n-i}^l \end{pmatrix}.$$

Then $S_i^u \prec (R_i^u)'$, $S_{n-i}^l \prec (R_{n-i}^l)'$, and $S_i^u + S_{n-i}^l = S$. Conversely let $S_j^u \prec (R_j^u)'$, $S_{n-j}^l \prec (R_{n-j}^l)'$, and $S_j^u + S_{n-j}^l = S$, where the components of the vectors are nonnegative integers. Then there exists an A_j^u with row sum vector R_j^u, column sum vector S_j^u, and an A_{n-j}^l with row sum vector R_{n-j}^l, column sum vector S_{n-j}^l such that

$$A = \begin{pmatrix} A_j^u \\ A_{n-j}^l \end{pmatrix}$$

is an element of $\mathfrak{A}$.

PROOF: This is an immediate consequence of the existence theorem.

LEMMA 3.2. If $(a_1, \ldots, a_r) < (b_1, \ldots, b_r)$, $(c_1, \ldots, c_s) < (d_1, \ldots, d_s)$, and $U = (a_1, \ldots, a_r, c_1, \ldots, c_s)$, $V = (b_1, \ldots, b_r, d_1, \ldots, d_s)$, then $U < V$.

PROFF: We many assume of course that $a_1 \geq \ldots \geq a_r$, $b_1 \geq \ldots \geq b_r$, $c_1 \geq \ldots \geq c_s$, $d_1 \geq \ldots \geq d_s$. Clearly

$$\sum_{i=1}^{r} a_i + \sum_{i=1}^{s} c_i = \sum_{i=1}^{r} b_i + \sum_{i=1}^{s} d_i .$$

Suppose the h largest components of U are $a_1, \ldots, a_\alpha, c_1, \ldots, c_\beta$, and the h largest of V are $b_1, \ldots, b_\gamma, d_1, \ldots, d_\delta$. Here $h = \alpha + \beta = \gamma + \delta$ and $h = 1, \ldots, r + s - 1$. Then

$$\sum_{i=1}^{\alpha} a_i + \sum_{i=1}^{\beta} c_i \leq \sum_{i=1}^{\alpha} b_i + \sum_{i=1}^{\beta} d_i \leq \sum_{i=1}^{\gamma} b_i + \sum_{i=1}^{\delta} d_i . \tag{3.2}$$

Hence $U < V$.

LEMMA 3.3. If $(a_1, \ldots, a_r) < (b_1, \ldots, b_r)$, then $(a_1, \ldots, a_r, c_1, \ldots, c_s) < (b_1, \ldots, b_r, c_1, \ldots, c_s)$.

PROOF: This is a special case of Lemma 3.2.

LEMMA 3.4. If $a \geq b+j$ and $j \geq 0$, then $(a-j, b+j) < (a, b)$.

PROOF: This is immediate from the definition of $<$.

THEOREM 3.1. Let A be an element of $\mathfrak{A}(R, S)$. We assume $s_1 \geq \ldots \geq s_m$, but we do not assume any ordering for R. Suppose

$$A = \begin{pmatrix} A_i^u \\ A_{n-i}^l \end{pmatrix}.$$

Then there is an $^*A$ which is an element of $\mathfrak{A}(R, S)$, and such that if

$$^*A = \begin{pmatrix} ^*A_i^u \\ ^*A_{n-i}^l \end{pmatrix},$$

then $^*S_{n-i}^l$ has the same components (in a different order) as S_{n-i}^l and furthermore the components of $^*S_{n-i}^l$ are in nonincreasing order.

PROOF: Let $S_{n-i}^l = (h_1, \dots, h_m)$. Suppose $h_j > h_k$ with $j > k$. Define $^{++}A_{n-i}^l$ to be a matrix with row sum vector the same as A_{n-i}^l and column sum vector $^{++}S_{n-i}^l$ the same as S_{n-i}^l but with h_j and h_k interchanged, *e·g,* take $^{++}A_{n-i}^l$ the same as A_{n-i}^l with the $j-th$ and $k-th$ columns interchanged. Let $^{++}S_i^u = S - {}^{++}S_{n-i}^l$. Now $^{++}S_i^u$ and S_i^u agree except for two positions. These two in $^{++}S_i^u$ are $(s_h - h_j, s_j - h_k)$ and in S_i^u are $(s_k - h_k, s_j - h_j)$. Now by hypothesis $j > k$ implies $s_k \geq s_j$. Thus

$$0 \leq s_j - h_j \leq s_k - h_j < s_k - h_k, \tag{3.3}$$

and

$$0 \leq s_j - h_j < s_j - h_k \leq s_k - h_k. \tag{3.4}$$

But (3.3) and (3.4) imply $(s_k - h_j, s_j - h_k) \prec (s_k - h_k, s_j - h_j)$, and by Lemma 3.3, $^{++}S_i^u \prec S_i^u$. By Lemma 3.1, $S_i^u \prec (R_i^u)'$, whence $^{++}S_i^u \prec (R_i^u)'$. Hence there exists an $^{++}A_i^u$ with row sum vector R_i^u and column sum vector $^{++}S_i^u$. Now clearly

$$\begin{pmatrix} ^{++}A_i^u \\ ^{++}A_{n-i}^l \end{pmatrix} \text{ is an element of } \mathfrak{A}(R, S).$$

Continuing in this manner we obtain the desired $^*A$.

COROLLARY. In addition to the hypotheses of Theorem 3.1, let $R = (r_1, \dots, r_n)$, where $r_1 \geq \dots \geq r_n$. Let A be an element of $\mathfrak{A}$ such that rows $i_1, \dots, i_t$ and columns $j_1, \dots, j_u$ exhaust all 1's. Then there is an A^* in $\mathfrak{A}$ such that rows $1, \dots, t$ and columns $1, \dots, u$ exhaust all 1's.

PROOF: By Theorem 3.1 there is an A_1 in $\mathfrak{A}$ such that rows $i_1, \dots, i_t$ and columns $1, \dots, u$ exhaust all $1's$. Consider A_1^T (i.e. the transpose of A_1). A_1^T need not, of course, be an element of $\mathcal{A}$. In A_1^T columns $i_1, \dots, i_t$ and rows $1, \dots, u$ exhaust all $1's$. Again by Theorem 3.1 there exists an A_2 with A_2^T in $\mathcal{A}$ and where columns $1, \dots, t$ and rows $1, \dots, u$ exhaust all $1's$ of A_2. Then in A_2^T rows $1, \dots, t$ and columns $1, \dots, u$ exhaust all $1's$ and A_2^T is the required A^* of the corollary.

