RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

OTTON MARTIN NIKODÝM

Critical remarks on some basic notions in Boolean lattices. II

Rendiconti del Seminario Matematico della Università di Padova, tome 27 (1957), p. 193-217

<http://www.numdam.org/item?id=RSMUP_1957__27__193_0>

CRITICAL REMARKS ON SOME BASIC NOTIONS IN BOOLEAN LATTICES. II.

Nota (*) *di* OTTON MARTIN NIKODÝM (*Gambier, Ohio (U.S.A.)*)

The present paper is the continuation and termination of the other one by the author: « Critical remarks [1]) on some basic notions in Boolean lattices I ».

In the part I the role of the notion of equality is analyzed and the notion of homomorphism is clarified. This part II deals with ideals and with the notion of genuine extension in Boolean tribes. There are added examples concerning various behaviour of non negative measures.

These two papers have not an expository character, though neither new theorems nor problems have been aimed at. As mentioned in part I, the above important notions were rather confused in the literature and therefore they should be clarified through a deeper analysis and precise definitions. The author endevoured to do this to the extent he needed for his own research work. The author believes that the papers I and II will be useful for mathematicians interested in Boolean tribes.

Since this Part II is the continuation of the Part I and since both parts should be considered as a totality, we refer continuously to I, and even conserve the current

---

(*) Pervenuta in Redazione il 3 maggio 1957.

Indirizzo dell'A.: Kenyon College, Gambier, Ohio (U.S.A.).

(1) Anais da Academia Brasileira de Ciencias. Vol. 24, (1952), p. 113-136. Both papers represent part of the work of the author under a cooperative contract between the U. S. Atomic Energy Commission and Kenyon College, (Ohio). The manuscript of I and II was submitted, March 1951, to the A. E. C. and afterward sent, upon invitation. te the Anais where it was accepted. However Part II could not there appear for reasons till now (30/VI 1957) unknown to the author. This Part II appears now under the grant of the U. S. National Science Foundation.

numbers of chapters and sections. Numbers in [] refer to the list of papers at the end. Only those papers are quoted which have been essentially used by the author, and no complete bibliography of the topic has been intended.

§ 3. - Ideals.

18. - We recollect the fundamental notions and properties of ideals in a Boolean tribe, for this may be useful in giving precise proofs of several statements involved in examples sketched in this paper.

Since a tribe (A) can be conceived as a Stone's ring, we con consider ideals in (A), as we use to do in any ring. The ordinary definition of an ideal, taken from algebra [10] is, in the case of a tribe, equivalent to the following one:

By an *ideal in* (A) we understand any not empty set J of somata such that:

1) if $a, b \in J$, then $a + b \in J$,

2) if $a \in J$, $b \leq a$, then $b \in J$ (²).

The condition 1) can be replaced by:

1') if $a, b \in J$, then $a \dot{+} b \in J$.

The ideal J generates the notion of *equivalence modulo* J for somata, $a =^J b$, defined by $a \dot{+} b \in J$.

The obvious equalities:

$$a = b + (a - b) - (b - a), \quad a = b - (b - a) + (a - b),$$
$$a + (a \dot{+} b) = b + (a \dot{+} b), \quad a - (a \dot{+} b) = b - (a \dot{+} b)$$

give a proof of the equivalence of several following statements: I) $a =^J b$; II) there exists $p, q \in J$ with $a = b + p - q$; III) there exists $p, q \in J$ with $a = b - p + q$; IV) there exists $p \in J$ with $a + p = b + p$; V) there exists $p \in J$ with $a - p = b - p$.

The equivalence-relation modulo J possesses the formal properties of the identity, and is a relation whose domain, range and field coincide with A. The operations $a + b$, $a \cdot b$,

---

²) Of course, since J is a set of somata, J aught to be equality-invariant.

co a, $a - b$, $a \dot{+} b$ are invariant with respect to « $=^J$ » e. g. if $a =^J a'$, $b =^J b'$, then $a \cdot b =^J a' \cdot b'$. The relation $a =^J b$ is equality-invariant, but $a = b$ and $a \leq b$ are not invariant with respect to « $=^J$ ».

If we define on A a new ordering relation $x \leq^J y$ defined by $x \cdot y =^J x$, the collection A will be organized into a new Boolean tribe (A_J), which we shall call the *equivalence-tribe of* (A) *modulo* J. The notion of equality in (A_J), generated by $\leq^J$, is precisely the equivalence $=^J$ introduced above.

The statements $x +^J y =^J z$, $\text{co}^J x =^J y$ etc. are equivalent to $x + y =^J z$, $\text{co}\, x =^J y$ etc. respectively, hence we can use old operation symbols $-$, $\cdot$, co, etc. instead of the new ones $+^J$, $\cdot^J$, co^J, etc. when dealing with (A_J).

The statement $x \leq^J y$ is equivalent to $x - y \in J$. We also notice that if $a + b =^J c$, there exist a_1, b_1 such that $a =^J a_1$, $b =^J b_1$ $a_1 + b_1 = c$, and similarly for other operations. If S is a relation with $\text{(|}\, S \equiv \text{|)}\, S \equiv \text{(|)}\, S \equiv A$, having the formal properties of identity, being equality invariant, and such that the operations $a + b$, $a \cdot b$, are invariant with respect to S, then there exists a well determined ideal J in (A) such that aSb is equivalent to $a =^J b$.

19. - An ideal J splits the set A of all somata into mutually disjoint subsets, called *equivalence-classes modulo J*. We define:

Given a soma a, the set of all somata x such that $x =^J a$ is termed the *equivalence class modulo J determined by a*, is denoted by $[a]$, and a is said to be a *representative of* $[a]$. The following are equivalent: I) $[a] \cap [b] \neq 0$; II) $a =^J b$; III) $[a] \equiv [b]$. We have $J \equiv [0]$.

If we introduce for the equivalence classes ξ, η, ... the ordering relation

$$\xi \leq' \eta \cdot \underset{df}{=} \cdot a - b \in J,$$

where a, b are representatives of ξ, η respectively, the relation thus defined is independent of the choice of the representatives a, b of ξ, η, and it organizes the class $\boldsymbol{A}$ of all equivalence classes into a Boolean tribe, called the *quotient* *tribe of* (A) *modulo* J, and denoted by $(A)/J$ or $[A_J]$. The equality $='$ on $[A_J]$ is the identity of classes.

