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PROJECTIVE CLASSIFICATION OF THE PLANE TRILINEARITIES OF DIMENSION FOUR

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Contributions to the general theory of *plurilinearities* can be found in Comessatti [4] and Severi [9]. Complete studies of particular cases are given in Gherardelli [5, 6], Comessatti [4], Morgantini [7, 8], Thrall-Chanler [10], Benedicty [1, 2, 3]. Several other cases are considered, from a different point of view, by Enriques (*Rend. Acc. Lincei,* 1890) and Togliatti (*Atti Acc. Torino*, 1916-17).

This paper will focus attention on a study of plurilinear correspondences of dimension four, defined as in (1), between three projective planes over the field of the complex numbers. The classification of such trilinearities culminates in the theorem of Section 9.

1. - As a definition of *plane trilinearity* $T = T_4$ of dimension 4 we adopt the following one:

(i) T is an algebraic pure correspondence of dimension 4 between three complex projective planes X, Y, Z; i.e. T is an algebraic pure subvariety of dimension 4 of the Segre's variety $W = X \times Y \times Z$;

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(ii) T is non-degenerate an any one of the planes X, Y, Z; i.e. every point of each one of the planes belongs to at least one triad of T;

(iii) for a generic point $P \in X$ $[Q \in Y;\ R \in Z]$ the set of all the pairs Q, R $[R, P;\ P, Q]$ such that $P \times Q \times R \in T$ is a collineation (singular or not) of dimension 2 between Y and Z $[Z$ and X, X and $Y]$.

2. - (a) According to part (i) of DEF. 1, any trilinearity T can be represented by a system of equations of the type

$$F_t(x_1y_1z_1,\ x_1y_1z_2,\ \ldots,\ x_iy_jz_k,\ \ldots,\ x_3y_3z_3) = 0,$$

where F_t are homogeneous polynomials.

According to part (iii) of the same definition, a generic choice of $x = (x_1, x_2, x_3)$ determines a collineation of dimension 2 between Y and, Z, which is, at least in one way, generically single-valued. Therefore, after possibly a change of the letters y and z, a generic choice of x and y must determine z as a single-valued algebric function of the pair x, y:

2.1 $$\rho z_k = \varphi_k(x,\ y) \qquad (k = 1,\ 2,\ 3)$$

where φ_k are homogeneous polynomials, of degree independent of k, with respect to each one of the series of variables x and y.

Since a generic choice of x $[y]$ makes 2.1 be a collineation, the polynomials φ_k can be chosen to be bilinear with respect to the x' s and the y' s.

We have thereby proved that: *The triads of T_4 which arise from a generic choice of x and y satisfy a set of equations*

2.2 $$\rho z_k = \Sigma_{ij} a_{ijk} x_i y_j \qquad (k = 1,\ 2,\ 3),$$

where the constants a_{ijk} are not all zero.

(b) By *singular collineation* (of dimension 2) between two planes X, Y, mentioned in 1 (iii), is meant a pure bilinearity C between X and Y, of dimension 2, as defined in [3], and different from a non-singular collineation. According to the classification given in [3], such bilinearities turn out to be, with respect to suitable frames of reference, and if they are not degenerate, of the following projective types:

A. $x_i y_j = 0 \qquad (i,\ j = 1,2)$

B. $x_1 y_3 = x_2 y_3 = x_1 y_1 + x_2 y_2 = 0.$

It is immediately checked that both are reducible and that each one of their components is degenerate.

(c) EXAMPLE. Let C be a non-degenerate collineation (singular or not) between X and Y. Let U be the set of the triads $P \times Q \times R$, where $P \times Q \in C$ and R is arbitrary in Z. Then U is a tricorrespondence of dimension 4 between X, Y, Z.

If C is not singular, then U is a trilinearity which can be represented by

2.3 $$\rho_k y_j = x_j z_k \qquad (j,\ k = 1,\ 2,\ 3).$$

The last remark in (b) implies that this one is the only case where U is irreducible and non-degenerate.

(d) REMARK: We write conventionally $\rho s_k = t_k$ $(k = 1, 2, ...)$ to express the fact that the matrix

$$\begin{Vmatrix} s_1 & s_2 \ldots \\ t_1 & t_2 \ldots \end{Vmatrix}$$

has rank not greater than 1.

3. - (a) Let us consider now an irreducible T_4. Its generic points are represented by 2.2, therefore T_4 coincides with a component of dimension 4 of the variety V represented by 2.2 (with the convention stated in 2 (d)). Thus, if V is irreducible, 2.2 represent already T_4; if not, a representation of T_4 can be obtained by adding to 2.2 a suitable set of equations. In any case a general choice of x, y determines, by means of 2.2, a general point of T_4.

(b) For a generic choice of z in 2.2 we must obtain a collineation (singular or not) between X and Y. It follows that for the system

3.1 $$\rho z_k = \Sigma_j (\Sigma_i a_{ijk} x_i) y_j \qquad (k = 1, 2, 3)$$

(i) either

$$\Delta = \mathrm{Det} \, || \, \Sigma_i a_{ijk} x_i \, ||_{jk} = 0,$$

(ii) or

3.2 $$\rho y_j = \begin{vmatrix} z_1 & \Sigma_i a_{i,j+1,1} x_i & \Sigma_i a_{i,j+2,1} x_i \\ z_2 & \Sigma_i a_{i,j+1,2} x_i & \Sigma_i a_{i,j+2,2} x_i \\ z_3 & \Sigma_i a_{i,j+1,3} x_i & \Sigma_i a_{i,j+2,3} x_i \end{vmatrix} = \Sigma_k b_{jk} z_k,$$

where b_{jk} is the algebraic complement of the element (jk) in the matrix $|| \, \Sigma_i a_{ijk} x_i \, ||_{jk}$.

Since 3.2 must represent a collineation between X and Y, a factor F, linear in the x' s and non-identically zero with respect to the z' s, must divide the right hand members of 3.2. If F does not depend on the z' s, it divides all the polynomials b_{jk} $(j, k = 1, 2, 3)$. If it does, since 3.2 must represent, for a generic x, a collineation D between Y and Z, it must be linear in the z' s and D does not depend on z.