The preceding corollary gives the following canonical form for a matrix $A_{\tilde{\rho}}$ in $\mathcal{A}$ with minimal term rank.

$$A_{\tilde{\rho}} = \begin{pmatrix} W & X \\ Y & 0 \end{pmatrix},$$

where W is of size $e \times f$ and $\tilde{\rho} = e + f$.

We now proceed to develop an algorithm for determining $\tilde{\rho}$. Let $U = (u_1, \dots, u_m)$, where the $u's$ are integers as usual. Let k_1 be the smallest subscript (if any) such that there exists an $l < k_1$ satisying $u_{k_1} > u_l + 1$. That is u_{k_1} is the first component with a component as much as two smaller to the left of it. With this fixed k_1 let l_1 be the largest of the subscripts l. That is u_{l_1} is the component as far to the right as possible but still to the left of u_{k_1} which satisfies $u_{l_1} + 1 < u_{k_1}$.

Define

$$(3.5) \qquad \sigma U = (u_1, \dots, u_{l_1} + 1, u_{l_1+1}, \dots, u_{k_1} - 1, u_{k_1+1}, \dots, u_m).$$

If no k_1 exists define $\sigma U = U$. σ is then a « smoothing » operator. σ^i will denote σ applied i times. We write

$$(3.6) \qquad \sigma(U) = (\sigma(u_1), \dots, \sigma(u_n)).$$

For clarity we consider the following example. Let $U = (5, 3, 4, 5, 1, 7)$. Then $u_{k_1} = u_4 = 5$, $u_{l_1} = u_2 = 3$, so that $\sigma U = (5, 4, 4, 4, 1, 7)$ and $\sigma^2 U = (5, 4, 4, 4, 2, 6)$.

LEMMA 3.5. $\sigma U \prec U$.

PROOF: Lemma 3.3 and Lemma 3.4.

LEMMA 3.6. $\sigma U \rhd U$, or equivalently, $C \rhd U$ implies $\sigma C \rhd U$.

PROOF: Immediate from definition.

LEMMA 3.7. If $U = (u_1, \ldots, u_n)$ and $u_1 \geq \ldots \geq u_n$, then $C \rhd U$ implies $C > U$.

PROOF: Immediate from definition.

LEMMA 3.8. Let $H = (h_1, \ldots, h_\mu, \ldots, h_\nu, \ldots, h_m)$. Suppose $\sigma(h_\mu) = h_\mu + 1$, $\sigma(h_\nu) = h_\nu - 1$. Then $\mu \leq \lambda < \nu$ implies $h_\mu + 1 \geq h_\lambda$.

PROOF: Suppose to the contrary, $h_\mu + 1 < h_\lambda$. Then $h_\lambda \geq h_\mu + 2$. But then $\sigma(h_\nu) = h_\nu$ contrary to hypothesis.

LEMMA 3.9. With the same hypothesis as in Lemma 3.8, $\mu < \lambda < \nu$ implies $h_\mu \leq h_\lambda - 1$.

PROOF: For suppose $h_\mu \geq h_\lambda$. Then $\sigma(h_\mu) = h_\mu$ which is contrary to assumption.

LEMMA 3.10. Let $H = (h_1, \ldots, h_m)$. Suppose $\sigma(h_\mu) = h_\mu + 1$, $\sigma(h_\nu) = h_\nu - 1$. Then $\mu < \lambda < \nu$ implies $\sigma(h_\mu) = h_\mu + 1 = h_\lambda$ or equivalently

$$\mu \leq \lambda \leq \rho < \nu \quad \text{implies} \quad \sigma(h_\lambda) = \sigma(h_\rho) \leq \sigma(h_\nu).$$

PROOF: Lemma 3.8 and Lemma 3.9.

LEMMA 3.11. Let $H = (h_1, \ldots, h_m)$. Let $\sigma^t H = (h_1^t, \ldots, h_m^t)$. Let $j_1^t, \ldots, j_s^t$ ($j_1^t \leq j_2^t \leq \ldots \leq j_s^t$) be the subscripts for which

$$\sum_{v=1}^{j_k^t} h_v^t = \sum_{v=1}^{j_k^t} h_v \tag{3.7}$$

(Note that we always have $\sum_{v=1}^{\tau} h_v^t \geq \sum_{v=1}^{\tau} h_v$.)

If j_i^t and j_{i+1}^t are two consecutive subscripts for which equality holds, then

$$j_i^t < \lambda < j_{i+1}^t \quad \text{implies} \quad h_\lambda^t \leq h_{\lambda+1}^t. \tag{3.8}$$

PROOF: The proof is by induction on t. The lemma is certainly true if $t = 0$ and is true for $t = 1$ by Lemma 3.10. Suppose then it is true for $t - 1$. Suppose that after the next application of σ component α is increased by 1 and component β is decreased by 1. (If σ has no effect we are, of course, through.) Let j_γ^{t-1} be the largest subscript of the j_i^{t-1} such that $j_i^{t-1} < \alpha$, if such exists. Let j_δ^{t-1} be the smallest subscript of the $j_i^{t-1} \geq \beta$. In this case one always exists since m is among the j_i^{t-1}. Now the subscripts for which (3.7) holds are

$$j_1^{t-1}, \dots, j_\gamma^{t-1}, j_\delta^{t-1}, j_{\delta+1}^{t-1}, \dots, j_{t_k}^{t-1}. \tag{3.9}$$

Now by the induction hypothesis the only ones to worry about are j_γ^{t-1} and j_δ^{t-1} and again there is nothing more to prove if j_γ^{t-1} does not appear.

Let $j_\gamma^{t-1} < \lambda < j_\delta^{t-1}$. Consider first the case

$$\alpha \leq \lambda < \beta. \tag{3.10}$$

Then by Lemma 3.10, $h_\lambda^t \leq h_{\lambda+1}^t$. Consider next the case

$$j_\gamma^{t-1} < \lambda < \alpha \leq j_{\gamma+1}^{t-1}. \tag{3.11}$$

Then by the induction hypothesis, $h_\lambda^t \leq h_{\lambda+1}^t$. Consider finally the case

$$j_{\delta-1}^{t-1} < \beta \leq \lambda < j_\delta^{t-1}. \tag{3.12}$$

By the induction hypothesis, $h_\lambda^t \leq h_{\lambda+1}^t$.