For the operations in $[A_J]$ we have

$$[a] +' [b] =' [a + b].\ 1' =' [1], \text{ etc.}$$

The set $[a] +' [b]$ is identical with the set of all $a_1 + b_1$, where $a =^J a_1$, $b =^J b_1$, and analogously for other operations.

Both tribes (A_J) and $[A_J]$ are finitely isomorphic, hence even strongly completely isomorphic, the isomorphism R from (A_J) onto $[A_J]$ being defined by

$$aR\xi \cdot \underset{df}{=} \cdot \xi = [a] \text{ [3]}.$$

19. - There is a hemimorphism S from (A) onto (A_J), defined by

$$aSa' \cdot \underset{df}{=} \cdot a =^J a',$$

and hence, there is also a hemimorphism T from (A) onto $[A_J]$ defined by

$$aT\xi \cdot \underset{df}{=} \cdot a \in \xi.$$

Conversely, if Q is a hemimorphism from (A) onto (A'), then the set of all somata $x \in A$, such that xQO', is an ideal J in (A), and Q satisfies the conditions

1) aQa', bQa' imply $a =^J b$.
2) if aQa', $a =^J b$, then bQa'.

20. - For some purposes we use to introduce denumerably additive ideals, even in finitely additive Boolean tribes.

We have a double choice in definition:

(I′) If $a_1, a_2, \dots, a_n, \dots \in J$, then $\sum_{n=1}^{\infty} a_n$ is meaningful and $\in J$;

(I'_w) If $a_1, a_2, \dots, a_n, \dots \in J$, and $\sum_{n=1}^{\infty} a_n$ is meaningful, then $\sum_{n=1}^{\infty} a_n \in J$.

---

[3]) Usually no discrimination is made between (A_J) and $[A_J]$, and both are called « the quotient tribe modulo J » though they are really different.

An ideal satisfying (I′) will be termed ***denumerably additive ideal in*** (A), while such one which satisfies (I'_w), will be termed ***weak denumerably additive ideal in*** (A). An ideal will be termed ***finitely additive*** if we shall want to emphasize that no one of the conditions (I′), (I'_w) is supposed to take place.

20.1 - An ideal may not satisfy neither (I′) nor (I'_w). A denumerably additive ideal is also a weak denumerably additive ideal, but the converse is not true.

1) E. g. Let (A) be the tribe of all subsets of $\langle 0, 1\rangle$ with set–inclusion as ordering relation. The class J of all at most finite point–sets in $\langle 0, 1\rangle$ is an ideal but it does not satisfy neither (I′) nor (I'_w).

2)E. g. Let A be the smallest collection of subsets of $\langle 0, 1)$ containing each single–point set and every half–open interval $\langle\alpha, \beta)$ where $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, and besides, such that if $a, b \in A$, then $a \cup b \in A$, $\langle 0, 1) \sim a \in A$. The class of all at most finite point–sets is an ideal satisfying (I'_w) but not (I′).

21. - We shall sketch a proof of the following theorem:

If (A) is a tribe, (J) a denumerably additive ideal in it, $a_1, a_2, \dots, a_n, \dots \in A$, $\sum\limits_{n=1}^{\infty} a_n$ is meaningful, $a_n =^J b_n$, $(n = 1, 2, \dots)$, then $\sum\limits_{n=1}^{\infty} b_n$ is also meaningful, and $\left(\sum\limits_{n=1}^{\infty} a_n\right) =^J \left(\sum\limits_{n=1}^{\infty} b_n\right)$.

Proof. There exist $p_n \in J$ such that $a_n + p_n = b_n + p_n$.

Since J is a denumerably additive ideal, the sum $\sum\limits_{n=1}^{\infty} p_n$ is meaningful. Hence

$$\sum_{n=1}^{\infty} (a_n + p_n) = \sum_{n=5}^{\infty} a_n + \sum_{n=1}^{\infty} p_n, \quad \text{(see N. 9)}.$$

Since

$$\sum_{n=1}^{\infty} p_n \in J,$$

we have

$$b_n \cdot \sum_{m=1}^{\infty} p_m \in J,$$

hence

$$\sum_{n=1}^{\infty} \left[b_n \cdot \left(\sum_{m=1}^{\infty} p_m \right) \right] \in J.$$

Put

$$c \underset{df}{=} \sum_{n=1}^{\infty} (b_n + p_n) - \sum_{n=1}^{\infty} p_n + \sum_{n=1}^{\infty} \left[b_n \cdot \sum_{m=1}^{\infty} p_m \right].$$

We get:

$$c = \sum_{n=1}^{\infty} \left[b_n \cdot \mathrm{co} \sum_{m=1}^{\infty} p_m \right] + \sum_{n=1}^{\infty} \left[b_n \cdot \sum_{m=1}^{\infty} p_m \right] = \sum_{n=1}^{\infty} b_n .$$

hence $\sum\limits_{n=1}^{\infty} b_n$ is meaningful.

Now we have $b_n = a_n - r_n + q_n$, where r_n, q_n are some elements of J. Put $d_n \underset{df}{=} a_n - r_n$. The sum $\sum\limits_{n=1}^{\infty} d_n = \sum\limits_{n=1}^{\infty} (a_n - r_n)$ is meaningful.

Now we have

$$\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} (a_n - r_n) = \sum_{n=1}^{\infty} \left[a_n \cdot \prod_{m=1}^{\infty} \mathrm{co}\,(a_m - r_m) \right].$$

Since

$$a_n \cdot \prod_{m=1}^{\infty} \mathrm{co}\,(a_m - r_m) \leq a_n \cdot (\mathrm{co}\, a_n + r_n) \leq r_n ,$$

we get

$$\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} (a_n - r_n) \in J.$$

On the other hand

$$\sum_{n=1}^{\infty} (a_n - r_n) - \sum_{n=1}^{\infty} a_n = 0,$$

which gives

(1) $$\sum_{n=1}^{\infty} d_n \overset{J}{=} \sum_{n=1}^{\infty} a_n .$$

Having this, consider

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (d_n + q_n).$$

We prove similarly as before that

$$\sum_{n=1}^{\infty} (d_n + q_n) - \sum_{n=1}^{\infty} d_n \in J$$

which gives

(2) $$\sum_{n=1}^{\infty} d_n =^J \sum_{n=1}^{\infty} b_n.$$

The results (1) and (2) complete the proof.