In the last case any generic z gives rise to the same fixed collineation between X and Y; therefore the trilinearity is of the type 2.3.

(c) Hence: *All irreducible trilinearities of dimension* 4 *between X, Y, Z are irreducible components of dimension* 4 *of varieties represented by one of the following sets of equations*

(i) Equations 2.2, where $\Delta = \mathrm{Det} \, || \, \Sigma_j a_{ijk} x_i \, ||_{jk} = 0$;

(ii) Equations 2.2, where the b_{jk}' s mentioned above have a linear factor in common, which does not depend on the z' s; we can assume $\Delta \not\equiv 0$;

(iii) Example 2(c).

4. - Consider the equations of a trilinearity of type 3(c)(i) or (ii)

4.1 $$\rho z_k = \Sigma_{ij} a_{ijk} x_i y_j \qquad (k = 1, 2, 3)$$

and let the coefficients of each row be arranged in the matrices

4.2 $$A_k = \| a_{ijk} \|_{ji} \qquad (k = 1, 2, 3).$$

Let us form three linearly independent linear combinations of them, giving the three new matrices

4.3 $$\Sigma_k \gamma_{hk} A_k \qquad (h = 1, 2, 3),$$

where Det $\| \gamma_{hk} \| \neq 0$. These linear combinations correspond to a change of coordinates in the Z plane

$$\rho z'_h = \Sigma_k \gamma_{hk} z_k \qquad (h = 1, 2, 3),$$

which reduces the trilinearity to the form:

4.4 $$\rho z'_h = \Sigma_{ij} (\Sigma_k \gamma_{hk} a_{ijk}) x_i y_j ,$$

where the matrices 4.3 are the matrices of the coefficients of each row of 4.4.

Let R be the maximum rank of the matrices $\Sigma_k \gamma_{\cdot k} A_k$ as $(\gamma_{\cdot 1}, \gamma_{\cdot 2}, \gamma_{\cdot 3})$ describes the set of all the triads, except (0, 0, 0). Then we have the distinct cases: (a) $R = 3$, (b) $R = 2$ and (c) $R = 1$.

In each case, the maximum rank will be given by a generic choice of the γ' s.

5. - $R = 3$ (Case 4(a)).

Let the reference in Z be generic; i.e., the rank of A_k is 3 ($k = 1, 2, 3$). If we write the trilinearity in vector notation we have

$$\rho z_k = y_{-1} A_k x \qquad (k = 1, 2, 3)$$

where A_k is given by 4.2 and x is the vertical matrix whose elements are x_1, x_2, x_3.

The representation of the trilinearity can be much simplified if we operate a suitable change of reference $x' = Mx$ in X and $y' = Ny$ in Y so that in

$$\rho z_k = y_{-1} N_{-1} A_k M x$$

$\boldsymbol{N}_{-1}\boldsymbol{A}_1\boldsymbol{M}$ reduces to $\boldsymbol{E}$, the identity matrix, and the matrix $\boldsymbol{N}_{-1}\boldsymbol{A}_2\boldsymbol{M}$ reduces to $\boldsymbol{S}$, some standard form. The problem is solved in the following way. Let the coefficients of $\boldsymbol{A}_1$ be the coefficients of a (non-singular) collineation C_1 and let those of $\boldsymbol{A}_2$ be the coefficients of another collineation C_2, both between the planes Y and $\bar{X}$, the dual of X. Then $C = C_1^{-1}C_2$ is a non-singular collineation of Y onto itself. Change the reference in Y so that the collineation takes one of the classical standard forms. Now consider a change in the reference of the X plane so that the matrix $\boldsymbol{A}_1 = \boldsymbol{E}$ and therefore $\boldsymbol{A}_1\boldsymbol{A}_2 = \boldsymbol{A}_2$. This means that the matrix $\boldsymbol{A}_2$ itself becomes one of the following

$$5.1 \left\| \begin{matrix} a_{112} & 0 & 0 \\ 0 & a_{222} & 0 \\ 0 & 0 & a_{332} \end{matrix} \right\|, \quad 5.2 \left\| \begin{matrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & a_{332} \end{matrix} \right\|, \quad 5.3 \left\| \begin{matrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{matrix} \right\|$$

where $a_{112}, a_{222}, a_{332} \neq 0$.

(b) Consider 5.1. When we fix a generic point in Z the equations of the trilinearity 4.1 must represent a collineation (singular or not) between X and Y. In particular consider the point (0, 0, 1); one obtains:

$$x_1y_1 + x_2y_2 + x_3y_3 = 0$$

$$a_{112}x_1y_1 + a_{222}x_2y_2 + a_{332}x_3y_3 = 0.$$

There are two cases: Either

$$5.1.1 \qquad y_1 : y_2 : y_3 = \left\| \begin{matrix} x_1 & x_2 & x_3 \\ a_{112}x_1 & a_{222}x_2 & a_{332}x_3 \end{matrix} \right\|$$

or

$$5.1.2 \qquad \left\| \begin{matrix} x_1 & x_2 & x_3 \\ a_{112}x_1 & a_{222}x_2 & a_{332}x_3 \end{matrix} \right\| = 0$$

identically with respect to x.

In 5.1.1 we have

$$\sigma y_1 = (a_{332} - a_{222})x_2 x_3 ,$$

$$\sigma y_2 = (a_{112} - a_{332})x_1 x_3 ,$$

$$\sigma y_3 = (a_{222} - a_{112})x_1 x_2 .$$

The only possible form of the common factor F referred to previously is either hx_1, hx_2, or hx_3, each of which divides out if and only if two of the a_{ii2}'s ($i = 1, 2, 3$) are equal, say $a_{112} = a_{222} \neq a_{332}$, but for the reordering of the variables. We have the form:

$$\rho z_1 = x_1 y_1 + x_2 y_2 + x_3 y_3$$

$$\rho z_2 = a_{112} x_1 y_1 + a_{112} x_2 y_2 + a_{332} x_3 y_3$$

$$\rho z_3 = \tau_1 y_1 + \tau_2 y_2 + \tau_3 y_3$$

(where $\tau_i = a_{1i3}x_1 + a_{2i3}x_2 + a_{3i3}x_3$), which, after a non-singular collineation in the z's

$$\sigma z_1' = z_1 - (z_2 - a_{112}z_1)/(a_{332} - a_{112})$$

$$\sigma z_2' = (z_2 - a_{112}z_1)/(a_{332} - a_{112})$$

$$\sigma z_3' = z_3 - a_{113}z_1 - a_{333}(z_2 - a_{112}z_1)/(a_{332} - a_{112})$$

becomes:

$$\rho z_1 = x_1 y_1 + x_2 y_2 ,$$

$$\rho z_2 = \qquad\qquad x_3 y_3$$

$$\rho z_3 = \tau_1 y_1 + \tau_2 y_2 + \tau_3 y_3 ,$$

with $a_{113} = a_{333} = 0$.