LEMMA 3.12. Under the same hypothesis as Lemma 3.11,

$$1 \leq \lambda < j_1^t \quad \text{implies} \quad h_\lambda^t \leq h_{\lambda+1}^t. \tag{3.13}$$

PROOF: The proof is by induction on t. The theorem is valid for $t = 0$ and $t = 1$. Suppose the theorem valid for $t - 1$. Suppose that after the next application of σ component α is increased by 1 and component β is decreased by 1. If $\alpha > j_1^{t-1}$, then $j_1^{t-1} = j_1^t$ and the result follows by the induction hypo thesis. Suppose $\beta \leq j_1^{t-1}$. Then once again $j_1^{t-1} = j_1^t$ and the result follows from the induction hypothesis and Lemma 3.10. Suppose then that

$$\alpha \leq j_1^{t-1} < \beta.$$

Let j_δ^{t-1} be the first subscript among the j_k^{t-1} such that $j_\delta^{t-1} \geq \beta$. Now $j_{\delta-1}^{t-1} < \beta$ and we have $j_1^t = j_\delta^{t-1}$. Let

$$1 \leq \lambda < j_1^t = j_\delta^{t-1}.$$

Now if

$$1 \leq \lambda < \alpha - 1,$$

the conclusion follows by the induction hypothesis. If

$$\alpha \leq \lambda \leq \beta - 1,$$

the conclusion follows by Lemma 3.10. If

$$j_{\delta-1}^{t-1} < \beta \leq \lambda < j_{\delta}^{t-1},$$

then the conclusion follows by Lemma 3.11.

LEMMA 3.13. Let $H \lhd G$. Suppose $\sigma^{t+1}H = \sigma^t H$. Then $\sigma^t H \prec G$.

PROOF: We many suppose G has nonincreasing components $g_1 \geq g_2 \geq \dots \geq g_n$. Let $\gamma_1, \dots, \gamma_s$ be the subscripts $j_1^t, \dots, j_s^t$ of Lemma 3.11 for which equality holds in (3.7). Now define

$$\rho_j = \sum_{i=1}^{\gamma_j} g_i - \sum_{i=1}^{\gamma_j} h_i^t = \sum_{i=1}^{\gamma_j} g_i - \sum_{i=1}^{\gamma_j} h_i, \tag{3.14}$$

and note that

$$\rho_j \geq 0 \quad (j = 1, \dots, s) \tag{3.15}$$

Since $\sigma^{t+1}H = \sigma^t H$ by Lemmas 3.11 and 3.12, the components between two γ_i differ by at most 1. Hence

$$\begin{aligned} &(h_1^t, \dots, h_{\gamma_1}^t) \lhd (g_1, \dots, g_{\gamma_1-1}, g_{\gamma_1} - \rho_1) = G_1 \\ &(h_{\gamma_1+1}^t, \dots, h_{\gamma_2}^t) \lhd (g_{\gamma_1+1} + \rho_1, g_{\gamma_1+2}, \dots, g_{\gamma_2-1}, g_{\gamma_2} - \rho_2) = G_2 \\ &\qquad\vdots \\ &(h_{\gamma_{s-1}+1}^t, \dots, h_m^t) \lhd (g_{\gamma_{s-1}+1} + \rho_{s-1}, g_{\gamma_{s-1}+2}, \dots, g_m) = G_s \end{aligned} \tag{3.16}$$

Thus

$$\sigma^t H \lhd (G_1, \dots, G_s) \lhd G \tag{3.17}$$

Since σ has no effect on $\sigma^t H$, $\sigma^t H$ can be made to have nonincreasing components by interchanging adjacent elements which differ by 1. Hence by Lemma 2.1,

$$\sigma^t H \prec G. \tag{3.18}$$

Suppose $H \lhd G$. Let $\mathcal{L} = \mathcal{L}(H, G)$ be the class of vectors V with integral components satisfying

$$H \lhd V \prec G. \tag{3.19}$$

Note that G is in $\mathcal{L}$, and by Lemma 3.13, there is a t with $\sigma^t H$ in $\mathcal{L}$. Let $V = (v_1, \dots, v_m)$ and $H = (h_1, \dots, h_m)$. Suppose that

$$v_l \neq h_l, \ v_{l+1} = h_{l+1}, \ v_{l+2} = h_{l+2}, \dots, \ v_m = h_m.$$

Then define

$$n(V) = m - l. \tag{3.20}$$

LEMMA 3.14. Suppose V is in $\mathcal{L}$. Then σV is in $\mathcal{L}$ and, moreover, $n(V) \geq n(\sigma V)$.

PROOF: σV is in $\mathcal{L}$ by Lemma 3.5 and Lemma 3.6. Suppose $n(V) = \alpha$. Let $\beta = m - \alpha$. Then $h_\beta \neq v_\beta$, $h_{\beta+1} = v_{\beta+1}, \dots, h_m = v_m$. Since $V \rhd H$,

$$\sum_{i=0}^{\alpha} h_{i+\beta} \geq \sum_{i=0}^{\alpha} v_{i+\beta} \tag{3.21}$$

so that

$h_\beta > v_\beta$. Now suppose that $n(\sigma V) > n(V)$. Then $\sigma(v_\beta) = h_\beta > v_\beta$. Hence there is a $\gamma > \beta$ such that $\sigma(v_\gamma) < v_\gamma = h_\gamma$. This implies $n(\sigma V) \leq m - \gamma < m - \beta = \alpha$, which is a contradiction.

LEMMA 3.15. Suppose $H \lhd G$. Suppose $\sigma^{t-1} H \not\prec G$ but $\sigma^t H \prec G$. Then $\sigma^t H \in \mathcal{L}(H, G)$ and $n(\sigma^t H)$ satisfies

$$V \in \mathcal{L}(H, G) \quad \text{implies} \quad n(V) \leq n(\sigma^t H). \tag{3.22}$$

PROOF: Suppose that $U \in \mathcal{L}$ with $n(U) = \alpha$ maximal. We apply σ as often as possible to the first $m - \alpha$ components of H. These, by the definition of σ, are truly the first applications of σ to all of H. Suppose this takes λ applications of σ. We assert

$$\sigma^\lambda H \prec G.$$

For let $\sigma^\lambda H = (h_1^\lambda, \dots, h_m^\lambda)$ and let $U = (u_1, \dots, u_m)$. Let $\beta = m - \alpha$. Now

$$(u_1, \dots, u_\beta) \rhd (h_1, \dots, h_\beta),$$

and $\sigma^{\lambda+1}(h_1, \dots, h_\beta) = \sigma^\lambda(h_1, \dots, h_\beta)$, so by Lemma 3.13, $\sigma^\lambda(h_1, \dots, h_\beta) \prec (u_1, \dots, u_\beta)$. Hence by Lemma 3.2,

$$\sigma^\lambda H \prec U \prec G.$$

Now of course $\lambda \geq t$ and since by Lemma 3.14, $i < j$ implies $n(\sigma^i H) \geq n(\sigma^j H)$, we have

$$n(\sigma^t H) \geq n(\sigma^\lambda H) \geq \alpha.$$

LEMMA 3.16. $h_i \leq h_{i+1}$ implies $\sigma(h_i) \leq \sigma(h_{i+1})$.

