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ON IRREGULAR VARIETIES WHICH CONTAIN CYCLIC INVOLUTIONS

Nota (*) *di* LEONARD ROTH (*a Londra*)

1. *Introduction.* - The present note generalises some familiar results of De Franchis and Comessatti concerning irregular multiple planes. In the first place, a classical theorem of De Franchis [4, 5] states that, on any double plane of irregularity $q > 0$, the branch curve is reducible, consisting of a number of curves belonging to a pencil; it follows from this that any surface V_2 which is a simple model of the double plane must contain a pencil, of genus q, of curves.

The theorem in question is established by computing the simple integrals of the first kind attached to V_2. Actually, it is the second of the above results which is significant, for it means that V_2 cannot possess a proper model V_2^* on its Picard-Severi variety V_q (see [9]). Thus we may conclude that the existence on V_2 of a *rational* involution I_2 of order 2 implies the non-existence of V_2^* and hence, by a theorem of Severi [10], that V_2 contains a pencil of genus q. From this the result concerning the branch curve can be deduced.

Now it appears that the De Franchis theorem is merely a special case of a proposition about superficially irregular algebraic varieties V_r of any dimension $r \geqslant 2$ which carry superficially *regular* involutions I_2. Denoting by $g_k(I_2)$ the number of linearly independent differential forms of the first kind and of degree $k (k = 1, 2, ..., r)$ attached to the image variety of I_2, we show that:

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If a variety $V_r(r \geqslant 2)$ *of superficial irregularity* $q > 0$ *carries an involution* I_2 *such that* $g_1(I_2) = g_2(I_2) = 0$, *then* V_r *must contain a congruence of subvarieties (of some dimension* $\geqslant 1$), *the congruence having superficial irregularity* q.

Moreover, we readily see that the coincidence locus of I_2 always belongs to the congruence in question.

We show further that this result is itself a special case of the following theorem:

If a variety $V_r(r \geqslant 2)$ *of superficial irregularity* $q > 0$ *carries an involution* I_2 *such that* $g_1(I_2) = 0$, *while for any even value of* $k(\leqslant r)$, $g_k(I_2) < \binom{q}{k}$, *then* V_r *must contain an irregular congruence (of superficial irregularity* $\leqslant$ q).

The limitation $q \geqslant r$ is required for the method of proof; the restriction is, however, inessential, since whenever $q < r$, V_r must contain some congruence of superficial irregularity q ([10]).

We then indicate how the same methods can be applied to the case where V_r carries a superficially regular cyclic involution I_m of any order $m \geqslant 3$. Here, however, the results are less precise, since — in constrast with the case $m = 2$ — there are now various types of associated involution of order m on the Picardian V_q of V_r. Moreover, unlike the case $m = 2$, we have now always to deal with singular transformations of V_q, and these necessarily give rise to problems of existence. Examples of irregular cyclic planes quoted by Comessatti [2] demonstrate that general results analogous to those obtained in the case $m = 2$ cannot be established.

Finally we remark that the previous considerations may be extended to the case where the involution I_m is non cyclic, provided that the associated involution on V_q is generable by a (finite) group of automorphisms of V_q; the results then obtained are exactly similar to those mentioned above.

2. Generalities. - Consider a non-singular algebraic variety $V_r(r \geqslant 2)$ having superficial irregularity $q \geqslant r$; in all that follows the case $q < r$ can be set aside since we know that V_r will then contain a congruence of superficial irregularity q ([10]). Assuming that V_r does not contain such a congruence we may obtain for V_r a model (simple or multiple) of dimension r on the second Picardian or Picard-Severi variety V_q constructed with the period matrix associated with the linearly independent simple integrals $u_i (i=1, 2, \ldots, q)$ of the first kind attached to V_r (see [1, 9, 13]). Denoting by x a point current on V_r, we write $u_i(x)=u_i$, and take u_i for coordinates on V_q; then the locus of the corresponding point (u) on V_q is an irreducible algebraic variety V_r^*. This will be a simple model of V_r if and only if the congruences

$$(1) \qquad u_i(x) \equiv u_i(y) \qquad \text{(mod. periods)}$$

where x, y are points of V_r, in general admit a single solution. If instead for arbitrary x, the equations (1) admit $\nu (> 1)$ solutions, it follows that V_r^* is a ν-ple model of V_r: to a point of V_r^* there then corresponds a set of ν distinct points on V_r, belonging to the *fundamental involution* I_ν. In either case we shall assume that V_r^* is non-singular; such a hypothesis may possibly be restrictive.

