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On higher differences. Nota II

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ON HIGHER DIFFERENCES

Nota II (*)

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II. - The Relations between the Differential Coefficients and the Higher Differences of a Function.

1. Introduction. This paper is in continuation of Note I on the subject of Higher Differences. We make use of the same notations as employed in Note I. The object of this paper is to study the relations between the operators F and the operators in Differential Calculus.

2. Lemma (i).

If (1)

$$Z_{3r} = \begin{vmatrix} {}^{r}O^{-}_{r-1} & 1 & \\ {}^{r+1}O^{-}_{r-1} & {}^{r+1}O^{-}_{r} & 1 \\ {}^{r+2}O^{-}_{r-1} & {}^{r+2}O^{-}_{r} & {}^{r+2}O^{-}_{r+1} \end{vmatrix}_{3}$$

then similarly formed

$$Z_{nr} = {}^{n+r-1}S_{n} \,.$$

$$[z_{0r} = 1]$$

[Z_{3r} evidently develops into

$$\sum_{p=0}^{2} (-)^{p} \; {}^{r+2}O^{-}_{r+1-p} Z_{2-p,\, r}$$

Similarly develop Z_{nr} and then prove the theorem by induction].

(*) Pervenuta in Redazione il 26 novembre 1953.

LEMMA (ii). If (2)

$$P_{4c} = \begin{vmatrix} C & 1 & & \\ C^2 & {}^2O_1^- & 1 & \\ C^3 & {}^3O_1^- & {}^3O_2^- & 1 \\ C^4 + (-)^4\, {}^4S_4 & {}^4O_1^- & {}^4O_2^- & {}^4O_3^- \end{vmatrix}_4$$

then

$$P_{nc} = (-)^{n-1} (C)_n {}^nS_n$$

[Here develop P_{nc} in terms of C, C^2 etc. and then apply (1) above and Th. (5), Note I].

LEMMA (iii). — If (3)

$$Q_{4r} = \begin{vmatrix} 1 & 1 & & \\ a^r & {}^2O_1^- & 1 & \\ a^{2r} & {}^3O_1^- & {}^3O_2^- & 1 \\ a^{3r} & {}^4O_1^- & {}^4O_2^- & {}^4O_3^- \end{vmatrix}_4$$

then

$$Q_{nr} = (-)^{n-1} (a^r)_n a^{-r}\, {}^nS_n + a^{-r}\, {}^nS_n\,,\ Q_{n0} = {}^nS_n$$

[Put $C = a^r$ in Lemma (ii)].

3. By Differential Calculus and by § II, Note I,

$$\left.\begin{aligned} F^n u_x &= u_{x+n} = e^{n\frac{d}{dx}} u_x \\ \text{and}\qquad F_n u_x &= u_{a^n x} = e^{a^n x \frac{d}{dx}} u_0 \end{aligned}\right\} \quad (4)$$

where $\frac{d^m}{dx^m} u_0$ stands for the value of $\frac{d^m}{dx^m} u_x$ when x is replaced by 0.

Thus the operators F are related to the operators in Differential Calculus.

4. $A^n u_x$ may be expressed in terms of the differential coefficients of u_x.

THEOREM. If u_x be a rational and integral function of x of degree l in x, then

$$A^n u_x = \sum_{p=0}^{l-1} \frac{1}{(1+p)!} A^n 0^{1+p} \frac{d^{1+p} u_x}{dx^{1+p}} \tag{5}$$

where $A^n O^m$ stands for the value of $A^n x^m$ when $x = 0$.

By Th. (26), Note I, we have

$$(\sum_{p=0}^{n} {}^n O_p^- A^p) u_x = e^{n\frac{d}{dx}} u_x$$

In this equation, putting $n = 1$, 2, 3 and 4, we have four equations from which eliminating Au_x, $A^2 u_x$ and $A^3 u_x$, we have

$$(-)^4 A^4 u_x = \begin{vmatrix} 1-e^{\frac{d}{dx}} & 1 & & \\ 1-e^{\frac{2d}{dx}} & {}^2O_1^- & 1 & \\ 1-e^{\frac{3d}{dx}} & {}^3O_1^- & {}^3O_2^- & 1 \\ 1-e^{\frac{4d}{dx}} & {}^4O_1^- & {}^4O_2^- & {}^4O_3^- \end{vmatrix}_4 u_x$$

$\therefore$

$$(-)^3 A^4 u_x = \sum_{p=0}^{l-1} \frac{1}{(1+p)!} \begin{vmatrix} 1^{1+p} & 1 & & \\ 2^{1+p} & {}^2O_1^- & 1 & \\ 3^{1+p} & {}^3O_1^- & {}^3O_2^- & 1 \\ 4^{1+p} & {}^4O_1^- & {}^4O_2^- & {}^4O_3^- \end{vmatrix}_4 \frac{d^{1+p} u_x}{dx^{1+p}}$$

$$= \sum_{p=0}^{l-1} \frac{1}{(1+p)!} \left\{ \sum_{m=0}^{3} (-)^m (1+m)^{p+1} Z_{3-m,\, 2+m} \right\} \frac{d^{1+p} u_x}{dx^{1+p}},$$