PROOF: There are six easy cases to dispose of.

CASE 1. $\sigma(h_{i+1}) = h_{i+1} + 1$.
Then $\sigma(h_i) \leq h_i + 1 \leq h_{i+1} + 1 = \sigma(h_{i+1})$.

CASE 2. $\sigma(h_{i+1}) = h_{i+1}$ and $\sigma(h_i) = h_i + 1$.
In this case $h_{i+1} > h_i$ so $\sigma(h_i) = h_i + 1 \leq h_{i+1} = \sigma(h_{i+1})$.

CASE 3. $\sigma(h_{i+1}) = h_{i+1}$ and $\sigma(h_i) \leq h_i$.
Then $\sigma(h_i) \leq h_i \leq h_{i+1} = \sigma(h_{i+1})$.

CASE 4. $\sigma(h_{i+1}) = h_{i+1} - 1$ and $\sigma(h_i) = h_i$.
In this case $h_{i+1} > h_i$ so $\sigma(h_i) = h_i \leq h_{t+1} - 1 = \sigma(h_{i+1})$.

CASE 5. $\sigma(h_{i+1}) = h_{i+1} - 1$ and $\sigma(h_i) = h_i + 1$.
In this case $h_{i+1} \geq h_i + 2$ so $\sigma(h_i) = h_i + 1 \leq h_{i+1} - 1 = \sigma(h_{i+1})$.

CASE 6. $\sigma(h_i) = h_i - 1$.
Then $\sigma(h_i) = h_i - 1 \leq h_{i+1} - 1 \leq \sigma(h_{i+1})$.

LEMMA 3.17. $h_i > h_{i+1}$ implies $\sigma(h_i) - \sigma(h_{i+1}) \leq h_i - h_{i+1}$.

PROOF: $h_i > h_{i+1}$ implies $\sigma(h_{i+1}) \geq h_{i+1}$ and $\sigma(h_i) \leq h_i$. Hence $\sigma(h_i) - \sigma(h_{i+1}) \leq h_i - h_{i+1}$.

LEMMA 3.18. Let S be a vector with nonincreasing integral components. If $S - H$ is nonincreasing, then $S - \sigma^t H$ is nonincreasing.

PROOF: $S - H$ is nonincreasing so that

$$s_i - h_i \geq s_{i+1} - h_{i+1}. \qquad (3.23)$$

Then if $h_i \leq h_{i+1}$, by Lemma 3.16, $\sigma(h_i) \leq \sigma(h_{i+1})$ so that

$$s_i - \sigma(h_i) \geq s_i - \sigma(h_{i+1}) \geq s_{i+1} - \sigma(h_{i+1}). \tag{3.24}$$

If $h_i > h_{i+1}$, by Lemma 3.17, $\sigma(h_i) - \sigma(h_{i+1}) \leq h_i - h_{i+1}$, so that

$$s_i - s_{i+1} \geq h_i - h_{i+1} \geq \sigma(h_i) - \sigma(h_{i+1}) \tag{3.25}$$

and

$$s_i - \sigma(h_i) \geq s_{i+1} - \sigma(h_{i+1}). \tag{3.26}$$

Hence $S - \sigma H$ is nonincreasing, and repeating the proof, gives the desired result.

Let $\mathfrak{A}$ be the class of 0,1 matrices with row sum vector $R = (r_1, \ldots, r_n)$ $(r_1 \geq \ldots \geq r_n)$ and column sum vector $S = (s_1, \ldots, s_m)$ $(s_1 \geq \ldots \geq s_m)$. Let

$$A = \begin{pmatrix} A_i^u \\ A_{n-i}^l \end{pmatrix},$$

where A_i^u has row sum vector $(r_1, \ldots, r_i)$ and column sum vector S_i^u, and where A_{n-i}^l has row sum vector $(r_{i+1}, \ldots, r_n)$ and column sum vector S_{n-l}^l. Here

$$S = S_i^u + S_{n-i}^l.$$

Let G be the family of vectors S_i^u, where S_i^u is the column sum vector of some A_i^u and where $S - S_i^u = S_{n-i}^l$ is non-increasing. Let $\psi_i(S_i^u)$ equal the number of final components of S_i^u equal to the corresponding components of S. Define

$$\psi_i = \max_{S_i^u \in G} \psi_i(S_i^u). \tag{3 27}$$

Let

$$\psi = \min_{1 \leq i \leq n-1} (i + m - \psi_i). \tag{3.28}$$

Then by Theorem 3.1 and its corollary,

$$\tilde{\rho} = \min \{ m, n, \psi \}. \tag{3.29}$$

We proceed to evaluate ψ_i, and thereby ψ, and $\tilde{\rho}$. Let S_i^u be in G. Define

(3.30) $$T = R' - S,$$

(3.31) $$H_i = S - (R_{n-1}^l)' = (R_i^u)' - T.$$

Then since

$$S_{n-i} < (R_{n-i}^l)',$$

we must have

(3.32) $$S_i^u = S - S_{n-i}^l \rhd S - (R_{n-i}^l)' = H_i .$$

But (3.32) implies

(3.33) $$H_i \lhd S_i^u < (R_i^u)',$$

whence

(3.34) $$G \subseteq \mathfrak{L}(H_i, (R_i^u)').$$

Now Let V_i be in $\mathfrak{L}(H_i, (R_i^u)')$. Then $H_i = (h_1, \dots, h_m) \lhd V_i = (v_1, \dots, v_m)$, so that

(3.35) $$\sum_{j=0}^{\gamma} v_{m-j} \leq \sum_{j=0}^{\gamma} h_{m-j} \qquad (\gamma = 0, \dots, m-1).$$

Moreover, by (3.31), every component of H_i is less than or equal to the corresponding component of S, so that

$$h_j \leq s_j .$$

Now if $v_m = s_m$, then $h_m = s_m$. If also $v_{m-1} = s_{m-1}$, then $h_{m-1} = s_{m-1}$, and so on. Thus if $\psi_i(V_i)$ equals the number of final components of V_i equal to the corresponding components of S, then