We observe that a *necessary and sufficient condition* for the existence of V_r^* on V_q is that V_r *should not contain any congruence of superficial irregularity* q ([10]).

Suppose now that V_r carries an involution I of order 2; this generates an automorphism between points P, P' of V_r, under which all the integrals u_i must be invariant. Hence we have a transformation from $u_i(P)$ to $u_i'(P')$ of the form

$$(2) \qquad u_i' = \sum_{j=1}^{q} \lambda_{ij} u_j + \mu_i \qquad (i=1, 2, \ldots, q)$$

where λ_{ij}, μ_i are constants.

Evidently the transformation (2), applied to V_r^*, is subordinate to a transformation of the entire variety V_q which generates an involution J, likewise of order 2, on V_q. From the theory of Picard varieties it is known ([7]) that J can be represented by the canonical form

$$(3_1) \qquad u_i = u_i + a_i \qquad (i=1, 2, \ldots, p;\ p \geqslant 0)$$

$$(3_2) \qquad u_j = -u_j \quad (j=p+1,\ p+2, \ldots, q)$$

where the a_i are constants (possibly zero) and where p is the superficial irregularity of J. We note that if V_q has general moduli, there are just two possibilities: either $p=0$ or $p=q$. In all other cases we have a singular transformation of V_q which can exist only for particular values of the moduli of V_q ([3]).

It is clear, by comparison of equations (1) and (3), that *if* $\nu>1$, *I cannot belong to the fundamental involution I_ν.*

In the case $\nu=1$, the sets of I are in birational correspondence with the sets of an involution I^* of order 2 on V_r^*, which is subordinate to J. When $\nu>1$, we have instead that *I is mapped on a ν-fold involution I^* (likewise of order 2) on V_r^*, which is subordinate to J*; this follows by comparing equations (1) and (3). In particular, when $q=r$, $V_r{}^*$ coincides with V_q and I^* with J.

3. On the characters g_k, q_k. - We denote by $g_k(k=1, 2, \dots, r)$ the number of linearly independent differential forms of the first kind and of degree k which are attached to V_r. Here g_r is the geometric genus $P_g(V_r)$, while g_1 is the superficial irregularity q of V_r. The arithmetic genus $P_a(V_r)$ is then given by the Severi-Kodaira relation ([11])

$$P_a=g_r-g_{r-1}+\dots+(-1)^r g_1 . \tag{4}$$

Defining the r-dimensional irregularity q_r as the difference P_g-P_a, we then introduce the set of k-dimensional irregularities $q_k(k=2, 3, \dots, r-1)$ by taking appropriate linear sections of V_r and applying (4) to each in turn. We thus obtain the relations ([11, 13]):

$$\begin{cases} g_k=q_k+q_{k+1} & (k=2, 3, \dots, r-1) \\ g_1=q_2 . \end{cases} \tag{5}$$

We say that V_r is *completely regular* if and only if $q_k=0$ $(k=2. 3, \dots, r-1)$. Clearly a necessary and sufficient condition for the complete regularity of V_r is $g_s=0$ $(s=1, 2, \dots, r-1)$.

Suppose now that V_r is mapped on a multiple non-singular variety V_r'; in that case we have the inequalities

$$(6) \qquad g_k(V_r) \geqslant g_k(V_r') \qquad (k = 1, 2, \ldots, r).$$

It follows from (5) and (6) that, *if V_r is completely regular, then so also is V_r'.* For, if for some $s(1 \leqslant s \leqslant r-1)$ we had $g_s(V_r') > 0$, then we should have $g_s(V_r) > 0$, whence V_r could not be completely regular.

One last preliminary remark: suppose that V_r contains a congruence Γ of some positive superficial irregularity $(\leqslant q_2)$; then Γ will be mapped by a congruence Γ' of subvarieties on V_r'. Now, in the case where V_r' is *superficially regular,* Γ' will perforce be superficially regular: this means that Γ' cannot correspond birationally, element for element, to Γ. Applying this result to the case we have to consider, let V_r' denote a birational image of the involution I on V_r; if I is *superficially regular,* we deduce that to a member of Γ' there will correspond *two* members of Γ, in general distinct. Moreover, the coincidence locus of I must belong to Γ, and the branch locus on V_r' must belong to Γ'.

4. On the Wirtinger involution. - Returning to the Picard variety V_q, we consider the case where the involution J is *superficially regular*; the involution, represented by equations (3_2), then has for image a generalised Wirtinger variety [1]) (in the case $r = 2$, a generalised Kummer surface) which we shall denote by W_q.