Consider now a generic z and the case $\Delta \neq 0$. A linear polynomial F must factor all the second order minors of $\| \Sigma_i a_{ijk} x_i \|$ as previously stated. The non-zero ones are

$$x_1 x_3 ,\ x_2 x_3 ,\ x_1 \tau_2 - x_2 \tau_1 ,\ x_1 \tau_3 ,\ x_2 \tau_3 ,\ x_3 \tau_1 , \text{ and } x_3 \tau_2 .$$

Thus the only possible F is of the form ax_3 which divides each one of the minors if and only if:

$$a_{123} = a_{223} = a_{213} = a_{133} = a_{233} = 0$$

and the trilinearity is of the form:

5.1.1.1
$$\begin{aligned} \rho z_1 &= x_1 y_1 + x_2 y_2 , \\ \rho z_2 &= \qquad\qquad x_3 y_3 \\ \rho z_3 &= a_{313} x_3 y_1 + a_{323} x_3 y_2 \end{aligned}$$

with the requirement a_{313}, $a_{323} \neq 0$, since we are in the case $\Delta \neq 0$. Now take the case 5.1.1 $\Delta = 0$ (identically with respect to x). That is

$$x_3(x_1 \tau_2 - x_2 \tau_1) = 0$$

therefore

$$a_{213} = a_{313} = a_{123} = a_{223} = a_{323} = 0$$

and the last line of the trilinearity takes the form

$$\rho z_3 = a_{133} x_1 y_3 + a_{233} x_2 y_3 ,$$

which reduces to 5.1.1.1 by interchanging the x' s and y' s, this means nothing more than changing the name of coordinates to y_j for plane X and x_i for plane Y.

In case 5.1.2 there exists a linear combination of the right hand members of 4.1, which is identically zero. Consequently, after a change of the reference in the Z-plane the trilinearity takes on the form

5.1.2 $\quad \rho z_k = \Sigma_{ij}\, a_{ijk} x_i y_j \quad (k = 1,\ 2), \qquad \rho z_3 = 0.$

A component of T, that one whose points one originated by a generic pair in $X \times Y$, is evidently degenerate on Z; the other component(s) is (are) represented by $\Sigma_{ij}\, a_{ijk} x_i y_j = 0$ $(k = 1,\ 2)$ and belong(s) therefore to case 2.3 or to a dimension different from 4.

(c) Let us pass onto the consideration of case 5.2.

The trilinearity takes on the form

$$\rho z_1 = x_1 y_1 + x_2 y_2 + x_3 y_3$$

$$\rho z_2 = (x_1 + x_2) y_1 + x_2 y_2 + a_{332} x_3 y_3$$

$$\rho z_3 = \tau_1 y_1 + \tau_2 y_2 + \tau_3 y_3$$

with $a_{332} \neq 0$. By a non-singular collineation in z' s:

$$\sigma z_1' = z_1$$

$$\sigma z_2' = z_2 - z_1$$

$$\sigma z_3' = z_3 - a_{113} z_1 - a_{213}(z_2 - z_1),$$

we get the form:

$$\sigma z_1 = x_1 y_1 + x_2 y_2 + x_3 y_3 ,$$

$$\sigma z_2 = x_2 y_1 \qquad + a_{332} x_3 y_3 ,$$

$$\sigma z_3 = a_{313} x_3 y_1 + \tau_2 y_2 + \tau_3 y_3 ,$$

where $a_{332} \neq 1$.

Consider the point (0, 0, 1) in Z; it gives the correspondences:

$$x_1 y_1 + x_2 y_2 + x_3 y_3 = 0$$

$$x_2 y_1 + a_{332} x_3 y_3 = 0.$$

Since it is impossible for the second order minors of

$$\begin{Vmatrix} x_1 & x_2 & x_3 \\ x_2 & 0 & a_{332} x_3 \end{Vmatrix}$$

to be identically zero with respect to x, we have

$$y_1 : y_2 : y_3 = a_{332} x_2 x_3 : (x_2 - a_{332} x_1) x_3 : (- x_2^2).$$

The only possible factor F is $h x_2$. Then $a_{332} = 0$ and we have the form

$$\rho z_1 = x_1 y_1 + x_2 y_2 + x_3 y_3 ,$$

$$\rho z_2 = x_2 y_1 ,$$

$$\rho z_3 = a_{313} x_3 y_1 + \tau_2 y_2 + \tau_3 y_3 .$$

For a generic z, either

5.2.1 $$\Delta \equiv \begin{vmatrix} x_1 & x_2 & x_3 \\ x_2 & 0 & 0 \\ a_{313}x_3 & \tau_2 & \tau_3 \end{vmatrix} \neq 0, \text{ or}$$

5.2.2. $\Delta \equiv 0$ identically with respect to x.

Consider case 5.2.1. Then F must factor all the second order minors of Δ. By inspection the only possible factor is $F = hx_2$. Thus x_2 must divide each of $a_{313}x_3$, τ_2, and τ_3. That is $a_{313} = a_{123} = a_{323} = a_{133} = a_{333} = 0$ and we have the form:

5.2.1.0
$$\begin{aligned} \rho z_1 &= x_1y_1 + x_2y_2 + x_3y_3 \\ \rho z_2 &= x_2y_1 \\ \rho z_3 &= \qquad a_{223}x_2y_2 + a_{233}x_2y_3. \end{aligned}$$

If $a_{223} \neq 0$, then the substitution of x_3 for $x_3 - \frac{a_{233}}{a_{223}}x_2$ and of y_2 for $y_2 + \frac{a_{233}}{a_{223}}y_3$ reduces 5.2.1.0 to a form which is easily recognized as a subcase of 5.1.1.1.