(3.36) $$n(V_i) \geq \psi_i(V_i).$$

Now let V_i have $n(V_i)$ maximal for the vectors in $\mathfrak{L}(H_i, (R_i^u)')$. Then V_i also has $\psi_i(V_i)$ maximal for the vectors in $\mathfrak{L}(H_i, (R_i^u)')$. Let t be such that $\sigma^{t-1}H_i \not< (R_i^u)'$ but $\sigma^t H_i < (R_i^u)'$. Then by Lemma 3.15, $\sigma^t H_i$ is in $\mathcal{L}(H_i, (R_i^u)')$. By Lemma 3.15 and (3.27), (3.34),

$$\psi_i \leq \psi_i(\sigma^t H_i). \tag{3.37}$$

We next assert that $\sigma^t H_i$ is in G. This will give us an effective procedure to calculate the ψ_i defined by (3.27), and thereby, $\tilde{\rho}$. To show that $\sigma^t H_i$ is in G, we must show that $\sigma^t H_i \prec (R_i^u)'$, $S - \sigma^t H_i$ is nonincreasing, and $S - \sigma^t H_i \prec (R_{n-i}^l)'$. Now

$$\sigma^t H_i \in \mathcal{L}(H_i, (R_i^u)'),$$

so that $\sigma^t H_i \prec (R_i^u)'$. Moreover,

$$S - \sigma^t H_i \lhd S - H_i = (R_{u-i}^l)'.$$

Thus we need only show that $S - \sigma^t H_i$ is nonincreasing. But $S - H_i$ is nonincreasing, so the last conclusion follows by Lemma 3.18.

EXAMPLE 1.

Let $\mathfrak{A}$ be the class of 0,1 matrices of order 11 with row sum vector $R = (9, 9, 9, 5, 5, 1, 1, 1, 1, 1, 1)$ and column sum vector $S = (8, 6, 5, 5, 4, 4, 4, 4, 1, 1, 1)$. Then

$$T = (3, -1, 0, 0, 1, -1 \; -1, -1, 2, -1, -1).$$

We have the following table:

$H_i = (R_i^u)' - T$	$(R_i^u)'$
(−2, 2, 1, 1, 0, 2, 2, 2, −1, 1, 1)	(1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0)
(−1, 3, 2, 2, 1, 3, 3, 3, 0, 1, 1)	(2, 2, 2, 2, 2, 2, 2, 2, 2, 0, 0)
(0, 4, 3, 3, 2, 4, 4, 4, 1, 1, 1)	(3, 3, 3, 3, 3, 3, 3, 3, 3, 0, 0)
(1, 5, 4, 4, 3, 4, 4, 4, 1, 1, 1)	(4, 4, 4, 4, 4, 3, 3, 3, 3, 0, 0)
(2, 6, 5, 5, 4, 4, 4, 4 1, 1, 1)	(5, 5, 5, 5, 5, 3, 3, 3, 3, 0, 0)
(3, 6, 5, 5, 4, 4, 4, 4, 1, 1, 1)	(6, 5, 5, 5, 5, 3, 3, 3, 3, 0, 0)
(4, 6, 5, 5, 4, 4, 4, 4, 1, 1, 1)	(7, 5, 5, 5, 5, 3, 3, 3, 3, 0, 0)
(5, 6, 5, 5, 4, 4, 4, 4, 1, 1, 1)	(8, 5, 5, 5, 5, 3, 3, 3, 3, 0, 0)
(6, 6, 5, 5, 4, 4, 4, 4, 1, 1, 1)	(9, 5, 5, 5, 5, 3, 3, 3, 3, 0, 0)
(7, 6, 5, 5, 4, 4, 4, 4, 1, 1, 1)	(10, 5, 5, 5, 5, 3, 3, 3, 3, 0, 0)

$\sigma^t H_i$	$i + m - \psi_i(\sigma^t H_i)$
(1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 1)	$1 + 10 = 11$
(2, 2, 2, 2, 2, 2, 2, 2. 0, 1, 1)	$2 + 9 = 11$
(3, 3, 3, 3, 3, 3. 3, 3, 1, 1, 1)	$3 + 8 = 11$
(3, 3, 4, 4, 3, 4, 4, 4, 1, 1, 1)	$4 + 5 = 9$
(3, 5, 5, 5, 4, 4, 4, 4, 1, 1, 1)	$5 + 2 = 7$*
Same as corresponding entry for $H_i = (R_i^*)' - T$.	$6 + 1 = 7$
	$7 + 1 = 8$
	$8 + 1 = 10$
	$8 + 1 = 9$
	$9 + 1 = 10$
	$10 + 1 = 11$

Thus $\tilde{\rho} = 7$. We note for computational purposes once an entry in column $\sigma^t(H_i)$ is the same as the corresponding entry in column $H_i = (R_i^*)' - T$, then this is true for every subsequent entry of column $\sigma^t H_i$.

EXAMPLE 2. Let $\mathfrak{A}_{rs}$ be the class of 0,1 matrices of size $n \times m$, with r 1's in every row and s 1's in every column. Suppose that $r \geq s$. Then since $rn = ms$, $m \geq n$. It is well known that for $\mathfrak{A}_{rs}$, $\tilde{\rho} = n$. We can use the algorithm to obtain this result. We many assume $1 < r < n$ and $1 < s < m$.

Let $R = (r, \ldots, r)$ (n components r) and let $S = (s, \ldots, s)$ (m components s). Then $R' = (n, \ldots, n, 0, \ldots, 0)$ (r components n) and $T = R' - S = (n - s, \ldots, n - s, -s, \ldots, -s)$ (r components $n - s$ and $m - r$ components $-s$). Now $R_i^* = (r, \ldots, r)$ (i components), so that $(R_i^*)' = (i, \ldots, i, 0, \ldots, 0)$ (r components i). Then $H_i = (R_i^*)' - T = (i - n + s, \ldots, i - n + s, s, \ldots, s)$ (r components $i - n + s$ and $m - r$ components s).

Now we must apply σ to H_i until it is majorized by $(R_i^*)'$.

CASE 1. Suppose $s > i > 0$. Then $\psi_i(\sigma^t H_i) = 0$, and $i + m - \psi_i(\sigma^t H_i) = i + m \geq n$.

CASE 2. Suppose $s \leq i < n$ and $-n + s + i \geq 0$. Then $H_i \prec (R_i^*)'$, so that $H_i = \sigma^t H_i$ and $\psi_i(\sigma^t H_i) = m - r$. Then $i + m - \psi_i(\sigma^t H_i) = r + i \geq s + i \geq n$.