Now every differential form of the first kind and of degree k attached to W_q must arise from an analogous form attached to V_q; and it is known ([12]) that every such form is given by an expression of the type $du_1 du_2 \ldots du_k$. Evidently this furnishes a corresponding differential form on W_q if and only if it is invariant under the transformation (3_2). We thus

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(1) The name of Wirtinger variety is usually restricted to the case where V_q has all its divisors equal to unity; we may call this the ordinary Wirtinger variety (for $r = 2$, the ordinary Kummer surface).

obtain the results

$$(7) \qquad \begin{cases} g_k(W_q) = \binom{q}{k} & (k \text{ even}) \\ g_k(W_q) = 0 & (k \text{ odd}). \end{cases}$$

In these formulae, k takes the values 1, 2, ..., r. It now follows from (4) that the arithmetic genus P_a of W_q is given by

$$P_a(W_q) = (-1)^q(2^{q-1} - 1).$$

This result was obtained by Gröbner [3] for the ordinary Wirtinger variety by computing the Hilbert characteristic function for the manifold in question and then applying the Severi postulation formula.

5. First applications. - With the notation of n. 2, suppose that the involution I carried by V_r has superficial irregularity $p(0 < p \leqslant q)$; this means that precisely p linearly independent differential forms of the first kind and first degree — say $du_1, du_2, \dots, du_p$ — take the same values at corresponding points P, P' of I. The equations (2) for I assume the form (3).

If V_q has general moduli, and thus admits only ordinary transformations, we must have $p = q$. If instead $p < q$, we have a singular transformation; evidently the involution J is now pseudo- Abelian of type p ([8]). Hence V_r^* must contain a superficially irregular congruence, and thus so also must V_r. Whence the result: *If I has superficial irregularity $p(0 < p < q)$, V_r must contain a superficially irregular congruence.*

Suppose next that I is superficially regular and — as usual — that V_r does not contain any congruence of superficial irregularity $q(= g_1)$; in this case the model V_r^* on V_q certainly exists and if $q = r$, coincides with V_q. The equations (1) now take the form (3_2), so that J is a generalised Wirtinger (or Kummer) involution, whose characters $g_k(J)$ are given by (7); in particular, then, we have $g_2(J) = \binom{q}{2}$.

Now the number $g_2(I^*)$ will equal $g_2(J)$ provided that V_r^* does not contain a superficially irregular congruence, for in that case none of the differential forms $du_i du_j$ can vanish identically on V_r^* ([12]). In any event, since we know that V_r^* certainly does not contain any congruence of superficial irregularity q (n. 2), not all the integrals u_i attached to V_r^* can be functions of one integral alone ([12]), and therefore we must have $g_2(I^*) > 0$; hence, whatever the value of ν, it follows that $g_2(I) > 0$, by equations (6). Thus

If V_r carries an involution I of the second order such that $g_1(I) = 0$, $g_2(I) = 0$, then V_r must contain a congruence of superficial irregularity q.

As remarked in n. 3, *the members of the congruence are conjugate in I, and the coincidence locus of I belongs to the congruence.*

In the case where $r \geqslant 3$, it follows from (5) that, if the characters g_1 and g_2 are both zero, then q_2 and q_3 are also zero, and vice-versa. Thus

If $V_r (r \geqslant 3)$ carries an involution I of the second order which is bidimensionally and also tridimensionally regular, then V_r must contain a congruence of superficial irregularity q.

In particular, then, if I is *unirational* or *birational*, V_r must contain a congruence of superficial irregularity q.

6. The double space S_r. - Consider first the case $r = 2$; suppose that V_2 contains a rational involution I, which means that V_2 can be mapped on a double plane S_2 of irregularity q. By the previous theorem, V_2 must contain an irrational pencil, of genus q, and the coincidence locus of I must consist of curves belonging to the pencil. Hence the branch curve consists of a number curves of curves belonging to a pencil in S_2, and the general curve of this pencil maps a pair of curves of V_2; this result is due to De Franchis [4].

Next, let $r = 3$; then the double planes in the corresponding double space S_3 are « generic » surfaces, having irregularity q. Hence, by the previous result, the branch surface in S_3 consists of a number of surfaces of a pencil, from which it follows that V_3 must contain a pencil, of genus q, of surfaces.

