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A NOTE ON THE DANIELL INTEGRAL

Nota (*) *di* IAN RICHARDS (*a Cambridge, Mass.*)

INTRODUCTION

The object of this paper is to show that by making certain simple modifications in the classical formulation of the Daniell theory of integration it is possible to derive the theorems of Fubini, Helly-Bray, and others from the existence and uniqueness theorems which are the core of the Daniell theory.

The Daniell theory of integration proves the existence and uniqueness of an extension of an « integral » defined on a « lattice » of functions to a « complete integral » defined on a larger « lattice ». (A lattice L of functions is a vector space satisfying the additional condition that $f \varepsilon L$ implies $|f| \varepsilon L$).

We show here that it is possible to prove the uniqueness (but not the existence) of such an extension under slightly weaker conditions, and that this enables us to derive Fubini's theorem as a corollary of our uniqueness theorem. The trick in this and the other applications is to consider vector spaces of functions instead of lattices whenever possibile; since integration is a linear process, the set of functions for which some statement concerning integration is true is usually a vector

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space, but it is often not so clear that it is a lattice. Therefore we try to formulate our basic abstract theorems in terms of vector spaces instead of lattices in order to facilitate their application to the proof of classical identities.

BASIC DEFINITIONS

DEFINITION 1: An *improper integral* I on a real vector space V of real valued functions is a positive linear functional defined on V such that $f_n \in V$, $f_n \downarrow 0 \Rightarrow I(f_n) \downarrow 0$. ($f_n \downarrow f$ means that the sequence of functions $\{f_n\}$ converges pointwise to the function f, and $f_{n+1} \leq f_n$ for all n. Similarly for $f_n \uparrow f$).

DEFINITION 2: An improper integral I on a vector space V is *complete* if & only if $f_n \in V$, $f_n \uparrow f$, $\lim I(f_n) < \infty \Rightarrow f \in V$. (Then by the definition above, $I(f) = \lim I(f_n)$).

DEFINITION 3: A *lattice* L of functions is a real vector space of real valued functions such that $f \in L \Rightarrow |f| \in L$. (Then $f \in L$, $g \in L \Rightarrow \max(f, g) \in L$, $\min(f, g) \in L$).

DEFINITION 4: An *integral* is an improper integral which is defined on a lattice of functions.

DEFINITION 5: The *completion* L^* of a lattice L is the intersection of all sets L' of real valued functions such that

a) $L \subset L'$;

b) $f_n \in L'$, $f_n \uparrow f \Rightarrow f \in L'$;

c) $g_n \in L'$, $g_n \downarrow g \Rightarrow g \in L'$.

EXAMPLES

EXAMPLE 1: Let L be the lattice of all continuous functions with compact support on $(-\infty, \infty)$, and let $I(f) = \int_{-\infty}^{\infty} f(x)dx$ for $f \in L$.

EXAMPLE 2: Let V be the vector space of continuous functions on $(-\infty, \infty)$ such that $I(f) = \lim_{y \to \infty} \frac{1}{y} \int_0^y \int_{-x}^x f(t)dtdx$ exists.

To show that these examples satisfy defintion 1 we use:

LEMMA 1 (Dini's theorem): If $\{f_n\}$ is a sequence of continuous functions with compact support, and $f_n \downarrow 0$, then $f_n(x) \to 0$ uniformly in x.

FUNDAMENTAL THEOREMS

PROPOSITION 1: If L is a lattice, then its completion L^* is a lattice which is closed under pointwise convergence.

Proof: Let M be the set of functions $f \in L^*$ such that $(f+g) \in L^*$ for all $g \in L$. Then $L \subset M$, and M is closed under monotone increasing and decreasing convergence. Hence $M = L^*$.

Now let M' be the set of all functions $f \in L^*$ such that $(f+g) \in L^*$, $af \in L^*$, $|f| \in L^*$ for all $g \in L^*$ and all real a. By the above $L \subset M'$, and thus as before $M' = L^*$. Hence L^* is a lattice.

Now $f_n \in L^*$, $f_n \to f \Rightarrow \max(f_k, \ldots, f_n) \uparrow g_k$ as $n \to \infty$, $g_k = \sup(f_n \mid n \geq k)$, and $g_k \downarrow f$. Hence $f \in L^*$. q. e. d.

THEOREM 1: Let L be a lattice, and let I be a complete improper integral on a vector space V where $L \subset V \subset L^*$. Then $f \in L^*, f \geq 0 \Rightarrow$ either $f \in V$ or $\sup(I(g) \mid g \in V, 0 \leq g \leq f) = \infty$.

COROLLARY: $f \in L^*$, $g \in V$, $0 \leq f \leq g \Rightarrow f \in V$.