CASE 3. Suppose $s \leq i < n$ and $-n+s+i \leq 0$. In this case we need not have $H_i < (R_i^u)'$. Now

$$\begin{aligned}(i-n+s)r+s(n-r-i) &= ir-nr+sr+sn-sr-si \\ &= s(n-i)-r(n-i) \\ &= (s-r)(n-i) \leq 0.\end{aligned}$$

Thus we must smooth at least $n-r-i$ of the s's in H_i in order to obtain $\sigma^t H_i < (R_i^u)'$. Hence

$$\begin{aligned}\psi_i(\sigma^t H_i) &\leq m-r-(n-r-(n-r-i)) \\ &= m-n+i,\end{aligned}$$

and

$$i+m-\psi_i(\sigma^t H_i) \geq i+m-m+n-i=n.$$

Hence we conclude $\tilde{\rho}=n$.

4. Constructions.

(I) Construction of a matrix in $\mathfrak{A}$.

Let the class $\mathfrak{A}$ have roy sum vector $R=(r_1, \ldots, r_n)$ and column sum vector $S=(s_1, \ldots, s_m)$, with $s_1 \geq \ldots \geq s_m$. We may place 1's in row 1 and in the 1st r_1 columns. This follows upon noting that since column sums are nonincreasing, 1's may be shifted to the left by interchanges until they occupy the 1st r_1 position. Now applying the same argument to the class $\mathfrak{A}_1$ with row sum vector $R_1=(r_2, \ldots, r_n)$ and column sum vector $S_1=(s_1-1, \ldots, s_{r_1}-1, s_{r_1}+1, \ldots, s_m)$, we can put 1's in the r_2 columns where S_1 has the largest components. Continuing in this way we construct an A in $\mathfrak{A}$. We remark that the proof of the existence theorem in Section 2 uses this construction with respect to columns. We have been unable to determine the term rank of the matrix A constructed by this device in the general case. This would be a matter of some interest.

EXAMPLE 3. Let A have row sum vector $R=(3, 1, 2, 2)$ and column sum vector $S=(3, 3, 2)$. Then following our construction we have:

$$\begin{aligned} S &= (3,\ 3,\ 2), & R &= (3,\ 1,\ 2,\ 2), \\ S_1 &= (2,\ 2,\ 1), & R_1 &= (1,\ 2,\ 2), \\ S_2 &= (1,\ 2,\ 1), & R_2 &= (2,\ 2), \\ S_3 &= (0,\ 1,\ 1), & R_3 &= (2). \end{aligned}$$

Thus we construct

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

(II) Construction of a matrix $A\tilde{\rho}$ in $\mathfrak{A}$ with minimal term rank.

The algorithm of Section 3 enables one to get $A\tilde{\rho}$ in the form

$$A\tilde{\rho} = \begin{pmatrix} A_i^u \\ A_{n-i}^l \end{pmatrix},$$

where the row and column sum vectors of the submatrices are determined by the algorithm. Furthermore, these row and column sum vectors determine $\tilde{\rho}$. Thus we need only construct a matrix in each class determined by each of the submatrices. This can be done by the preceding construction.

EXAMPLE 4. Let $\mathfrak{A}$ be the class of Example 1. Then A_5^u determines the class with row sum vector (9, 9, 9, 5, 5) and column sum vector (3, 5, 5, 5, 4, 4, 4, 4, 1, 1, 1). A_6^l determines the class with row sum vector (1, 1, 1, 1, 1, 1) and column sum vector (5, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0). Hence we construct

$$A_5^u = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$A_6^l = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

and, thereby,

$$A\tilde{\rho} = \begin{pmatrix} A_5^u \\ A_6^l \end{pmatrix}.$$

(III) Construction of a matrix $A\bar{\rho}$ in $\mathfrak{A}$ with maximal term rank.

LEMMA 4.1. Let $\mathfrak{A}$ be the class of square 0,1 matrices with row sum vector $R = (r_1, \dots, r_n)$ $(r_1 \geq \dots \geq r_n)$ and column sum vector $S = (s_1, \dots, s_n)$ $(s_1 \geq \dots \geq s_n)$. Suppose that $\bar{\rho} = n$. Then there exists an $A\bar{\rho}$ in $\mathfrak{A}$ with n $1's$ on the diagonal from the top righ to the lower left, which we will call the *off diagonal.*

PROOF: Consider any A in $\mathfrak{A}$ with term rank n. Clearly then there is a permutation of the rows of A which will give $1's$ on the off diagonal. Suppose that after this permutation row j has fewer $1's$ than row k, with $j < k$. We consider the 2×2 submatrix of the permuted A composed of the entries from positions $(j, n-j+1)$, $(j, n-k+1)$, $(k, n-k+1)$, and $(k, n-j+1)$. The following are the possibilities for this 2×2 submatrix:

$$B_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

If we have B_1, interchange B_1 to $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and permute rows j and k. If B_4 occurs, permute rows j and k. If B_2 occurs, then since we have assumed row j has fewer ones than row k, there must be an interchange which changes B_2 to $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Then permute rows j and k. If B_3 occurs, since $s_1 \geq \dots \geq s_n$, there must be an interchange which changes B_3 to $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Then permute rows j and k. The preceding manipulations still leave $1's$ on the off diagonal but now row j has more $1's$ than row k. Continuing in this manner we obtain the desired A.

We remark this lemma does not hold for the main diagonal. Maximal matrices may be used to construct simple counterexamples.

Let $R = (r_1, \ldots, r_n)$, $S = (s_1, \ldots, s_m)$, where $r_1 \geq \ldots \geq r_n > 0$ and $s_1 \geq \ldots \geq s_m > 0$. Let

(4 1) $$R^* = R - Q_n,$$

(4.2) $$S^* = S - Q_m,$$

where Q_n, Q_m are vectors of n and m $1's$ respectively. Define

(4.3) $$T^* = S - (R^*)' = (t_1^*, \ldots, t_m^*)$$

and

(4.4) $$U^* = (u_1^*, \ldots, u_m^*), \quad u_k^* = \sum_{i=1}^{k} t_i^* .$$

Now let

(4.5) $$M^* = \max(u_i^*) \quad (i = 1, \ldots, m)$$

and

(4.6) $$N^* = \max(0, M^*).$$

Then the formula for $\bar{\rho}$ of the class $\mathfrak{A}(R, S)$ established in [4] is given by

(4.7) $$\bar{\rho} = n - N^* .$$

Note that for $m = n$, $M^* = N^*$.