Proceeding by induction, we thus obtain the result: *Every double space $S_r (r \geqslant 2)$ of superficial irregularity $q > 0$ contains a pencil, of genus q, of hypersurfaces; and the branch locus in S_r consists of a number of primals belonging to a pencil.*

It is clear that the image of the pencil on V_r is a hyperelliptic curve, since to a member of the (linear) pencil in S_r which maps it there corresponds a pair of hypersurfaces, in general distinct. This type of double S_r has been studied by Gallarati [6], who has calculated the invariants $g_k(V_r)$ in the case where the base of the pencil in S_r is irreducible and non-singular.

7. Extension of previous results.

I. - Let $q = r$; in this case, if $\nu = 1$, the involution I is coincident with the generalised Wirtinger involution J, and its characters $g_k(I)$ are given by (7). If $\nu > 1$, we have $g_k(I) \geqslant g_k(J)$. It follows that, in order that the model $V_r^* (= V_q)$ should exist, the inequalities $g_k(I) \geqslant \binom{q}{k}$ must be satisfied for every even value of k. Hence,

If, when $q = r$, the variety V_r carries a superficially regular involution I of the second order such that, for any even value of k, $g_k(I) < \binom{k}{q}$, then V_r must contain a congruence of superficial irregularity q.

II. - In the case where $q > r$, we can obtain a result which is more general than that of n. 5. Previously we have allowed V_r to contain some irregular congruence (necessarily of superficial irregularity $< q$). Suppose now that V_r *contains no superficially irregular congruence whatever;* this entails that V_r^* also can contain no such congruence. On this hypothesis the differential forms of the first kind of any degree $k \leqslant r$ attached to J must give rise to precisely the same number of differential forms of the first kind and of like degree attached to I^*. We thus have, for every even value of $k \leqslant r$,

$$g_k(I^*) = \binom{q}{k}, \qquad \text{whence} \qquad g_k(I) \geqslant \binom{q}{k}.$$

Therefore, *if V_r carries a superficially regular involution I of the second order such that, for any even value of $k (\leqslant r)$, $g_k(I) < \binom{q}{k}$, then V_r must contain an irregular congruence (of superficial irregularity $\leqslant q$).*

As remarked, in n. 3, the members of this congruence must be conjugate in I, so that the coincidence locus of I belongs to the congruence in question.

8. Notes and examples. - We add a few comments upon the preceding results.

In the first place we remark that the conditions of n. 5 are not necessary in order that V_r should contain a congruence of superficial irregularity q. Thus, consider a product variety $V_r = V_t \times V_{r-t}$ $(t \geqslant 1)$, where V_t is the simple model of a double space S_t; in particular, when $t = 1$, V_t is a hyperelliptic curve. In this case V_r carries an involution I which is mapped by the product $S_t \times V_{r-t}$; hence, if we assume that $g_1(V_{r-t}) = 0$, we shall have $g_1(I) = 0$. Now in this case, $g_2(I) = g_2(V_{r-t})$, from which it follows that the character $g_2(I)$ can have any non-negative value whatever. Evidently the variety V_r contains a congruence of varieties V_{r-t}, which is mapped by the points of V_t, and which has maximum superficial irregularity $g_1(V_t) = q$.

Returning to the general case we observe that, from the correspondence between V_r and I we have (n. 3), for every $k (1 \leqslant k \leqslant r)$, $g_k(I) \leqslant g_k(V_r)$. For the particular double spaces S_r considered by Gallarati [6], we have $g_k(V_r) = 0$ $(k = 2, 3, \ldots, r-1)$. This suggests an interesting problem: what are the most general conditions of validity for this last result?

In the second place, since for any birational involution I on V_r we have $g_k(I) = 0$ (all k), it follows that, in the previous example, $g_k(I) = g_k(V_r)$ $(k = 2, 3, \dots, r-1)$. This suggests another problem: under what conditions can we assert that this set of relations will hold? An analogous question can of course be raised for any involution, superficially regular or not, carried by a given variety V_r; but the answer is unknown even in the relatively simple case just considered, at any rate for a variety V_r of general character.

A certain amount is, however, known concerning involutions on a Picard variety V_q ([9]). Thus, for an involution I of any order on V_q, the sole condition $g_1(I) = q$ ensures that the image of I should also be a Picard variety. But the effect of other analogous conditions on the nature of I has not yet been investigated. The cyclic involutions — to which we now turn — on V_q have been studied by Lefschetz [7].