21.1. - Under similar circumstances, if $\prod_{n=1}^{\infty} a_n$ is meaningful, then

$$\prod_{n=1}^{\infty} a_n =^J \prod_{n=1}^{\infty} b_n.$$

The above theorems show that, even for finitely additive tribes, if the ideal is denumerably additive, then denumerable operations are $=^J$-invariant. This is obviously not the case when J is only a weak denumerably additive ideal, because it may happen that $a_n \in J$, $\sum_{n=1}^{\infty} a_n$ is not meaningful, though $a_n =^J 0$.

Denumerably additive ideals are especially important in the case the tribe itself is denumerably additive. In this case the proof of theorem n. 21 may be replaced by a very simple one.

22. - Let (A) be a finitely additive tribe and J a denumerably additive ideal in it; if $\sum_{n=1}^{\infty} a_n =^J b$, then there exist b_n such that $a_n =^J b_n$ and $\sum_{n=1}^{\infty} b_n = b$. An analogous property holds also for denumerable products.

Let J be a denumerably additive ideal in (A), and $[A_J]$

the tribe of equivalence classess modulo J; if $\sum_{n=1}^{\infty} a_n$ is meaningful, then $\left[\sum_{n=1}^{\infty} a_n\right] =' \sum_{n=1}^{\infty}{}' [a_n]$, and $\left[\sum_{n=1}^{\infty} a_n\right]$ coincides with the set of all $\sum_{n=1}^{\infty} b_n$ where $b_n =^J a_n$, $(n = 1, 2, \ldots)$.

Let us remark that it may happen, even if J is a denumerably additive ideal, that $\sum_{n=1}^{\infty}{}' [a_n]$ is meaningful. but $\sum_{n=1}^{\infty} a_n$ does not exist at all.

E. g. Let (A) be the tribe of all subsets of $(0, 1)$ having the form

$$(\bigcup_i (\alpha_i \beta_i) \cup E) \backsim F,$$

where E, F are at most denumerable sets of points, where the union is finite and where $0 \leq \alpha_i \leq 1$, $0 \leq \beta_i \leq 1$; we take set-inclusion as ordering relation on A.

The class of all at most denumerable sets of points is a denumerably additive ideal in (A).

Let Φ be a perfect non dense set in $(0, 1)$ with $0 \in \Phi$. $1 \in \Phi$, and let (p_i, q_i), $(i = 1, 2, \ldots)$ be all corresponding free intervals. If we put $a_n \underset{df}{=} (p_n, q_n)$, the sum $\sum_{n=1}^{\infty} a_n$ will be not meaningful, though $\sum_{n=1}^{\infty}{}' [a_n] =' [1]$.

If S is a general « equality »-relation, with $\mathrm{C}\!\!\mid S \equiv \mathrm{D} S \equiv A$. and such that finite and infinite operations on somata of A are S-invariant in the following sense: « if $\sum_{n=1}^{\infty} a_n$ is meaningful, $a_n S b_n$, $(n = 1, 2, \ldots)$, then $\left(\sum_{n=1}^{\infty} b_n\right) S \left(\sum_{n=1}^{\infty} a_n\right)$ », then S generates a denumerably additive ideal J in A, defined as the set of all somata x, for which $xS0$.

23. - If J is a denumerably additive ideal in (A), then the known homomorphism from (A) onto (A_J) is a denumerable-operation homomorphism.

The converse is not true, but we have the following theorems:

If T is a denumerable-operation homomorphism from (A) into (A'), and if J is the set of all somata $a \in A$ such that $aT0'$, then J is a weak denumerably additive ideal.

If T is a strong denumerable-operation homomorphism, then the above J is a denumerably additive ideal.

E. g. Let (A) be the tribe of all subsets of $(0, 1)$, $(\bigcup_i (\alpha_i, \beta_i) \cup \cup p) \sim q$ where p, q are at most finite sets of points, where the set-union is finite, and $0 \leq \alpha_i \leq 1$, $0 \leq \beta_i \leq 1$; let (A') be the tribe of all $\bigcup_i (\alpha_i, \beta_i)$ as before, and let the ordering relations on (A) and (A') be set-inclusions.

Let T be the correspondence which attaches $\bigcup_i (\alpha_i, \beta_i)$ to every $(\bigcup_i (\alpha_i, \beta_i) \cup p) \sim q$. The relation T is a denumerable operation homomorphism from (A) onto (A').

The corresponding ideal J is the class of all at most finite sets of points, hence it is a weak denumerably additive ideal.

§ 4. - Genuine extension of a tribe.

24. - Let (A), (A') be two tribes. Notions related to (A') will be denoted with primed symbols.

(A) is said to be a *finitely-genuine subtribe of* (A'), and (A') a *finitely genuine extension of* (A), if and only if the following conditions are satisfied for all somata $a, b, \ldots$ of (A):

I° $a + b = c$ is equivalent to $a +' b =' c$,

II° $a \cdot b = c$ is equivalent to $a \cdot' b =' c$.

III° $1 =' 1'$.

IV° $0 =' 0'$.

It follows that $A \subseteq A'$.

In the case the above conditions are satisfied and besides for somata a, b of (A)

V° $\sum_{n=1}^{\infty} a_n = b$ is equivalent to $\sum_{n=1}^{\infty}{}' a_n =' b$.

the tribe (A) is said to be a *denumerably genuine subtribe of* (A'), and (A') a *denumerably genuine extension of* (A).

If besides I° - IV° we also have for somata of (A)

V'. $\sum_{\alpha \in M} a_\alpha = b$ is equivalent to $\sum'_{\alpha \in M} a_\alpha =' b$,

(for every not empty set M of indices),

we say that (A) is a *completely genuine subtribe of* (A'), and (A') is a *completely genuine extension of* (A).

24.1. - If (A) is a finitely genuine subtribe of (A'), then the following statements on somata $a, b, \ldots$ of (A) are equivalent:

$$\text{I)}\ a \leq b \quad \text{and} \quad \text{I')}\ a \leq' b,$$

$$\text{II)}\ a = b \quad \text{and} \quad \text{II')}\ a =' b.$$

The equivalence of I) and I'), and also of II) and II') follows from I° only, and also from II° only.