We need only consider:

5.2.1.2
$$\begin{aligned} \rho z_1 &= x_1y_1 + x_2y_2 + x_3y_3, \\ \rho z_2 &= x_2y_1, \\ \rho z_3 &= x_2y_2, \end{aligned}$$

corresponding to 5.2.1.0 with $a_{233} = 0$.

Now consider case 5.2.2. In order for the determinant to vanish for all x, we must have $x_2\tau_3 - x_3\tau_2 \equiv 0$, that is $a_{123} = a_{233} = a_{133} = a_{333} = 0$ and $a_{223} = a_{333}$, or the form:

$$\begin{aligned} \rho z_1 &= x_1y_1 + x_2y_2 + x_3y_3 \\ \rho z_2 &= x_2y_1 \\ \rho z_3 &= a_{313}x_3y_1 + a_{223}x_2y_2 + a_{223}x_3y_3 \end{aligned}$$

which, after a simple change of reference in the z' s, becomes:

$$\rho z_1 = x_1 y_1 + x_2 y_2 + x_3 y_3$$

$$\rho z_2 = x_2 y_1$$

$$\rho z_3 = a_{313} x_3 y_1 - a_{223} x_1 y_1 .$$

However, this result is the case 5.2.1, but for renaming the variables.

(d) The final matrix to consider is 5.3 or the trilinearity

$$\rho z_1 = x_1 y_1 + x_2 y_2 + x_3 y_3$$

$$\rho z_2 = (x_1 + x_2) y_1 + (x_2 + x_3) y_2 + x_3 y_3$$

$$\rho z_3 = \tau_1 y_1 + \tau_2 y_2 + \tau_3 y_3 .$$

Consider the change of reference in Z:

$$\sigma z_1' = z_1$$

$$\sigma z_2' = z_2 - z_1$$

$$\sigma z_3' = z_3 .$$

The first two lines then become:

$$\rho z_1 = x_1 y_1 + x_2 y_2 + x_3 y_3$$

$$\rho z_2 = x_2 y_1 + x_3 y_2 .$$

At (0, 0, 1) in Z we have:

$$x_1 y_1 + x_2 y_2 + x_3 y_3 = 0$$

$$x_2 y_1 + x_3 y_2 = 0$$

where, since

$$\begin{Vmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & 0 \end{Vmatrix} \not\equiv 0$$

we have

$$y_1 : y_2 : y_3 = x_3^2 : (-x_2 x_3) : (x_2^2 - x_1 x_3).$$

No factor F exists. Hence at the particular point (0, 0, 1) in Z the transformation refuses to be linear, and this case gives no trilinearity.

(e) We have thus completed case 4 (a). Collecting all the results of this section, with suitable choices of references, we obtain the forms 2.3, 5.1.1.1, 5.2.1.2.

6. - $R = 2$ (Case 4 b).

(a) Let us choose the system of reference generic in Z, i.e. in such a way that $\|a_{ij1}\|$ and $\|a_{ij2}\|$ each have rank exactly equal to two. At the point (0, 0, 1) in Z we have the two equations:

$$C_k : \Sigma_{ij}\, a_{ijk} x_i y_j = 0 \qquad (k = 1,\ 2).$$

We then have two correlations of rank two between X and Y. They correspond to two non-singular collineations between two pencils. There are three major cases:

6.1 The two pencils have the same centers in their respective planes.

6.2 The pencils have the same center in X but distinct centers in Y.

6.3 The pencils have distinct centers in both X and Y.

(b) Consider case 6.1. One needs but to recognize that we have here two distinct collineations between the points of two distinct lines, provided one considers in each correlation the correspondence between the intersections of the lines of the pencil in X with a predetermined line r in X and those of the corresponding lines of the pencil in Y with some fixed line s in Y.

By changing the reference in X and Y, we can take (0, 0, 1) at the center of the pencil and assume that r (s) be the line $x_3 = 0$ ($y_3 = 0$). The collineations between r and s, i. e. the correlations between X and Y, will be represented by the equations

$$\Sigma_{ij}\, a_{ijk} x_i y_j = 0 \qquad (i,\ j,\ k = 1,\ 2).$$

Their matrices $B_k = \| a_{ijk} \|_{ij}$ $(i, j, k = 1, 2)$ can be reduced, by a suitable choice of the reference on r, s, to the following standard forms:

$$B_1 = E \text{ in any case,}$$

and the following for B_2:

6.1.1 $\left\| \begin{matrix} 1 & 0 \\ 0 & a \end{matrix} \right\|$, $a \neq 0,1$; 6.1.2 $\left\| \begin{matrix} 1 & 0 \\ 1 & 1 \end{matrix} \right\|$; 6.1.3 $\left\| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right\|$

Let us take case 6.1.1. The trilinearity then reduces to:

$$\begin{aligned} \rho z_1 &= x_1 y_1 + x_2 y_2 \\ \rho z_2 &= x_1 y_1 + a x_2 y_2 \\ \rho z_3 &= \tau_1 y_1 + \tau_2 y_2 + \tau_3 y_3 \end{aligned}$$

where the τ_i are the same as before. By a simple non-singular change of reference in Z one obtains

$$\begin{aligned} \rho z_1 &= x_1 y_1 \\ \rho z_2 &= \qquad x_2 y_2 \\ \rho z_3 &= \tau_1 y_1 + \tau_2 y_2 + \tau_3 y_3 \end{aligned}$$

wehere, in τ_1 and τ_2, $a_{113} = a_{223} = 0$. Now if $\Delta \equiv\!\!|\!\!\equiv 0$ (case 6.1.1.1), then an F must divide each of

$$x_1 x_2, \; x_1 \tau_2, \; x_1 \tau_3, \; x_2 \tau_1, \quad \text{and} \quad x_2 \tau_3 .$$

Either $F = h x_1$ and

$$a_{213} = a_{313} = a_{233} = a_{333} = 0$$

or $F = h x_2$ and $a_{123} = a_{323} = a_{133} = a_{333} = 0$. The trilinearity becomes respectively:

6.1.1.1.1
$$\begin{aligned} \rho z_1 &= x_1 y_1 \\ \rho z_2 &= \qquad x_2 y_2 \\ \rho z_3 &= (a_{123} x_1 + a_{323} x_3) y_2 + a_{133} x_1 y_3 \end{aligned}$$

6.1.1.1.2
$$\rho z_1 = x_1 y_1$$
$$\rho z_2 = \qquad x_2 y_2$$
$$\rho z_3 = (a_{213}x_2 + a_{313}x_3)y_1 + a_{233}x_2 y_3$$

which reduce to each other by interchanging the x' s and y' s.

