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A NOTE CONCERNING THE INTEGRALS INVOLVING THE DERIVATIVES OF LEGENDRE POLYNOMIALS

Nota (*) *di* BLAGOJ SAZDOV POPOV (*a Skopje*)

Here is an alternative, shorter, proof of the result proved by S. K. Chatterjee [1]).

From the well known formula

$$\int_{-1}^{1} uv^{(n)}dx = \left[\sum_{k=1}^{n} (-1)^{k-1} u^{(k-1)} v^{(n-k)}\right]_{-1}^{1} + (-1)^n \int_{-1}^{1} vu^{(n)}dx,$$

it follows that

$$\int_{-1}^{1} \frac{d^r P_m(x)}{dx^r} \frac{d^s P_n(x)}{dx^s} dx = \left| \sum_{k=1}^{r} (-1)^{k-1} \frac{d^{s+k-1} P_n(x)}{dx^{s+k-1}} \frac{d^{r-k} P_m(x)}{dx^{r-k}} \right|_{-1}^{1} +$$

$$+ (-1)^r \int_{-1}^{1} P_m(x) \frac{d^{r+s} P_n(x)}{dx^{r+s}} dx, \qquad (1 \leq r \leq m,\ 1 \leq s \leq n).$$

Since

$$\left|\frac{d^r P_n(x)}{dx^r}\right|_{x=1} = P_n^{(r)}(1) = (-1)^{n-r} P_n^{(r)}(-1) = (2r-1)!!\binom{n+r}{n-r},$$

$$(2r-1)!! = 1.3 \ldots (2r-1),$$

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1) S. K. CHATTERJEE, *On certain definite Integrals involving Legendre's Polynomials*, Rend. Sem. Mat. Univ. Padova, vol. XXVII, 1957, pp. 144-148.

and

$$\int_{-1}^{1} P_m(x) \frac{d^{r+s}P_n(x)}{dx^{r+s}} dx = 0, \qquad m > n - r - s,$$

we have

$$\int_{-1}^{1} \frac{d^r P_m(x)}{dx^r} \frac{d^s P_n(x)}{dx^s} dx = [1 + (-1)^{m+n-r-s}] 2^{1-s-r} \sum_{k=1}^{r} (-1)^{k-1} (r-k)!$$

$$\times (s+k-1)! \binom{n+s+k-1}{s+k-1} \binom{n}{s+k-1} \binom{m+r-k}{r-k} \binom{m}{r-k},$$
$$(m - r \geq n - s).$$

The same method can be employed to obtain some other more general integral formulae for the product of the derivatives of Legendre polynomials.

We find

$$\int_{-1}^{1} \frac{d^p P_l(x)}{dx^p} \frac{d^r P_m(x)}{dx^r} \frac{d^s P_n(x)}{dx^s} dx =$$

$$= K \sum_{k=1}^{p} \sum_{\nu=0}^{k-1} (-1)^{k-1} \binom{k-1}{\nu} (p-k)! (r+k-\nu-1)! (s+\nu)!$$

$$\times \binom{l+p-k}{p-k} \binom{l}{p-k} \binom{m+r+k-\nu-1}{r+k-\nu-1} \binom{m}{r+k-\nu-1} \binom{n+s+\nu}{s+\nu} \binom{n}{s+\nu}$$

$$K = [1 + (-1)^{m+n+l-p-r-s}] 2^{1-p-r-s}, \quad l-p \geq (m-r)+(n-s),$$

$$1 \leq p \leq l, \quad 1 \leq r \leq m, \quad 1 \leq s \leq n.$$