Proof of theorem: We first show that $f \in L^*$, $g \in L \Rightarrow \max[\min(f, g), 0] \in V$. To do this we simply observe that the set of functions f for which the above holds contains L and, by definiton 2, is closed under monotone increasing and decreasing convergence. We now use the following:

LEMMA 2: For any $f \in L^*$ there exists a sequence $\{g_n\}$ such that $g_n \in L$, $g_n \geq 0$, $g_n \uparrow$, and $f \leq \lim g_n$. ($\lim g_n$ may not be finite).

Proof of lemma: If $f_n \quad f$, $f_n \leq \lim_{k \to \infty} g_{nk}$, and $g_{nk} \in L$, then $h_n = \max(g_{1n}, \ldots, g_{nn}) \in L$, and $f \leq \lim h_n$. Hence the set of functions for which the lemma holds satisfies the conditions of definition 5. q.e.d.

To prove the theorem, for any $f \geq 0$ in L^* we choose a sequence $\{g_n\}$ so that $g_n \in L$, $g_n \geq 0$, $g_n \uparrow$, and $f \leq \lim g_n$. Then, by the above, if $f_n = \min(f, g_n)$, then $f_n \in V$, $f_n \uparrow f$, and hence by definition 2 either $f \in V$ or $\lim I(f_n) = \infty$. q.e.d.

THEOREM 2 (Uniqueness theorem): Let L be a lattice, and let I and I' be complete improper integrals on vector spaces V and V' respectively where $L \subset V \subset L^*$ and $L \subset V' \subset L^*$. Suppose that $I(f) = I'(f)$ for all $f \in L$. Then $f \in V$, $f \geq 0 \Rightarrow f \in V'$ and $I(f) = I'(f)$.

Proof: Let V'' be the vector space of functions $f \in V \cap V'$ for which $I(f) = I'(f)$. Then $L \subset V''$, and I and I' define a complete improper integral on V''. Furthurmore, $\sup(I(g) | g \in V'', 0 \leq g \leq f) \leq I(f)$. Thus by theorem 1, $f \in V$, $f \geq 0 \Rightarrow f \in V''$. q.e.d.

THEOREM 3 (Existence theorem): An integral I on a lattice L can be extended to a complete integral I^* on a lattice L^1. (In other words, $L \subset L^1$ and $I^*(f) = I(f)$ for all $f \in L$).

Proof: Since this construction is classical, we only outline the main steps; for more detail see the reference (1) by Loomis given below.

For any $f \in L^*$, we shall say that a sequence $\{g_n\}$ is an « upper covering » of f if $g_n \in L$, $g_n \uparrow$, and $f \leq \lim g_n$. Similarly we define lower coverings. (By lemma 2, every $f \in L^*$ has upper and lower coverings).

Now we let $I^+(f) = \inf[\lim I(g_n)]$ for all upper coverings $\{g_n\}$ of f. Similarly we define $I^-(f)$.

We let L^1 be the set of functions $f \in L^*$ such that $I^+(f) = I^-(f) \neq \pm\infty$, and let $I^*(f) = I^+(f)$ for all $f \in L^1$.

To show that I^* is a complete integral on L^1 it is sufficient to prove (a) — (g) below: ((a) — (d) show that I^* is linear; (e) shows that I^* is positive; (f) shows that L^1 is a lattice; given that I^* is linear, the continuity and completeness conditions of definitions 1 & 2 follow from (a) and (g). Since $I^+(f) \leq I(f) \leq I^-(f)$ for all $f \in L$, $I^*(f) = I(f)$ for all $f \in L$).

a) $I^+(f) \geq I^-(f)$;

b) $I^+(f) = -I^-(-f)$;

c) $I^+(f+g) \leq I^+(f) + I^+(g)$;

d) $I^+(af) = a \cdot I^+(f)$ for $a > 0$;

e) $f \geq g \Rightarrow I^+(f) \geq I^+(g)$ and $I^-(f) \geq I^-(g)$;

f) $I^+[\max(f, 0)] + I^+[\min(f, 0)] = I^+(f)$;

g) $f_n \geq 0$, $f = \Sigma f_n \Rightarrow I^+(f) \leq \Sigma I^+(f_n)$.

(a) follows from the continuity condition of definition 1. (g) is proved by the standard trick of covering each f_n by an upper covering $\{g_{nk}\}$ so that $\lim\limits_{k\to} I(g_{nk}) \leq I^+(f_n) + \epsilon/2^n$. The other statements are easily verified.

APPLICATIONS

A) (Lebesgue Dominated Convergence Theorem): If I is a complete integral on a lattice L, then $f_n \in L$, $g \in L$, $|f_n| \leq |g|$, $f = \lim f_n \Rightarrow f \in L$ and $I(f) = \lim I(f_n)$.