If a_{133}, or a_{323}, is not 0, then T belongs also to case 6.2 or 6.3, otherwise 6.1.1.1 becomes

$$\rho z_1 = x_1 y_1$$
$$\rho z_2 = x_2 y_2$$
$$\rho z_3 = x_1 y_2$$

for which $\Delta \equiv 0$; it can therefore be discarded.

Suppose $\Delta \equiv 0$. Then $x_1 x_2 \tau_3 = 0$ and $a_{133} = a_{233} = a_{333} = 0$ giving the form:

6 1.1.2
$$\rho z_1 = x_1 y_1$$
$$\rho y_2 = \qquad x_2 y_2$$
$$\rho z_3 = (a_{213}x_2 + a_{313}x_3)y_1 + (a_{123}x_1 + a_{323}x_3)y_2 \,.$$

If we interchange x' s and y' s, the second order minors of the new Δ are $a_{123}x_1^2$, $a_{213}x_2^2$, $x_2(a_{133}x_1 + a_{233}x_2)$, $x_1(a_{133}x_1 + a_{233}x_2)$, $x_1 x_2$. Therefore if $\Delta \not\equiv 0$ either $F = hx_1$ or $F = hx_2$ and, respectively, $a_{213} = a_{233} = 0$ or $a_{123} = a_{133} = 0$, and the trilinearity reduces to a particular case of 6.1.1.1. If $\Delta \equiv 0$ we have $a_{313} = a_{323} = 0$ and T becomes

6.1.1.2.1
$$\rho z_1 = x_1 y_1 ,$$
$$\rho z_2 = \qquad x_2 y_2$$
$$\rho z_3 = a_{213}x_2 y_1 + a_{123}x_1 y_2 \,.$$

As it can easily be checked, a generic z determines between X and Y a (singular) collineation, which does not depend on z; this case reduces therefore to 2.3.

Move on to 6.1.2:

$$\rho z_1 = x_1 y_1 + x_2 y_2$$
$$\rho z_2 = (x_1 + x_2)y_1 + x_2 y_2$$
$$\rho z_3 = \tau_1 y_1 + \tau_2 y_2 + \tau_3 y_3 ,$$

which admits the form:

$$\rho z_1 = x_1 y_1 + x_2 y_2$$
$$\rho z_2 = x_2 y_1$$
$$\rho z_3 = a_{313} x_3 y_1 + \tau_2 y_2 + \tau_3 y_3 .$$

Assume $\Delta \equiv\!\!\!|\; 0$, Then F divides each of x_2^2, $x_1\tau_2 - a_{313}x_2x_3$, and $x_1\tau_3$, and, since $F = hx_2$ is the only possible factor in common, then $a_{123} = a_{323} = a_{133} = a_{333} = 0$, and

6.1.2.1
$$\rho z_1 = x_1 y_1 + x_2 y_2, \quad \rho z_2 = x_2 y_1,$$
$$\rho z_3 = a_{313} x_3 y_1 + a_{223} x_2 y_2 + a_{233} x_2 y_3 .$$

If a_{313}, or a_{233}, is not zero, then T belongs also to type 6.2 or 6.3. Otherwise it reduces to a case for which $\Delta \equiv 0$.

If $\Delta \equiv 0$, then $x_2^2 \tau_3 = 0$ and $a_{133} = a_{233} = a_{333} = 0$:

6.1.2.2
$$\rho z_1 = x_1 y_1 + x_2 y_2$$
$$\rho z_2 = x_2 y_1$$
$$\rho z_3 = a_{313} x_3 y_1 + (a_{123} x_1 + a_{223} x_2 + a_{323} x_3) y_2 .$$

As in case 6.1.1.2, if we interchange x's and y's, x_1^2 is a second order minor of Δ; thus, if $\Delta \equiv\!\!\!|\; 0$, x_1 has to divide all such minors, which implies $a_{213} = a_{233} = 0$ and the trilinearity becomes a subcase of 6.1.2.1. If $\Delta \equiv 0$, then $a_{313} = a_{323} = 0$ and T becomes

6.1.2.2.1
$$\rho z_1 = x_1 y_1 + x_2 y_2,$$
$$\rho z_2 = x_2 y_1,$$
$$\rho z_3 = a_{123} x_1 y_2 + a_{223} x_2 y_2 .$$

This case can be discarded for the same reason we discarded 6.1.1.2.1. Also case 6.1.3 can be discarded, for the same reason as 5.1.2.

(c) We shall go on to 6.2, where the correlations C_1 and C_2 have the same centers in X and different centers in Y. To obtain the canonical form, let the common center of C_1 and C_2 in X be (0, 0, 1) and the centers in Y be (0, 0, 1) and (0, 1, 0) respectively. Then $a_{311} = a_{321} = a_{331} = a_{131} = a_{231} = 0$ and $a_{312} = a_{322} = a_{332} = a_{122} = a_{222} = 0$.

There are two possible situations: (6.2.1). The line s belonging to both centers in Y gives, by way of C_1^{-1} and C_2^{-1}, the same line in X; (6.2.2) the line s in Y gives two distinct lines in X.