If (A) is a finitely genuine subtribe of (A'), then the following couples of statements on somata of (A) are equivalent:

$$a = \operatorname{co} b \quad \text{and} \quad a =' \operatorname{co}' b,$$

$$a - b = c \quad \text{and} \quad a -' b =' c,$$

$$a \dot{+} b = c \quad \text{and} \quad a \dot{+}' b =' c.$$

If (A) is a denumerably genuine subtribe of (A'), then for somata of (A) the statement

$$\prod_{n=1}^{\infty} a_n = b$$

is equivalent to

$$\prod_{n=1}^{\infty}{}' a_n =' b.$$

If (A) is a completely genuine subtribe of (A'), then for somata of (A) the statement

$$\prod_{\alpha \in M} a_\alpha = b$$

is equivalent to $\prod'_{\alpha \in M} a_\alpha =' b$ for $M \neq \emptyset$.

25. - We shall prove the independence of the conditions I°, II°, III°, IV° [4]).

E. g. 1) where I°, III°, IV° are satisfied but II° is not satisfied.

Let A be the collection of the following subsets of $\langle 0, 1 \rangle$:

$$\emptyset, \quad \bigcup_i \langle \alpha_i, \beta_i \rangle$$

where the set-union is finite and where $0 \leq \alpha_i < \beta_i \leq 1$. Let the ordering relation on A be the inclusion of sets [5]).

Let A' be the smallest collection of subsets of $\langle 0, 1 \rangle$ containing every closed interval $\langle \alpha, \beta \rangle$, every single-point set, and such that if $a', b' \in A'$, then $a' \cup b' \in A'$, and if $a' \in A'$, then $\langle 0, 1 \rangle \sim a' \in A'$. Let the ordering relation on A' be the set-inclusion.

For $a, b \in A$ we have

$$a \cdot b = \overline{(a \cap b)^{0}}, \qquad \text{co}\, a = \overline{[\langle 0, 1 \rangle \sim a]^{0}},$$

where E^0 denotes the set of all inner points of E and $\bar{E}$ denotes the closure of E.

The following are equivalent for somata of (A):

$$a + b = c \quad \text{and} \quad a +' b =' c$$

and we also have

$$0 =' 0 \quad , \quad 1 =' 1'.$$

Nevertheless if we put $a \underset{df}{=} \left\langle 0, \frac{1}{2} \right\rangle$, $b \underset{df}{=} \left\langle \frac{1}{2}, 1 \right\rangle$ we have $a \cdot b = 0$ and $a \cdot' b ='$ the set composed of the single point $1/2$.

E. g. 2) To have an example where II°, III°, IV° are satisfied and I° is not satisfied, it suffices to take the same collections A, A' as in the preceding example 1), and define

---

4) The notion of genuine extension and the theorem stating the independence of I° - IV° were presented to the Intern. Congress of Mathem. in Cambridge (Mass.) (1950).

5) This tribe is used in the S. SAKS' book on integration [22], and the somata are termed « figures ».

on both the ordering relation as the converse of the set-inclusion.

E. g. 3) Where I°, II°, III°. are true but IV° is not true.

Let A be the class of all sets $\omega \cup F$, where $\omega \underset{df}{=} \langle -2, -1 \rangle$ and F is any subset of $\langle 0, 1 \rangle$. Let $\leq$ mean the set-inclusion.

Let A' be the smallest collection of subsets of $\langle 0, 1 \rangle \cup \omega$ containing A, and such that if $a' \in A'$, then

$$[\langle -2, -1 \rangle \cup \langle 0, 1 \rangle \sim a'] \in A'$$

and if $a', b' \in A'$, then $a' \cup b' \in A'$. The ordering relation on A' is defined as set-inclusion.

The following are equivalent for somata of A:

$$a + b = c \quad \text{and} \quad a +' b =' c :$$

$$c \cdot b = c \quad \text{and} \quad a \cdot' b =' c :$$

besides we have $1 =' 1'$, but $0 \equiv \omega$, $0' \equiv \emptyset$, hence we have $0 \neq' 0'$.

E. g. 4) An example where I°. II°. IV are satisfied but III° does not hold is given by the same collections A, A' as in the preceding example 3), if we define, on both, the ordering relations as the converse of the set-inclusion.

25.1. - Let us remark, that in all four above examples the following statements are equivalent for the somata of (A):

(1) $$a \leq b \quad \text{and} \quad a \leq' b .$$

It follows that if we admit for the two tribes (A) and (A') the only condition (1) none of the conditions I°, II°, III°, IV° will follow. We can even find an example where (1) is satisfied but where all four conditions I° - IV° will be not satisfied.

Let us denote by (A), (A'); (A), (B'); (C), (C'); (D), (D') the tribes considered in the examples n. 25, 1). 2), 3), 4), respectively and provide symbols referring to them by indices α, β, γ, δ respectively.

Let E be the class of all quadruples (a, b, c, d), where $a \in A$, $b \in B$, $c \in C$. $d \in D$, and define $(a, b, c, d) \leq (a_1\ b_1\ c_1\ d_1)$

by the conjunction of the following propositions

$$a \leq_\alpha a_1 , \ b \leq_\beta b_1 , \ c \leq_\gamma c_1 , \ d \leq_\delta d_1 .$$

E will be organized into a tribe (E).

Let E' be the class of all quadruples (a', b', c', d'), where $a' \in A'$, $b' \in B'$, $c' \in C'$, $d' \in D'$ with the ordering relation

$$(a' \ b' \ c' \ d') \leq' (a'_1, \ b'_1, \ c'_1, \ d'_1)$$

defined by

$$a' \leq'_\alpha a'_1, \ \ b \leq'_\beta b'_1, \ \ c' \leq'_\gamma c'_1, \ \ d' \leq'_\delta d'_1 .$$

We obtain a tribe (E').

We see that (1) is satisfied, on account of the remark made at the beginning of this n. 25, 1); but we also see that no one of the condition I°, II°, III°, IV° is satisfied [6]).

26. - Remark that if (A), (A') are tribes where $A \subseteq A'$, and if for the somata of (A) the statements

$$a \leq b, \ \ a \leq' b$$

are equivalent, then, for the somata of (A)

$$a +' b =' c \quad \text{implies} \quad a + b = c,$$

$$a \cdot' b =' c \quad \text{implies} \quad a \cdot b = c,$$

and even

$$\sum_{n=1}^{\infty}{}' \ a_n =' b \quad \text{implies} \quad \sum_{n=1}^{\infty} a_n = b,$$

$$\sum_{\alpha \in M}{}' \ a_\alpha =' b \quad \text{implies} \quad \sum_{\alpha \in M} a_\alpha = b,$$

and similarly for infinite products.

The above remarks show that, for the notion of subtribe, we do not need to consider analogous possibilities concerning the meaningfulness of infinite sums, as we did when dealing with homomorphism.