9. The general cyclic involution $I_m (m \geqslant 3)$. - Consider next the case where V_r carries a cyclic involution I_m of any order $m \geqslant 3$; such an involution is generated by an automorphism of V_r to which the remarks made in n. 2 apply. We have now a system of equations analogous to (3), which are of the form

$$\begin{cases} u_i' = u_i + a_i & (i = 1, 2, \dots, p;\ p \geqslant 0) \\ u_j' = \varepsilon_j u_j & (j = p+1, p+2, \dots, q) \end{cases} \tag{8}$$

where p is the superficial irregularity of the associated involution J on the Picard - Severi variety V_q, which certainly exists provided V_r contains no congruence of superficial irregularity q; and where ε_j denotes an m th root of unity, other than unity itself ([7]).

Precisely as in n. 5 we see that: *if I_m has superficial irregularity $p (0 < p < q)$, then V_r must contain a superficially irregular congruence.* Supposing instead that $g_1(I_m) = 0$, we have $g_1(I_m^*) = 0$, in which case $p = 0$ in equations (8).

On this hypothesis, we may proceed to calculate the characters $g_k(J)$, for $k = 1, 2, \dots, r$. To begin with, we have $g_1(J) = 0$. Next, $g_2(J)$ is equal to the number of products

$\varepsilon_k \varepsilon_l$, where ε_k, ε_l are *different* numbers $(k \neq l)$ occurring in (8), which are equal to unity. And similarly for the remaining characters $g_k(J)$.

An essential difference between the present case and the preceding is that, while for $m = 2$, $p = 0$, we have an ordinary transformation of V_q, for $m > 2$, $p = 0$, we always have a singular transformation. Such transformations can exist only on varieties V_q with particular moduli ([3]); and in every case which is *a priori* possible it must be shown that the corresponding V_q can effectively be constructed. Moreover, since there is now a number of different involutions J for any given value of m, the results are necessarily less precise. We have the following analogue of the previous theorems:

If V_r carries a cyclic involution $I_m (m \geqslant 3)$ such that $g_k(I_m) = 0$ (all k), then either there exists an associated involution J on V_q for which $g_k(J) = 0$ (all k), or else V_r contains a superficially irregular congruence.

The proof is exactly as before. It should be noted that, in the case where the above-mentioned involution J actually exists, no general conclusion can be drawn. Thus Comessatti [2], in his classification of the irregular cyclic triple planes $(r = 2,\ m = 3)$ has shown that all such surfaces contain irrational pencils, though not necessarily of genus q; this had been previously noticed by Bagnera and De Franchis in their study of the hyperelliptic surfaces. Comessatti also quotes an example of an irregular quintuple plane $(r = 2,\ m = 5)$ which contains no irrational pencil whatever.

In conclusion, we point out that the previous methods will apply also to the case where, instead of the cyclic involution J, we have on V_q any superficially regular involution $J_m (m \geqslant 3)$, provided always that it is generable by a group $\mathfrak{G}$ (of order m) of automorphisms of V_q. It is known ([9]) that a sufficient, but not a necessary, condition for J_m to be so generable is that the image variety of J_m should have some positive plurigenus. When the group $\mathfrak{G}$ exists, it may be represented analytically by a number of sets of equations such as (8); in that case the characters $g_k(J_m)$ may be calculated from these equations, and we may then deduce results similar to the preceding.

REFERENCES

[1] A. ANDREOTTI: Mem. Ac. roy. Belgique, 27 (1952), fasc. 4.
[2] A. COMESSATTI, Rend. Palermo, 31 (1911), 1.
[3] F. CONFORTO, *Abelsche Funktionen und algebraische Geometrie*, Berlin, 1956.
[4] M. DE FRANCHIS, Rend. Acc. Lincei, (5), 13 (1904), 688.
[5] — — Rend. Palermo, 20 (1905). 49.
[6] D. GALLARATI, Boll. U.M.I., (3), 14 (1959), 175.
[7] S. LEFSCHETZ, Trans. Amer. Math. Soc., 22 (1921), 327:407.
[8] L. ROTH, Rend. Sem. Mat. Padova, 23 (1954), 277.
[9] — — *Sulla varietà di Picard e le sue applicazioni*, Rend. Sem. Mat., Milano, 30 (1960).
[10] F. SEVERI, Annali di Mat., (4), 21 (1942), 1.
[11] — — Annali di Mat., (4), 41 (1956), 161.
[12] — — Rend. Acc. Lincei, (8), 20 (1956), 7.
[13] — — *Geometria dei sistemi algebrici sopra una superficie e sopra una varietà algebrica*, (vol. II), Roma, 1959.