To further reduce the form of C_1 and C_2, in case 6.2.1 take the correspondent of s in X as $x_2 = 0$ and some general line r as $x_1 = 0$. Let the correspondents of r in Y by way of C_1 and C_2 respectively be $y_2 = 0$ and $y_3 = 0$, and the correspondent of s is already determined by the centers as $y_1 = 0$. Finally, let $x_1 + x_2 = 0$ go into $y_1 - y_2 = 0$ by C_1 and $y_1 - y_3 = 0$ by C_2. In case 6.2.2 let the correspondents of s by way of C_2^{-1} and C_2^{-1} respectively be $x_2 = 0$ and $x_1 = 0$ in plane X. Let $C_2^{-1}(s)$ and $C_1^{-1}(s)$ in X have the respective correspondents in Y, by way of C_1 and C_2, $y_2 = 0$ and $y_3 = 0$. Let $x_1 + x_2 = 0$ go into $y_1 - y_2 = 0$ by C_1 and $y_1 - y_3 = 0$ by C_2.

The conditions in 6.2.1 give for the trilinearity the form:

$$\begin{aligned} \rho z_1 &= x_1 y_1 + x_2 y_2 \\ \rho z_2 &= x_1 y_1 \qquad + x_2 y_3 \\ \rho z_3 &= \tau_1 y_1 + \tau_2 y_2 + \tau_3 y_3 , \end{aligned}$$

which, after changing references in Z and in Y, becomes

6.2.1
$$\begin{aligned} z_1 &= x_1 y_1 + x_2 y_2 \\ z_2 &= \qquad\qquad x_2 y_3 \\ z_3 &= \tau_1 y_1 + \tau_2 y_2 + \tau_3 y_3 . \end{aligned}$$

If $\Delta \neq 0$, then F must factor out of each of: x_2^2, $x_1 \tau_3$, $x_1 \tau_2$. That factor F must be $h x_2$ and $a_{123} = a_{323} = a_{133} = a_{333} = 0$, with the result, after changing references in X and Z:

$$\begin{aligned} z_1 &= x_1 y_1 + x_2 y_2 \\ z_2 &= \quad x_2 y_3 \\ z_3 &= (a_{113} x_1 + a_{213} x_2 + a_{313} x_3) y_1 . \end{aligned}$$

If $a_{313} \neq 0$, then the trilinearity belongs also to case 6.3, because we could choose a point in Z which gives two singular correlations with distinct centers in both X and Y. We take therefore $a_{313} = 0$ and we have:

6.2.1.1 $\rho z_1 = x_1 y_1 + x_2 y_2,\ \rho z_2 = x_2 y_3,\ \rho z_3 = (a_{113}x_1 + a_{213}x_2)y_1.$

If $\Delta \equiv 0$, then $a_{123} = a_{313} = a_{213} = a_{323} = 0$, $a_{223} = a_{113}$. This gives the form: $\rho z_1 = x_1 y_1 + x_2 y_2,\ \rho z_2 = x_2 y_3,\ \rho z_3 = a_{223} x_1 y_1 + + a_{223} x_2 y_2 + (a_{133} x_1 + a_{233} x_2 + a_{333} x_3) y_3.$
A suitable change of reference in Z eliminates the first two terms in the expression of z_3; the condition $R < 3$ implies $a_{333} = 0$, and we obtain:

6.2.1 2 $$\rho z_1 = x_1 y_1 + x_2 y_2,\quad \rho z_2 = x_2 y_3,\quad \rho z_3 = x_1 y_3.$$

This case, however, can be discarded because of the same reason as 6.1.1.2.1.

Under case 6.2.2, T is:

$$z_1 = x_1 y_1 + x_2 y_2$$
$$z_2 = x_2 y_1 + x_1 y_3$$
$$z_3 = \tau_1 y_1 + \tau_2 y_2 + \tau_3 y_3.$$

If $\Delta \not\equiv 0$, an F must divide, among others, the polynomials x_1^2, x_2^2, which is obviously impossible; hence no result is given.

Suppose $\Delta \equiv 0$. This means $x_1^2 \tau_2 - x_1 x_2 \tau_1 + x_2^2 \tau_3 = 0$ and $a_{123} = a_{323} = a_{313} = a_{233} = a_{333} = 0$, $a_{223} = a_{113}$, and $a_{213} = a_{133}$. Thus the trilinearity is:

$$\rho z_1 = x_1 y_1 + x_2 y_2$$
$$\rho z_2 = x_2 y_1 \qquad + x_1 y_3$$
$$\rho z_3 = (a_{113} x_1 + a_{213} x_2) y_1 + a_{113} x_2 y_2 + a_{213} x_1 y_3$$

which reduces to

6.2.2.2 $$\rho z_1 = x_1 y_1 + x_2 y_2$$
$$\rho z_2 = x_2 y_1 + x_1 y_3$$
$$\rho z_3 = 0,$$

and can be discarded because of the reason given in case 5.1.2.

(d) Finally, consider the case 6.3 in which the two correlations have different centers in both planes. There are three distinct situations possible: (6.3.i) The line s belonging to both pencils in X goes into the same line by way of C_1 and C_2 in Y; (6.3.ii) the line s in X goes into a line in Y by way of C_1 which passes through the center of C_2, and a different line by way of C_2; (6.3.iii) the line s goes into distinct lines in Y by way of C_1 and C_2, neither of which passes through the center of the correlation which gave the other.

We again obtain the canonical form by taking the centers for C_1 and C_2 in the X plane as (0, 0, 1) and (0, 1, 0) respectively; and in the Y plane as (0, 0, 1), (1, 0, 0). We obtain $a_{311} = a_{321} = a_{331} = a_{131} = a_{231} = 0$, and $a_{212} = a_{222} = a_{232} = a_{112} = a_{312} = 0$.