---

6) To fit the constructions permitted in the Principia Mathem. [1], we may define a quadruple (a, b, c, d) as the ordered couple $[(a, b), (c, d)]$ of two ordered couples (a, b) and (c, d).

26.1. - It is easy to have an example where $A \subseteq A'$, but where the condition $a \leq' b$ does not imply $a \leq b$ for somata a, b of (A).

E. g. (A) is the tribe of all Lebesgue-measurable subsets of $\langle 0, 1 \rangle$ with set-inclusion as ordering relation.

(A') is the same collection as A, but with $a' \leq' b'$ defined by the condition

$$\text{meas}\,(a' \sim b') = 0.$$

We have $A \equiv A'$. We have

$$\left\langle 0, \frac{1}{2} \right\rangle +' \left(\frac{3}{4}\right) \leq' \left\langle 0, \frac{1}{2} \right\rangle,$$

but it is not true that

$$\left\langle 0, \frac{1}{2} \right\rangle + \left(\frac{3}{4}\right) \leq \left\langle 0, \frac{1}{2} \right\rangle,$$

because the inclusion

$$\left\langle 0, \frac{1}{2} \right\rangle \cup \left(\frac{3}{4}\right) \subseteq \left\langle 0, \frac{1}{2} \right\rangle$$

is not true.

An example where $A \subseteq A'$, but where $a \leq b$ does not imply $a \leq' b$, is given by the above one if we interchange A with A'.

27. - Let (A) be a tribe and J an ideal in (A). Consider (A) and $(A') \underset{df}{\equiv} (A_J)$ i. e. the equivalence tribe modulo J.

We have $A \equiv A'$, nevertheless (A) is not a finitely genuine subtribe of (A') and also (A') is not a finitely genuine subtribe of (A). Indeed the statements

$$a = b, \quad a =^J b$$

are not equivalent, unless J is the class composed of the single soma 0.

28. - A tribe (A) may be a finitely-genuine subtribe of (A') but not its denumerably-genuine subtribe.

E. g. Let (A) be the tribe of all finite set-unions

$$\bigcup_i (\alpha_i, \beta_i)$$

where $0 \leq \alpha^i \leq 1$, $0 \leq \beta^i \leq 1$, and where the ordering relation is the set-inclusion.

Let (A') be the tribe of all borelian subsets of $(0, 1)$ with set-inclusion as ordering relation.

(A) is obviously a finitely-genuine subtribe of (A'), but it is not its denumerably-genuine subtribe.

Indeed let E be a non dense perfect subset of $\langle 0, 1 \rangle$ with $0, 1 \in E$, and let (p_i, q_i), $(i = 1, 2, ...)$ be all corresponding free intervals. We have for $a_n \underset{df}{=} (p_n, q_n)$, $(n = 1, 2, ...)$

$$\sum_{n=1}^{\infty} a_n = 1,$$

but

$$\sum_{n=1}^{\infty}{}' \, a_n \neq' 1'.$$

28.1. - *A tribe (A) may be a denumerably-genuine subtribe of (A') but not its completely-genuine subtribe.*

E. g. Consider the set V of all elements each of which being an ordered couple (x, α) where $x \in (0, 1)$ and where α varies over the range of all denumerable ordinals. Let A be the smallest collection of sets of elements, such that

1. If α is a denumerable ordinal and E a borelian subset of $(0, 1)$, then the class of all elements (x, α) where $x \in E$, belongs to A;

2. If $\varphi_1, \varphi_2, \dots, \varphi_n, \dots$ is an infinite sequence of sets with $\varphi_n \in A$, then $\bigcup_{n=1}^{\infty} \varphi_n \in A$;

3. If $\varphi \in A$, then $(V \backsim \varphi) \in A$.

If we consider the set-inclusion as ordering relation on A, we obtain a denumerably additive Boolean tribe (A). The class J of all at most denumerable sets is a denumerably additive ideal in (A). Put

$$(B) \underset{df}{=} (A)/J.$$

Let (A') be the class of all subsets of V with set-inclusion as ordering relation, and put

$$(B) \underset{df}{=} (A')/J.$$

(B) is obviously a denumerably genuine subtribe of (B'). We shall prove that it is not a completely genuine subtribe.

Define, for all α, the set ψ_α as the set of all elements (x, α), where $x \in (0, 1)$, (open interval), and denote by a_α the equivalence class modulo J whose representative is ψ_α.

We have

$$\sum_{\alpha \geq 2} a_\alpha = \text{co}\ a_1 .$$

Nevertheless

$$\sum_{\alpha \geq 2}{}' a_\alpha \neq' \text{co}'\ a_1 .$$

for $\bigcup_{\alpha \geq 2} \psi_\alpha$ differs from co ψ_1 by a non denumerable set of elements.

29. - Let (A), (A') be two tribes, we say that R is a relation which *finitely embeds* (A) *into* (A') if R is an isomorphism from (A) onto a tribe (B) which is a finitely genuine subtribe of (A'). If such a relation R exists we say that (A) can be *finitely-embedded into* (A'), and that (A') is a *finite-operation extension of* (A) *through isomorphism.*

In the case where (B) is supposed to be a denumerably genuine subtribe [completely genuine subtribe] of A', we say that R *denumerably embeds* (A) *into* (A') [*completely embeds* (A) *into* (A')] *and that* (A') *is a denumerable-operation extension of* (A) [*complete extension of* (A)] *through isomorphism.*

The above considerations seem to clarify all ambiguities involved in the notions of homomorphism and embedding in Boolean tribes.

§ 5. - Measure.

30. - If we attach to every soma a of a finitely-additive Boolean tribe (A) a real number $\mu(a)$ such that

1° $a = a'$ implies $\mu(a) = \mu(a')$,

2° if $a \cdot b = 0$, then $\mu(a + b) = \mu(a) + \mu(b)$, we say that $\mu(a)$ is a *finitely-additive real measure on* (A) [7].

Usually is supposed that $\mu(a) \geq 0$. In this case we say that the measure *is non negative.*

In the sequel we shall confine ourselves to this case only.

The measure $\mu(a)$ is said to be *effective,* if $\mu(a) = 0$ implies $a = 0$.