LEMMA. 4.2. *Let $\mathfrak{A}(R, S)$ be given with $R = (r_1, r_2, \ldots, r_n)$ $(r_1 \geq r_2 \geq \ldots \geq r_n > 0)$ and $S = (s_1, \ldots, s_m)$ $(s_1 \geq s_2 \geq \ldots \geq s_m > 0)$. Let δ_1 and δ_2 be vectors with m components, t of which are 1 and $m - t$ which are 0, where $t = r_j$ for some j and where $S_1 = S - \delta_1$ and $S_2 = S - \delta_2$ have nonincreasing components.*

Let $R_1 = (r_1, \ldots, r_{j-1}, r_{j+1}, \ldots, r_n)$. Suppose that $\mathfrak{S}_1 \lhd \mathfrak{S}_2$. Then If $\mathfrak{A}(R_1, S_1)$ exists so does $\mathfrak{A}(R_1, S_2)$ and the maximal term rank for $\mathfrak{A}(R_1, S_1)$ is less than or equal to the maximal term rank for $\mathfrak{A}(R_1, S_2)$.

PROOF: Now $\mathfrak{S}_1 \lhd \mathfrak{S}_2$ implies $S_1 \rhd S_2$. Since S_1 and S_2 have nonincreasing components, this implies $S_1 \succ S_2$. Thus $R_1' \succ S_2$ and $\mathfrak{A}(R_1, S_2)$ exists. Suppose $S_1 = (s_{11}, \ldots, s_{1k}, 0, \ldots, 0)$ and $S_2 = (s_{21}, \ldots, s_{2l}, 0, \ldots, 0)$, where $s_{1k} > 0$ and $s_{2l} > 0$. Since $S_1 \succ S_2$ this implies $k \leq l$. Now define

$$R_1^* = R_1 - Q_{n-1}, \tag{4.8}$$

$$S_1^* = S_1 - Q_k \quad , \quad S_2^* = S_2 - \bar{Q}_l. \tag{4.9}$$

Here Q_{n-1} has $n-1$ 1's, and Q_k and $\bar{Q}_l$ have k and l 1's, respectively, in initial positions and 0's. elsewhere. Then

$$\sum_{i=1}^{\gamma} s_{1i}^* \geq \sum_{i=1}^{\gamma} (s_{1i} - 1) \geq \sum_{i=1}^{\gamma} (s_{2i} - 1) = \sum_{i=1}^{\gamma} s_{2i}^* \tag{4.10}$$

$$(\gamma = 1 \ldots, l).$$

Let us now consider the classes $\mathfrak{A}(R_1, S_1)$ and $\mathfrak{A}(R_1, S_2)$, possible zero columns deleted. We may apply the $\bar{\rho}$ formula described in (4.1) - (4.7) to each of these classes. Let U_1^* and U_2^* correspond to the U^* of (4.4) for the classes $\mathfrak{A}(R_1, S_1)$ and $\mathfrak{A}(R_1, S_2)$, respectively. Then (4.10) implies that the maximal component of U_1^* is $\geq$ the maximal component of U_2^*. Hence by the formula for maximal term rank, it follows that the maximal term rank of $\mathfrak{A}(R_1, S_1)$ is $\leq$ the maximal term rank of $\mathfrak{A}(R_1, S_2)$.

Note that the lemma may also be applied with the roles of R and S interchanged.

LEMMA 4.3. Let $R = (r_1, \ldots, r_n)$ $(r_1 \geq \ldots \geq r_n)$ and $S = (s_1, \ldots, s_m)$ $(s_1 \geq \ldots \geq s_m)$. Suppose the maximal term rank $\bar{\rho}$ for $\mathfrak{A}(R, S)$ satisfies $\bar{\rho} < m$. Then there is an A in $\mathfrak{A}(R, S)$ where the submatrix of the first $m-1$ columns of A has maximal term rank $\bar{\rho}$.

PROOF: Let A in $\mathfrak{A}(R, S)$ have maximal term rank $\bar{\rho}$.

Select $1's$ of A accounting for the term rank $\bar{\rho}$. Suppose one these $1's$ occurs in column m and row j. Suppose that column k has none of these $1's$. If there is a 1 in position (j, k), we may use this 1 instead of the 1 in position (j, m). If there is a 0, since the s_i are nonincreasing, an interchange will place a 1 in position (j, k) which can be used as one of the $1's$ accounting for $\bar{\rho}$.

We now show how to construct the matrix of Lemma 4.1. We are given $R = (r_1 \,. \ldots, r_n)$ and $S = (s_1, \ldots, s_n)$ nonincreasing. These vector determine a class $\mathfrak{A}$ with maximal term rank $\bar{\rho} = n$. We are to construct the matrix of order n with $1's$ in the off diagonal positions. By Lemma 4.1 there exists a matrix in $\mathfrak{A}$ of term rank n with a 1 in the $(1, n)$ position. The matrix obtained by deleting the first row is of term rank $n - 1$ and determines a class $\mathfrak{A}(R_1, S_1)$. But by Theorem 3.1 we may obtain a matrix A in $\mathfrak{A}$ and such that if the first row δ of this matrix is deleted, then the resulting $(n-1) \times n$ matrix has nonincreasing rows and columns. The $(n-1) \times n$ submatrix also determines the class $\mathfrak{A}(R_1, S_1)$ and so may be selected to be of term rank $n - 1$. Now consider a vector δ^* of r_1 $1's$ and $n - r$ $0's$ with a 1 in position n and defined so that $1's$ are placed to the left as far as possible provided only

$$S - \delta^*$$

is nonincreasing. We assert

$$\delta^* \rhd \delta.$$

By Lemma 3.2, this means that there exists a matrix in $\mathfrak{A}$ with first row δ^* and such that the $(n = 1) \times n$ submatrix with δ^* deleted has nonincreasing rows and columns and is of term rank $n - 1$.

Consider now the class

$$\mathfrak{A}(R_1, S - \delta^*).$$

This contains matrices of size $(n-1) \times n$ and the maximal term rank is $\rho = n - 1$. By Lemma 4.3, we know that there exists a matrix in $\mathfrak{A}(R_1, S - \delta^*)$ such that if its last column is deleted, then the term rank of the resulting submatrix is equal to $n-1$. Let the last column *of* this matrix be ε. Then by Theorem 3.1 there exists a matrix A_1 in $\mathfrak{A}(R_1, S-\mathfrak{d}^*)$ such that if its last column is deleted the resulting submatrix of order $n-1$ has nonincreasing rows and columns. It may be selected to be of term rank $n-1$. Now define ε^* to be the vector ε with 1'*s* placed to the top as far as possible provided only the transpose of R_1 minus ε^* is a column vector with nonincreasing components. Then

$$\varepsilon^* \rhd \varepsilon,$$

and by Lemma 4.2 there exists a matrix in $\mathfrak{A}(R_1, S-\mathfrak{d}^*)$ with last column ε^*. The submatrix obtained by deleting ε^* is of order $n-1$, has nonincreasing rows and columns, and is of term rank $n-1$.