Proof: Let $g_k = \sup(f_n \mid n \geq k)$; then $\max(f_k, \ldots, f_n) \uparrow g_k$, $|g_k| \leq |g|$, and $g_k \downarrow f$. Hence $f \in L$, and $\limsup I(f_n) \leq I(f)$. Similarly $\liminf I(f_n) \geq I(f)$. q.e.d.

DEFINITION 6: Let S be a space with a complete integral I on a lattice L of functions whose domain of definiton is S.

For any subset E of S and any real function $f(x)$ defined on S, we define the function $f_E(x)$ by: $f_E(x) = f(x)$ for all $x \in E$ and $f_E(x) = 0$ for all $x \in (S - E)$.

We say a subset E of S is *measurable* if for any function $f \geq 0$ on S such that $\min(f, g) \in L^*$ for all $g \in L^*$ we have $f_E \in L^*$. (Then, if E is measurable, $f \in L^* \Rightarrow f_E \in L^*$).

A measurable set E is a *null set* if $f \in L^* \Rightarrow f_E \in L$, $I(f_E) = 0$. A condition that holds on S except on a null set is said to hold *a.e.* (almost everywhere).

Note: It is clear that the class of measurable subsets of S is a σ-ring, and for any $f \geq 0$ such that $\min(f, g) \in L^*$ for all $g \in L^*$ the set function $m(E) = I(f_E)$ when $f_E \in L$, $m(E) = \infty$ when $f_E \in (L^* - L)$, is a countably additive measure on this σ-ring. (In most interesting cases, the function $f(x) = 1$ satisfies the condition $\min(f, g) \in L^*$ for all $g \in L^*$).

PROPOSITION 2: If $f_n \in L^*$ and $f_n \uparrow$, then the set $E = (x \mid \lim f_n(x) = \infty)$ is measurable. If f_n is integrable for each n, and $\lim I(f_n) < \infty$, then E is a null set.

COROLLARY: If $f \in L^*$, $f \geq 0$, then the set $E = (x \mid f(x) > 0)$ is measurable. If f is integrable, then $I(f) = 0$ if & only if E is a null set. (To prove this from the above we merely consider the sequence of functions $f_n = n \cdot f$).

Proof: Take any $f \geq 0$ such that $g \in L^* \Rightarrow \min(f, g) \in L^*$. Then $g_{nk} = \min(f, 1/k \cdot f_n) \in L^*$, and $f_E = \lim\limits_{k \to \infty} \lim\limits_{n \to \infty} g_{nk}$. Hence $f_E \in L^*$. If $I(f_n) \leq M$ for all n, then $I(g_{nk}) \leq M/k$ for all n and $I(f_E) = 0$. q.e.d.

B) (Fubini's theorem): Let S and T be spaces with lattices L_s and L_t of functions defined on S and T respectively.

Let I_s and I_t be complete integrals defined on L_s and L_t respectively.

Suppose that L is a lattice of functions $f(s, t)$ defined on $S \times T$ such that the «double integrals» $I_s[I_t(f(s, t))]$ and $I_t[I_s(f(s, t))]$ are defined and equal for all $f \in L$.

We let V_{st} be the vector space of functions $f(s, t) \in L^*$ such that $f(s, t) \in L_t$ a.e. (in s) and $I_t(f(s, t)) \in L_s$; for $f \in V_{st}$ we let $I_{st}(f) = I_s[I_t(f(s, t))]$. Similarly we define V_{ts} and I_{ts}.

Then $f \in V_{st}$, $f \geq 0 \Rightarrow f \in V_{ts}$ and $I_{st}(f) = I_{ts}(f)$.

Proof: Clearly I_{st} and I_{ts} define an integral on L. By proposition 2, I_{st} and I_{ts} are complete improper integrals on V_{st} and V_{ts}. The desired result now follows from theorem 2.

Note: V_{st} and V_{ts} are not lattices. Furthermore, there may exist functions f in V_{st} which do not belong to V_{ts} as well as f in both V_{st} and V_{ts} for which $I_{st}(f) \neq I_{ts}(f)$. (Of course in these cases we do not have $f \geq 0$).

APPLICATIONS TO FUNCTIONS OF A REAL VARIABLE

In this section we shall use the following notations: (By « function » we shall mean a function of a real variable).

C denotes the lattice of all continuous functions with compact support (i.e. vanishing outside of compact sets).

S denotes the lattice of « step functions » — i.e. functions which are constant on the interiors of a finite disjoint set of closed intervals and zero elsewhere.

S' denotes the lattice of « sawtooth functions » — i.e. continuous functions which are linear on each of a finite set of closed intervals and zero elsewhere.