If instead of 2°, the more general following condition 2°_d is fulfilled we say that the *measure is denumerably additive on* (A):

2°_d. if $a \in A$, a_n are all disjoint, $(n = 1, 2, \ldots)$, $\sum_{n=1}^{\infty} a_n$ is meaningful, then the series $\sum_{n=1}^{\infty} \mu(a_n)$ converges and

$$\mu\left(\sum_{n=1}^{\infty} a_n\right) = \sum_{n=1}^{\infty} \mu(a_n) \quad [8].$$

Besides the obvious trivial measure $\mu(a) = 0$ for all $a \in A$, there always exists, on a given not trivial tribe (A), a two-valued non negative finitely additive measure: Let J be a maximal ideal in (A), i. e. such that $J \neq A$, and if J'

---

7) The measure theory was founded by G. Jordan, H. Lebesgue and E. Borel for point sets in the euclidean space, and carefully studied by C. Carathéodory [21]. The notion of set function was introduced by H. Lebesgue [15], p. 380, and studied by Ch. de la Vallée Poussin [16] and M. Fréchet [17] who has considered set functions even for abstract sets. Measure on abstract Boolean tribes were introduced by C. Carathéodory [3] and by O. Nikodým [14]. For generalisation of measure to lattices see G. Birkhoff [4], p. 74 ff. Set functions are also treated in the known book by H. Hahn [18], [19] and constitute a basic notion in the Calculus of probability [20]. For more details see G. Birkhoff's book [4]. Basic theorems for abstract set-functions one can find by S. Saks [22]. See also the book by P. Halmos [24]. Recent deep studies are by e. g. Maharam, Pauc, Krickeberg, Hewitt, Yoshida and others.

8) We do not consider the generalisation of the notion of a non-negative measure, where the « value » ∞ is also admitted as a possible measure of a soma. Nevertheless to be more clear we shall use sometimes the term « *finite measure* ».

is an ideal in (A) with $J \subset J'$. then $J' \equiv A$. If we put for all $x \in J$

$$\mu(x) \underset{df}{=} 0$$

and for all $x \bar{\in} J$

$$\mu(x) \underset{df}{=} 1,$$

we get a finitely additive measure, which is however not effective unless (A) is composed of the only two somata 0 and 1. [25].

Since finitely additive measured tribes are nowadays important for the functional calculus e. g. [39], we allow ourselves to give some examples showing various possibilities which may occur in measures.

31. - There exists a denumerably additive Boolean tribe having no atoms, and which does not admit any finitely additive, finite, effective non negative measure. The following example was communicated to the author by J. Dixmier in a letter dated Sept. 6.1950 [9]).

Let V be a collection of some elements with cardinal $> \aleph_0$. Let (A) be the tribe of all subsets of V with set-inclusion as ordering relation. (A) is a completely additive tribe. The class J of all at most denumerable subsets of V is a denumerably additive ideal in (A). Put $(H) \underset{df}{=} (A)/J$. This is a denumerably additive tribe, having no atoms. We shall show that there exists no finitely additive, non negative, finite and effective measure. Supposed, by impossible, $\mu(\xi)$ be so. There exists $\xi \in H$ with $\mu(\xi) > 0$ for the cardinal of V is $> \aleph_0$. Take such a soma ξ. Since $\mu(\xi) > 0$, any representative p of ξ is a non denumerable set of elements of V. Let p be such a representative and denote by $\aleph$ its cardinal. We have $\aleph > \aleph_0$. Since $\aleph \cdot \aleph = \aleph$, there exists a class $\boldsymbol{M}$, of power $\aleph$, of sets N_α, where $N_\alpha \subseteq p$, and such that all N_α are mutually disjoint and where

$$\bigcup_{N_\alpha \in \boldsymbol{M}} N_\alpha = p.$$

---

[9]) An *atom of a tribe* (A) is such its soma a that $b \leq a$ implies that either $b=0$ or $b=a$. See [23], [9].

The set N_α is the representative of a soma s_α of (H). If $\alpha \neq \beta$, then $N_\alpha \cap N_\beta = \emptyset$ and hence $s_\alpha \cdot s_\beta = 0$. Since μ is an effective measure and each $s_\alpha \neq 0$, it follows that $\mu(s_\alpha) > 0$.

There exists $\rho > 0$ such that $\mu(s_\alpha) \geq \rho$ for a non denumerable number of indices α. Hence for any natural P there exist distinct $\alpha_1, \ldots, \alpha_P$ such that $\mu(s_{\alpha_1}) \geq \rho, \ldots, \mu(s_{\alpha_P}) \geq \rho$. Since $s_{\alpha_1} + s_{\alpha_2} + \ldots + s_{\alpha_P} \leq p$, it follows that $P \cdot \rho \leq \mu(p)$ for $P = 1, 2, \ldots$ which is a contradiction. The theorem is proved.

31.1. - By applying Mac Neille's embedding theorem ([2] p. 466) we get an example of *a completely additive tribe without atoms and not possessing any finite, non negative, finitely additive, effective measure.*

32. - There exists a denumerably additive tribe, without atoms, and which does not possess any finite, non negative, denumerably additive, effective measure.

The following example is known (See [4] p. 186). Let (A) be the tribe of all borelian subsets of $Q \underset{df}{=} (0, 1)$ with set -inclusion as ordering relation, and let J be its ideal composed of all borelian sets of the 1st category. Put $(B) \underset{df}{=} (A)/J$. The tribe (A) is denumerably additive, J is a denumerably additive ideal, and (B) is also a denumerably additive tribe, possessing no atoms.

There exists no finite, non negative, denumerably additive measure on (B).

The following proof has been kindly communicated to the author by S. Kakutani.

Suppose μ is such a measure. We may suppose $\mu(1) = 1$ Let $p_1, p_2, \ldots, p_n, \ldots$ be an everywhere dense set of points in Q. Denote by a_{ik} the set of all points x of Q whose distance from p_i is $\geq \frac{1}{k}$ $(k = 1, 2, \ldots)$. We have

$$a_{i1} \subseteq a_{i2} \subseteq \ldots \qquad (i = 1, 2, \ldots).$$

There sets are representatives of $A_{ik} \underset{df}{=} [a_{ik}]$ where $A_{ik} \in B$.

We have

$$A_{i1} \leq A_{i2} \leq \dots .$$

Let

$$\varepsilon = \varepsilon_1 + \varepsilon_2 + \dots \quad . \quad \varepsilon_n > 0, \ \varepsilon < \frac{1}{2}.$$

We have

$$\bigcup_{n=1}^{\infty} a_{in} = Q \backsim (p_i), \text{ (subtraction of sets)},$$

hence

$$\sum_{n=1}^{\infty} A_{in} = 1 \quad . \quad \prod_{n=1}^{\infty} \text{co } A_{in} = 0.$$

co $A_{i1} \geq$ co $A_{i2} \geq \dots$ on (B).