We may then proceed inductively and construct the desired matrix of Lemma 4.1.

EXAMPLE 5. We carry out the construction for the case $R=(3, 3, 3, 3, 2, 1, 1)$ and $S=(4, 4, 4, 1, 1, 1, 1)$. Following the lemmas we get

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We now proceed to construct a matrix $A\bar{\rho}$ of maximal term rank for an arbitrary $\mathfrak{A}(R, S)$. We assume R and S are nonincreasing. Suppose $\bar{\rho} < m$, and let ε^* a column vector of s_m 1'*s* and $n-s_m$ 0'*s*. Let the 1'*s* of ε^* be placed to the top as far as possible provided only the transpose of R minus ε^* is a column vector with nonincreasing components. Then as before, we may show that there exists a matrix in $\mathfrak{A}(R, S)$ with last column ε^*. Then $n \times (m-1)$ submatrix obtained by deleting ε^* has nonincreasing rows and columns and is of term rank $\bar{\rho}$. We continue filling in the last $m-\bar{\rho}$ columns by this procedure. Then we work on the resulting $n \times \bar{\rho}$ matrix.

If $\bar{\rho} < n$, we fill in the last $n - \bar{\rho}$ rows. This may be done so that we are left with a $\bar{\rho} \times \bar{\rho}$ submatrix with nonincreasing rows and columns of term rank $\bar{\rho}$. This matrix we fill in by the procedure described previously.

The following theorem is a byproduct of our discussion.

THEOREM 4.1. Let R and S be nonincreasing and let $\mathfrak{A}(R, S)$ have maximal term rank $\bar{\rho}$. Then there exists an A in $\mathfrak{A}(R, S)$ where the leading $\bar{\rho} \times \bar{\rho}$ minor has $1's$ on its off diagonal.

5. Invariant $1's$.

In [4] Ryser has proved that if A contains an invariant 1, then by permutations of rows and columns, A may be reduced to the form

$$\begin{pmatrix} S & X \\ Y & 0 \end{pmatrix}, \tag{5.1}$$

where S is *th* matrix of $1's$ and contains the invariant 1 of A.

It is easy to see that if the row sum vector and column sum vector have nonincreasing components, then A must be of the form (5.1) without permutations of rows or columns. It also follows from (5.1) that the class $\mathfrak{A} = \mathfrak{A}(R, S)$ contains only invariant $1's$ if and only if $\mathfrak{A}$ is maximal. Thus only the maximal class contains a single entry.

We begin by establishing a result containing (5.1).

THEOREM 5.1. Let A be in $\mathfrak{A}(R, S)$, where R and S have nonincreasing components. Suppose $a_{uv} = 1$ is invariant. Then A has the form

$$A = \begin{pmatrix} S & X \\ Y & 0 \end{pmatrix}.$$

Here S has all $1's$ and is of size $k \times j$, where $j \geq v$ is the number of invariant $1's$ of row u and $k \geq u$ is the number of rows with at least j invariant $1's$.

PROOF: Now $a_{uv} = 1$ an invariant 1 implies $a_{rs} = 1$ is inva riant for $1 \leq r \leq u$, $1 \leq s \leq v$. For otherwise an interchange would contradict the invariance of $a_{uv} = 1$. It then follows that we must have

$$A = \begin{pmatrix} S & X \\ Y & W \end{pmatrix},$$

where S is a matrix of $1's$ of size $k \times j$. All $1's$ of S are invariant. We may assume the entry in row k and column $j+1$ is 0. Suppose a 1 occurs in W in row t of A. Then we may apply an interchange if necessary and assume that a 1 occurs in row t and column $j+1$ of A. But then all entries in row t and columns 1, ..., j of A are also $1's$. Indeed, these are invariant $1's$, and this is not possible. Hence $W = 0$.

THEOREM 5.2. Let $R = (r_1, \ldots, r_n)$ and $S = (s_1, \ldots, s_m)$ have nonincreasing components. Let $r_1 = \lambda$. Then $\mathfrak{A}(R, S)$ has no invariant $1's$ if and only if

$$(s_1, s_2 - 1, \ldots, s_{\lambda+1} - 1, s_{\lambda+2}, \ldots, s_m)' \succ (r_2, \ldots, r_n).$$

PROOF: From (5.1) it is clear that $\mathfrak{A}$ has no invariant $1's$ if and only if we can put a 0 in the (1, 1) position of some A in $\mathfrak{A}$. Then by applying interchanges A has no invariant $1's$ if and only if (0, 1, ..., 1, 0, ..., 0) is a possibility for the first row of some A in $\mathfrak{A}$. The result now follows from the existence theorem.

THEOREM 5.3. Let R and S have nonincreasing components. Form $T = (t_1, \ldots, t_m) = S - R'$ and

$$U = (u_1, \ldots, u_m), \; u_k = \sum_{i=1}^{k} t_i$$

Then row i of a matrix in $\mathfrak{A}(R, S)$ has exactly j invariant $1's$ if and ondy if j is the largest subscript such that

$$u_j = 0 \tag{5.2}$$

and

$$r_i \geq j \tag{5.3}$$

If there is no i and j satisfying (5.2) and (5.3), then row i has no invariant $1's$.

PROOF: Suppose that row i has exactly j invariant $1's$.

Then by Theorem 5.1 a matrix A in $\mathfrak{A}$ is of the form

$$A = \begin{pmatrix} S & X \\ Y_1 & A_1 \\ Y_2 & 0 \end{pmatrix}.$$

Here S is a matrix of $1's$ of size i by j and Y_1 contains only invariant $1's$. It follows that $u_j = 0$. Also $r_i \geq j$. The integer j is the maximal integer with these properties.

Suppose that $u_j = 0$ and $r_i \geq j$. Then the first j $1's$ in row i are invariant. Otherwise we would deny $u_j = 0$.

In conclusion, we mention that invariant $1's$ are closely associated with certain properties of the integers $\tilde{\rho}$ and $\bar{\rho}$. Ryser [4] has shown that if $\mathfrak{A}$ is without an invariant 1 and if $\rho < m, n$, then $\tilde{\rho} < \bar{\rho}$. A topic deserving further study is the determination of necessary and sufficient conditions on the class $\mathfrak{A}$ in order that $\tilde{\rho} = \rho$. Such conditions could conceivably be developed by a study of the $\bar{\rho}$ formula and the $\tilde{\rho}$ algorithm. Our Example 2 is an instance of a class $\mathfrak{A}$ with this property.

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