Simplifying further, let us define the correspondences in the three cases as follows:

6.3.i

$$x_1 = 0 \rightarrow C_1 : y_2 = 0$$
$$x_1 = 0 \rightarrow C_2 : y_2 = 0$$
$$x_2 = 0 \rightarrow C_1 : y_1 = 0$$
$$x_3 = 0 \rightarrow C_2 : y_3 = 0$$
$$x_1 + x_2 = 0 \rightarrow C_1 : y_1 - y_2 = 0$$
$$x_1 + x_3 = 0 \rightarrow C_2 : y_2 - y_3 = 0.$$

Therefore C_1 and C_2 become

$$C_1 : x_1 y_1 + x_2 y_2 = 0$$
$$C_2 : x_1 y_3 + x_3 y_2 = 0$$

6.3.ii

$$x_1 = 0 \rightarrow C_1 : y_2 = 0$$
$$x_1 = 0 \rightarrow C_2 : y_3 = 0$$
$$x_2 = 0 \rightarrow C_1 : y_1 = 0$$
$$x_3 = 0 \rightarrow C_2 : y_2 = 0$$
$$x_1 + x_2 = 0 \rightarrow C_1 : y_1 - y_2 = 0$$
$$x_1 + x_3 = 0 \rightarrow C_1 : y_2 - y_3 = 0.$$

C_1 and C_2 become:

$$C_1 : x_1y_1 + x_2y_2 = 0$$
$$C_2 : x_1y_2 + x_3y_3 = 0.$$

Finally: 6.3.iii

$$x_1 = 0 \rightarrow C_1 : y_1 = 0$$
$$x_1 = 0 \rightarrow C_2 : y_3 = 0$$
$$x_2 = 0 \rightarrow C_1 : y_2 = 0$$
$$x_3 = 0 \rightarrow C_2 : y_2 = 0$$
$$x_1 + x_2 = 0 \rightarrow C_1 : y_1 - y_2 = 0$$
$$x_1 + x_3 = 0 \rightarrow C_2 : y_2 - y_3 = 0.$$

Therefore the correlations become:

$$C_1 : x_1y_1 + x_2y_2 = 0$$
$$C_2 : x_1y_1 + x_3y_3 = 0.$$

Before proceeding further, let us consider the rank of the linear combinations of the matrices A_k. In case 6.3.i it is already not greater than 2; in cases 6.3.ii, 6.3.iii the rank is always 3 for generic choice of γ_1, γ_2, γ_3. We have only to consider, therefore, case 6.3.i. which we rename 6.3.

The form of the trilinearity given by case 6.3. is:

$$\rho z_1 = x_1y_1 + x_2y_2$$
$$\rho z_2 = \qquad\qquad x_3y_2 + x_1y_2$$
$$\rho z_3 = \tau_1y_1 + \tau_2y_2 + \tau_3y_3.$$

If $\Delta \equiv\!|\!\equiv 0$, an F must divide each of

$$x_1^2,\ x_2\tau_1,\ x_2\tau_3,\ x_3\tau_1, \quad \text{and} \quad x_3\tau_3.$$

Therefore $F = hx_1$ and $a_{213} = a_{313} = a_{233} = a_{333} = 0$; the form is

$$\rho z_1 = x_1y_1 + x_2y_2$$
$$\rho z_2 = \qquad\qquad x_3y_2 + x_1y_3$$
$$\rho z_3 = a_{113}x_1y_1 + (a_{123}x_1 + a_{223}x_2 + a_{323}x_3)y_2 + a_{133}x_1y_3$$

which can be reduced to

6.3.1 $$\rho z_1 = x_1 y_1 + x_2 y_2, \quad \rho z_2 = x_3 y_2 + x_1 y_3.$$
$$\rho z_3 = (a_{123} x_1 + a_{223} x_2 + a_{323} x_3) y_2.$$

Suppose

$$\Delta \equiv - x_1^2 \tau_2 + x_1 x_2 \tau_1 + x_1 x_3 \tau_3 \equiv 0.$$

Then $a_{213} = a_{123} = a_{333} = 0$, $a_{113} = a_{223}$, $a_{313} = - a_{233}$, $a_{323} = a_{133}$ and we obtain:

$$\rho z_1 = x_1 y_1 + x_2 y_2$$
$$\rho z_2 = \qquad x_3 y_2 + x_1 y_3$$
$$\rho z_3 = (a_{113} x_1 + a_{313} x_3) y_1 + (a_{113} x_2 + a_{323} x_3) y_2 + (a_{323} x_2 - a_{313} x_3) y_3$$

which becomes, after changing the Z-references:

6.3.2 $$\begin{aligned} \rho z_1 &= x_1 y_1 + x_2 y_2 \\ \rho z_2 &= \qquad x_3 y_2 + x_1 y_3 \\ \rho z_3 &= a_{313} x_3 y_1 + a_{323} x_3 y_2 + (a_{323} x_2 - a_{313} x_3) y_3 . \end{aligned}$$

The rank of the linear combinations of the matrices 6.3.2 is three for any non-trivial choice of the a_{ijk} and for generic $\gamma_1, \gamma_2, \gamma_3$. Hence we discard this form.

(e) Case 4b gives, therefore, only the forms 6.2.1.1, 6.3.1.

7. - $R = 1$ (case 4c).

It can rather easily be verified that in this case a suitable choice of the references in the X, Y, Z planes gives rise to either of the following forms:

7.1 $$\rho z_1 = x_1 y_1, \quad \rho z_2 = 0, \quad \rho z_3 = 0$$

7.2 $$\rho z_1 = x_1 y_1, \quad \rho z_2 = x_1 y_2, \quad \rho z_3 = 0$$

7.3 $$\rho z_1 = x_1 y_1, \quad \rho z_2 = x_1 y_2, \quad \rho z_3 = x_1 y_3.$$

Now, the first two represent reducible trilinearities whose components are degenerate, or have dimension different from

4, or belong to case 2.3. The third one gives evidently, for a generic $x(x_1 \neq 0)$, a collineation between Y and Z, which does not depend on x; it has therefore a component of type 2.3 and a degenerate one. All three cases can therefore be discarded.

8. - (a) Let us now examine the only possible cases 5.1.1.1, 5.2.1.2, 6.2.1.1, 6.3.1, 2.3.

As it can easily be checked, in the cases 5 and 6 no plane is such that a generic point S of it determines between the others a collineation independent of S, therefore 5 and 6 are projectively distinct from 2.

In the case 6.3.1 a generic point in any one of the three planes determines between the other two a collineation which is not singular; in the cases 5 and 6.2.1.1 a generic point $Q[R]$ in $Y[Z]$ determines between X and $Z[Y]$ a singular collineation, while a generic point P in X determines between Y and Z a non singular collineation K. In these cases the X plane is therefore projectively distinguished from the other two.