It follows that there exists ν_i such that

$$\mu(\text{co } A_{i,\nu_i}) \leq \varepsilon_i .$$

Consider the somata A_{1,ν_1}, $A_{2,\nu_2}, \dots$ and the sets

(1) $$\dots \text{co } a_{1,\nu_1}, \ \text{co } a_{2,\nu_2}, \dots ,$$

where the set-complementary is taken with respect to Q, (1) are open intervals containing the points $p_1, p_2, \dots$ respectively. Hence $\bigcup_{i=1}^{\infty} \text{co } a_{i,\nu_i}$ is an open set which is everywhere dense in Q, and therefore co $\bigcup_{i=1}^{\infty} \text{co } a_{i,\nu_i}$ is nowhere dense and borelian, and then it belongs to J.

It follows

$$\text{co} \sum_{i=1}^{\infty} \text{co } A_{i,\nu_i} = 0,$$

hence

$$\prod_{i=1}^{\infty} A_{i,\nu_i} = 0, \quad \text{and then} \quad \mu\left(\prod_{i=1}^{\infty} A_{i,\nu_i}\right) = 0, \dots (2)$$

hence

$$\mu\left(\text{co} \prod_{i=1}^{\infty} A_{i,\nu_i}\right) = 1 \quad \text{i. e.} \quad \mu\left(\sum_{i=1}^{\infty} \text{co } A_{i,\nu_i}\right) = 1.$$

But

$$\mu\left(\sum_{i=1}^{\infty} \text{co } A_{i,\nu_i}\right) \leq \sum_{i=1}^{\infty} \mu' \text{co } A_{i,\nu_i}) \leq \sum_{i=1}^{\infty} \varepsilon_i = \varepsilon < \frac{1}{2}$$

and consequently $1 < \frac{1}{2}$ which is impossible.

32.1. - Now we shall define, on the above tribe (B), a finitely additive effective measure.

If $x \in (B)$, $x \neq 0$, then there exists an open interval (α, β) with $0 < \alpha < \beta < 1$ such that the equivalence class

$$[(\alpha, \beta)] \leq x.$$

We know that for every $x \neq 0$, $x \neq 1$ there exists a prime ideal J_x in (B) such that $x \bar{\in} J_x$, [9].

Take all intervals in $(0, 1)$ with rational extremities:

$$(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots . (0 < \alpha_n < \beta_n < 1).$$

Put

$$y_n \underset{df}{=} (\alpha_n, \beta_n),$$

and choose the corresponding prime ideals J_n, which do not contain y_n respectively.

Let $\mu_n(y)$ be the measure on (B) with values 0, 1 only, defined by

$$\mu_n(x) = 0 \quad \text{whenever} \quad y \in J_n,$$

$$\mu_n(y) = 1 \quad \text{whenever} \quad y \bar{\in} J_n. \qquad \text{(see [25])}.$$

Put

$$\mu(x) \underset{df}{=} \sum_{n=1}^{\infty} \frac{1}{2^n} \mu_n(x).$$

The measure $\mu(x)$ is additive and effective.

Thus we see that

There exists a denumerably additive Boolean tribe without atoms, possessing a finitely additive, effective, finite and non-negative measure, but which does not possess any denumerably additive, effective, finite, non negative measure.

33. - A known simple example of ***a completely additive tribe, without atoms, possessing an effective, denumerably additive, finite, non negative measure*** is the following one:

E. g. Let (A) be the tribe of all Lebesgue-measurable subsets of $\langle 0, 1\rangle$ with the ordering relation $a \leq b$ defined by

$$\text{meas}\,(a \backsim b) = 0.$$

It is isomorphic with the tribe of all Lebesgue-measurable subsets of $\langle 0, 1\rangle$ modulo the ideal composed of sets of measure 0. The Lebesguean measure is effective. The tribe has no atoms.

34. - A known example of ***a finitely additive tribe, without atoms, possessing an effective finitely additive measure*** is given by the tribe of all finite unions

$$\bigcup_i \langle \alpha_i, \beta_i\rangle$$

where $0 \leq \alpha_i \leq 1$, $0 \leq \beta_i \leq 1$, with set-inclusion as ordering relation, and the Lebesguean measure.

The measure is ***not denumerably additive,*** as may be easily seen by considering the free intervals of a non dense perfect set E with meas $E > 0$. The tribe ***is not*** denumerably additive, and has no atoms.

35. - It seems interesting to have the following example of ***a finitely additive tribe (but not denumerably additive) without atoms, and possessing a denumerably additive, effective, finite, non negative measure.***

E. g. Let A be the collection of all subsets of $\langle 0, 1\rangle$ having the form

$$\varphi = \bigcup_i \langle \alpha_i, \beta_i\rangle \quad , \quad 0 \leq \alpha_i \leq 1 \quad , \quad 0 \leq \beta_i \leq 1,$$

where the union is finite.

Let B be the collection of all sets

$$b = \varphi \dotplus E, \quad (\dotplus \text{ algebraic addition of sets})$$

where E is a borelian subset of $\langle 0, 1\rangle$ of the 1^o category.

If we define on B the ordering relation as the set-inclusion, w get a finitely additive tribe (B).

The class J of all borelian subsets of $(0, 1)$ which are of measure zero and, at the same time, of the 1° cathegory is a denumerably additive ideal in (B).

Define $(C) \underset{df}{=} (B)/J$. The tribe (C) is finitely additive but not denumerably additive.

Indeed if we put $c_n = \left(\frac{1}{2n}, \frac{1}{2n-1}\right\rangle$, $n = 1, 2, \ldots$, and consider the corresponding equivalence classes C_n, we see that $\sum_{n=1}^{\infty} C_n$ is not meaningful

Notice that if

$$a_1 = \varphi_1 \dot{+} E_1, \qquad a_2 = \varphi_2 \dot{+} E_2$$

are representatives of the same equivalence class, then $\varphi_1 \equiv \varphi_2$ i. e. all representatives of the same equivalence class have the same « A — part ».

Now, suppose that $P_1, P_2, \ldots, P_n, \ldots \in (C)$, with $P_i \cdot P_j = 0$ for $i \neq j$, and that

$$(1) \qquad 1 = \sum_{n=1}^{\infty} P_n. \quad \text{Let } p_1, p_2, \ldots, p_n, \ldots$$

be their representatives respectively, and $\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots$ the A-parts of them. we have $\varphi_i \cap \varphi_j \equiv \emptyset$, for $i \neq j$.