Proposition 3 (Riesz Representation Theorem): Let V be a vector space of functions with $S' \subset V \subset C$. Then any positive linear functional I on V is an improper integral and can be extended to a complete integral on a lattice L^1 containing C.

Proof: That I is an improper integral on V follows at once from lemma 1.

Let I' be the restriction of I to S'. By theorem 3, I' can be extended to a complete integral I^* on a lattice L^1.

For any $f \in C$ there is a sequence $g_n \in S'$ such that $g_n \uparrow f$.

Hence $C \subset L^1$, and $I^*(f) = I(f)$ for all $f \in V$. q.e.d.

Note: Proposition 3 gives a very simple construction of the Riemann integral and the Riemann-Stieltjes integral. First we can define the Riemann integral for functions in S' ; The extension of proposition 3 gives us the Lebesgue integral. The Riemann-Stieltjes integral of functions in S' can now be defined by the usual formula for integration by parts, since this replaces the Stieltjes integral by a Riemann integral. It is easily shown from the dominated convergence theorem (A) above that these definitions give the right result when the domain of integrable functions is extended to include the « step functions ».

C) (Integration by Parts): Let $f(x)$ and $g(x)$ be functions in L^1 and let $F(x) = \int_a^x f(t)dt$, $G(x) = \int_a^x g(t)dt$. Then $\int_a^b f(t)G(t)dt + \int_a^b g(t)F(t)dt = F(b)G(b) - F(a)G(a)$.

Proof: An elementary calculation shows that this holds when f and g are « step functions ». But if g is held fixed, then the expressions on both sides of the above identity define complete improper integrals over the vector spaces of functions f for which they exist. It is easily seen that the completions S^* and C^* of S and C are equal; then, by theorem 2, the identity holds for all $f \in L^1$, $g \in S$.

The same argument extends the proof to the case when both f and g are in L^1.

D) If $F(x)$ is a monotone increasing function, $g(x) \geq 0$ is L' with respect to $dF(x)$, and $G(x) = \int_a^x g(t)dF(t)$, then the two integrals, $I(h) = \int_a^b h(x)dG(x)$ and $I'(h) = \int_a^b h(x)g(x)dF(x)$ are identical.

The proof is the same as that for (C) above.

E) (Helly-Bray Convergence Theorem): Let $\{F_n(x)\}$ be a uniformly bounded sequence of monotone increasing functions on $[a, b]$ which converge to a monotone increasing function $F(x)$ at all points x where $F(x)$ is continuous. (We assume that $F(x)$ is continuous at $x=a$, $x=b$). Let $g(x)$ be any continuous function on $[a, b]$. Then $\int_a^b g(x)dF(x) = \lim \int_a^b g(x)dF_n(x)$.

Proof: Integrating by parts and using the Lebesgue dominated convergence theorem (A), we see that, since the discontinuities of $F(x)$ are countable, the Helly-Bray theorem holds for all $g(x)$ in S'. (The same argument could be used for any continuous $g(x)$ of bounded variation).

Now let V be the vector space of all functions $f(x)$ in C such that $I(f) = \lim \int_a^b f(x)dF_n(x)$ exists. Then by proposition 3, I can be extended to a complete integral on a lattice L^1 containing C. But we have seen that $I(f) = \int_a^b f(x)dF(x)$ for $f(x)$ in S'. Hence by theorem 2, $\int_a^b f(x)dF(x)$ is equal to $I(f)$ for all $f \in V$.

Now by the Bolzano-Weierstrass theorem, there are subsequences $\{F_{n_K}(x)\}$ such that $\lim_{k \to \infty} \int_a^b g(x)dF_{n_K}(x)$ exists. By what we have shown, this limit always equals $\int_a^b g(x)dF(x)$. q.e.d.

A COUNTEREXAMPLE

We ask whether an improper integral can always be extended to a complete improper integral. The answer is « no », and the uniqueness theorem (2) does not hold unless we consider complete improper integrals.

EXAMPLE 3: Let V be the vector space of continuous functions $f(x)$ on the open interval $(0, 1)$ such that $A(f) = \lim_{x \to 0} f(x)$ and $B(f) = \lim_{x \to 1} f(x)$ exist, and $A(f) = -B(f)$. Let $I(f) = A(f)$ for all $f \in V$.

Then I is an improper integral on V. (Note that $f \in V$, $f \geq 0 \Rightarrow I(f) = 0$). But there is a sequence $\{f_n\}$ such that

$f_n \in V$, $f_n \downarrow -1$, and $I(f_n) = 1$ for all n. Therefore I cannot be extended to a complete improper integral.

Furthurmore, if L is the lattice of functions f in V such that $I(f) = 0$, then $V \subset L^*$; but the restriction of I to L has another extension to an improper integral on V -namely $I(f) = 0$ for all $f \in V$.

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