The proof of all these properties is a matter of verification. It follows that 6.3.1 is projectively distinct from the others.

To compare 5.1.1.1, 5.2.1.2, 6.2.1.1 let r be a line in Z, given by $\Sigma_k w_k z_k = 0$; then r and T determine between X and Y a correlation L''': $\Sigma_{ijk} a_{ijk} x_i y_j w_k = 0$. The same can be said for a line s in Y, which gives a correlation L'' between X and Z.

It is easily checked that a line $r[s]$ of $Z[Y]$ gives rise to a correlation $L'''[L'']$ between X and $Y[Z]$, which behaves as follows:

5.1.1.1. L is generically (i. e. for a generic line) non-singular, it has rank 2 for the lines of *one* particular pencil;

5.2.1.2. L is generically non-singular, it never has rank 2;

6.2.1.1. L is generically singular.

This implies projective distinctness of the five considered cases.

(b) In the case 5.1.1.1, the symmetry of the right hand members with respect to the subscripts 1 and 2 and the fact that a_{312}, a_{323} are not simultaneously 0, allows us to use a slight change of the unit point in Z and, possibly, a change of the x' s and the y' s, to obtain $a_{313} = 1$. After that, if we consider

x_1, $x_2 - a_{323}x_1$, x_3, $y_1 + a_{323}y_2$, y_2, y_3, as new variables in X, Y, the trilinearity becomes:

I. $\rho z_1 = x_1y_1 + x_2y_2, \quad \rho z_2 = x_3y_3, \quad \rho z_3 = x_3y_1.$

Case 5.2.1.2 can be rewritten in the form:

II. $\rho z_1 = x_1y_1 + x_2y_2 + x_3y_3, \quad \rho z_2 = x_3y_1, \quad \rho z_3 = x_3y_2.$

In case 6.2.1.1 we have to distinguish two projectively distinct subcases: if $a_{113} \neq 0$ (and we can assume $a_{113} = 1$) the above mentioned correlation L''' has generally rank 2; it becomes 1 for the line $w_1 = w_2 = 0$ and for the pencil $w_1 + w_3 = 0$. If $a_{113} = 0$ (and we can assume $a_{213} = 1$) L''' has generically rank 2; it has rank 1 only if $w_1 = 0$. The same behavior holds for L'', between X and Z, in each of the mentioned cases.

The first case, if we choose $x_1 + a_{213}x_2$, x_2, x_3, y_1, $y_2 - a_{213}y_1$, y_3 as new variables in X, Y, and change slightly the Z-reference, gives rise to:

III. $\rho z_1 = x_1y_1, \quad \rho z_2 = x_2y_2, \quad \rho z_3 = x_2y_3.$

The second one becomes:

IV. $\rho z_1 = x_1y_1 + x_2y_2, \quad \rho z_2 = x_2y_1, \quad \rho z_3 = x_2y_3.$

In case 6.3.1 the above mentioned correlation L''', has generally rank 2. It has rank 1 if $w_1 = w_2 = 0$ or if $w_1 + a_{223}w_3 = w_2 + a_{323}w_3 = 0$. These two lines coincide if and only if $a_{223} = a_{323} = 0$. We distinguish therefore two cases: (i) $(a_{223}, a_{323}) \neq (0, 0)$, (ii) $a_{223} = a_{323} = 0$.

In case 6.3.1 (i), if we interchange the indices 2,3 for x, 1,3 for y and 1,2 for z, the standard form 6.3.1 remains unaltered. There is then no restriction in assuming $a_{323} = 1$. After that, the choice of x_1, x_2, $a_{123}x_1 + a_{223}x_2 + x_3$; y_1, y_2, $a_{223}y_1 - a_{123}y_2 + y_3$; z_1, $z_2 + a_{223}z_1$, z_3 as new variables, reduces 6.3.1 (i) to the form

V. $\rho z_1 = x_1y_1 + x_2y_2, \quad \rho z_2 = x_1y_3, \quad \rho z_3 = x_3y_2.$

Case 6.3.1 (ii) is immediately reduced to

VI. $\rho z_1 = x_1y_1 + x_2y_2, \quad \rho z_2 = x_3y_2 + x_1y_3, \quad \rho z_3 = x_1y_2.$

These two cases are actually distinct from a projective point of view, because: (V) In each one of the planes X, Y, Z there are two distinct lines which make L, L'', L''' become of rank 1; (VI) there is only one line which reduces L to rank 1. The proof is a simple matter of verification.

We already noticed that in case 2.3 the only irreducible non-degenerate trilinearity is:

VII. $$\rho_k x_i = y_i z_k \qquad (i,\ k = 1,\ 2,\ 3)$$

9. We can therefore state (still with the convention 2d):

THEOREM: *Any irreducible trilinearity of dimension* 4 *between three distinct complex projective planes, non-degenerate on any one of them, is projectively equivalent* (i.e.: can be reduced, by means of a suitable choice of the order of the planes and of the systems of reference) *to one and only one of the standard forms* I-VII *of section* 8.

COROLLARY: *In each of the above cases there are no projective invariants.*

BIBLIOGRAPHY

[1] M. BENEDICTY, Rendiconti di Mat. (5), 9, 1950.
[2] M. BENEDICTY, Rendiconti di Mat. (5), 14, 1955.
[3] M. BENEDICTY, Rendiconti del Circ. Mat. Palermo.
[4] A. COMESSATTI, Atti Acc. d'Italia, 14_1 (1944).
[5] G. GHERARDELLI, Rend. Acc. Lincei (6), 27 (1938).
[6] G. GHERARDELLI, Rend. Acc. Lincei (6), 29 (1939).
[7] E. MORGANTINI, Rend. Semin. Padova, 9 (1938).
[8] E. MORGANTINI, Rend. Semin. Padova, 17 (1948).
[9] F. SEVERI, *Serie, sistemi di equivalenza e corrispondenze algebriche sulle varietà algebriche*, Cremonese, Roma (1942).
[10] R. THRALL, T. H. CHANLER, Duke Math. J., 4 (1938).