The set $b \underset{df}{=} (0, 1) \backsim \bigcup_{n=1}^{\infty} \varphi_n$ cannot contain any segment, because this would contradict (1). Hence b is a non dense set. Its measure cannot be positive, because in this case (1) would be false too.

Now define for all $P \in (C)$:

$\mu(P) \underset{df}{=} \operatorname{meas} \varphi$, where φ is the A-part of P.

It follows that

$$\mu(1) = \sum_{n=1}^{\infty} \mu(P_n).$$

From this it is not difficult to deduce (see [14]) that, if $Q_1 . Q_2, \ldots$ are mutually disjoint somata of (C) with the meaningful sum, then

$$\mu\left(\sum_{n=1}^{\infty} Q_n\right) = \sum_{n=1}^{\infty} \mu(Q_n).$$

The theorem is established.

36. - F. Wecken has proved [13] that if a denumerably additive tribe admits a denumerably additive, effective, non negative and finite measure, it must be completely additive, so *there is no tribe which has a denumerably additive, effective etc. measure, and which would be denumerably additive without being completely additive.*

REFERENCES

[1] ALFRED NORTH WHITEHEAD and BERTRAND RUSSEL: *Principia Mathematica*. Vol. I. Cambridge (Engl.). University Press, 1910.

[2] H. M. MAC NEILLE: *Partially ordered sets*. Trans. Amer. Math. Soc., 42 (1937), p. 416-460.

[3] C. CARATHÉODORY: *Entwurf für eine Algebraisierung des Integralbegriffs*. Münch. Sitzber. (1938), p. 27-68.

[4] GARRET BIRKHOFF: *Lattice theory, revised edition*. (1948). New York City. (American Mathemat. Soc. Colloquium Publications). Vol. 25, XIII + 283 pp.

[5] M. H. STONE: *Linear transformations in Hilbert space and their applications to analysis*. New York (1932), (Amer. Math. Soc. Colloquium Publications, Vol. 15).

[6] O. NIKODÝM: *On Boolean fields of subspaces in an arbitrary Hilbert space*. Ann. Soc. Polon. Math., 17 (1938), pp. 138-165.

[7] O. NIKODÝM: C. R. Paris. 224 (1947, pp. 522-524, pp. 628-630, pp. 778-780.

[8] O. NIKODÝM: *Un nouvel appareil mathématique pour la théorie des quanta*. Annales le l'Institut Henri Poincaré. Tome II, pp. 49-112 (lectures 1947).

[9] M. H. STONE: *The theory of representation for Boolean algebras*. Trans. Amer. Math. Soc., 40 (1936), pp. 37-111.

[10] H. L. VAN der WAERDEN: *Moderne Algebra*. 2-d edition. F. Ungar Publish. Co. (New York). (1937, Leipzig edit.).

[11] RENÉ DE POSSEL: *Sur la dérivation abstraite des fonctions d'ensemble*. Journ. des Math., 101 (1936), pp. 391-409.

[12] N. BOURBAKI : *Topologie générale. Chap. I. Structure topologique.* Actuel. Sci. et Ind., n. 858. Hermann, Paris (1940).

[13] F. WECKEN : *Abstrakte Integrale und fastperiodische Funktionen.* Math. Zeitschr., 45 (1939), pp. 377-404.

[14] O. NIKODÝM : *Sur la mesure vectorielle parfaitement additive dans un corps obstrait de Boole.* Mém. de l'Acad. Roy. de Belgique (Classe des Sc.), t. 17 (1938).

[15] H. LEBESGUE : Ann. Ec. Norm. Sup. (3), 27 (1910), p. 380.

[16] CH. DE LA VALLÉE POUSSIN : *Intégrales de Lebesgue. Fonctions d'ensemble. Classes de Baire.* II éd. Paris, Gauthier-Villars Coll. Borel).

[17] M. FRÉCHET : *Des familles et fonctions additives d'ensemble abstraits.* I. Fund. Math., 4 (1923), pp. 329-365; II. Fund. Math., 5 (1924), pp. 207-251.

[18] H. HAHN : *Theorie der reellen Funktionen.* Leipzig & Berlin (1918).

[19] H. HAHN & A. ROSENTHAL. *Set functions.* Albuquerque (New Mexico), 1948.

[20] A. KOMOLGOROFF. *Grundbegriffe der Wahrscheinlichkeitsrechnung.* Ergebnisse d. Mathem., Vol. II (1933).

[21] C. CARATHÉODORY : *Vorlesungen uber reelle Funktionen.* Leipzig & Berlin (1918).

[22] S. SAKS : *Theory of the integral.* Warszawa (1937).

[23] A. TARSKI : *Zur Grundlegung der Boole'schen Algebra.* I. Fund. Math., 24 (1935), pp. 177-198.

[24] P. HALMOS : *Measure Theory.* D. Van Nostrand Comp. Inc. Toronto- New York-London, 1950, XI + 304 pp.

[25] A. TARSKI : *Une contribution à la théorie de la mesure.* Fund. Math., 15 (1930), pp. 42-56.

[26] LEOPOLDO NACHBIN : *Une propriété caractéristique des algèbres Boole'iennes.* Portugaliae Mathematica. Vol. 6 (1947), pp. 115-118.

[27] SHIZUO KAKUTANI : *Concrete reprentation of abstract L-spaces and the mean ergodic theorem.* Ann. of Mathem. 42 (1941), pp. 523-537.

[28] S. BOCHNER & R. S. PHILLIPS : *Additive set functions and vector lattices.* Ann. of. Math. 42 (1941), pp. 316-324.

[29] O. NIKODÝM : *Sur les fonctionnelles linéaires.* C. R. Paris, 229 (1949), pp. 16-18, pp. 169-171, pp. 288-289, pp. 863-865.

[30] O. NIKODÝM : *Sur les êtres fonctionoides.* C. R. Paris, 226 (1948), p. 375, pp. 458-460, pp. 541-543.

[31] O. NIKODÝM : *Tribus de Boole et fonctions mesurables. Tribu spectrale d'une fonction.* C. R. Paris, 228 (1949), pp. 37-38.

[32] O. NIKODÝM : *Tribus de Boole et functions mesurables. Transformations equimesurables.* C. R. Paris, 228 (1949), pp. 150-151.