[Lemma (i)]

$$= \sum_{p=0}^{l-1} \frac{1}{(1+p)!} \left\{ \sum_{m=}^{3} (-)^m (1+m)^{p+1}\, {}^4S_{3-m} \right\} \frac{d^{1+p}u_x}{dx^{1+p}}$$

$$= \sum_{p=0}^{l-1} \frac{1}{(1+p)!} \left\{ (-)^3 \sum_{m=0}^{4} (-)^m (4-m)^{1+p}\, {}^4S_m \right\} \frac{d^{1+p}u_x}{dx^{1+p}}$$

$$= (-)^3 \sum_{p=0}^{l-1} \frac{1}{(1+p)!} A^4 O^{1+p} \frac{d^{1+p}u_x}{dx^{1+p}}$$

$$\therefore\ A^4 u_x = \sum_{p=0}^{l-1} \frac{1}{(1+p)!} A^4 O^{1+p} \frac{d^{1+p}u_x}{dx^{1+p}}.$$

The general case may be similarly treated.

COR.

$$\Delta^n u_x = \frac{d^n u_x}{dx^n} + \frac{1}{(n+1)!} \Delta^n O^{n+1} \frac{d^{n+1}u_x}{dx^{n+1}}$$

$$+ \frac{1}{(n+2)!} \Delta^n O^{n+2} \frac{d^{n+2}u_x}{dx^{n+2}} + \dots$$

[This result of Finite Differences, follows from (5) when $a \to 1$, for $\Delta^n O^1 = \Delta^n O^2 = \Delta^n O^3 = \dots = \Delta^n O^{n-1} = 0$ and $\Delta^n O^n = n!$].

5. $A_n u_x$ may also be expressed in terms of the differential coefficients of u_x.

THEOREM. If u_x be a rational and integral function of x of degree l in x, then

$$A_n u_x = {}^nS_n \left\{ \sum_{p=0}^{l-n} \frac{x^{n+p}}{(n+p)!} (a^{n+p})_n \frac{d^{n+p}}{dx^{n+p}} \right\} u_0, \qquad (6)$$

where $\frac{d^m u_0}{dx^m}$ denotes what $\frac{d^m u_x}{dx^m}$ becomes when $x = 0$.

By Th. (27), Note I, we have

$$\left(\sum_{p=0}^{n} {}^nO_p^{-} A_p \right) u_x = e^{a^n x \frac{d}{dx}} u_0$$

If in this equation we put $n=1, 2, ..., n$, we have n equations from which eliminating

$$A_1u_x,\ A_2u_x, \dots A_{n-1}u_x, \quad \text{we have}$$

$$(-)^n A_n u_x = \begin{vmatrix} u_x - e^{ax\frac{d}{dx}}u_0 & 1 & & \\ u_x - e^{a^2x\frac{d}{dx}}u_0 & {}^2O_1^- & 1 & \\ \cdots & \cdots & \cdots & \cdots \\ u_x - e^{a^nx\frac{d}{dx}}u_0 & {}^nO_1^- & {}^nO_2^- \dots {}^nO_{n-1}^- \end{vmatrix}_n$$

$$= Q_{n0}u_x - \sum_{p=0}^{l} \frac{(ax)^p}{p!} Q_{np} \frac{d^pu_0}{dx^p} \qquad \text{[Lemma (iii)]}$$

$$= {}^nS_n u_x - \sum_{p=0}^{l} \frac{(ax)^p}{p!} {}^nS_n \{ (-)^{n-1}(a^p)_n a^{-p} + a^{-p} \} \frac{d^pu_0}{dx^p}$$

$$= {}^nS_n u_x - \sum_{p=0}^{l} \frac{x^p}{p!} {}^nS_n \frac{d^pu_0}{dx^p} + (-)^n\, {}^nS_n \sum_{p=0}^{l} \frac{x^p}{p!} (a^p)_n \frac{d^pu_0}{dx^p}$$

$$= (-)^n\, {}^nS_n \sum_{p=0}^{l} \frac{x^p}{p!} (a^p)_n \frac{d^pu_0}{dx^p}$$

$$\therefore\ A_n u_x = {}^nS_n \sum_{p=0}^{l-n} \frac{x^{n+p}}{(n+p)!} (a^{n+p})_n \frac{d^{n+p}u_0}{dx^{n+p}}$$

for $(a^p)_n = 0$ when $p < n$.

COR. (7)

$$\underset{a\to 1}{\text{Lt}} \frac{A_n u_x}{(a-1)^n} = x^n \sum_{p=0}^{l-n} \frac{x^p}{p!} \frac{d^{n+p}u_0}{dx^{n+p}}$$

This follows from (6) when $a \to 1$.

6. $\dfrac{du_x}{dx}$ may be expressed in terms of A^1, A^2, A^3, etc.

THEOREM. If u_x be a rational and integral function of x of degree n in x, then

$$\frac{du_x}{dx}=\sum_{r=1}^{n}\left\{\sum_{p=r}^{n}(-)^{p-1}\,{}^nC_p\frac{{}^pO_r^-}{p}A^r u_x\right\}. \tag{8}$$

By Th. (26), Note I and by (4), § 3, we have

$$\left(\sum_{p=0}^{n}{}^nO_p^-A^p\right)u_x=e^{n\frac{d}{dx}}u_x \tag{9}$$

Let us consider the particular case when u_x is a rational and integral function of x of the third degree. If we put $n=1$ in (9), we have

$$Au_x=\left(\frac{d}{dx}+\frac{1}{2\,!}\frac{d^2}{dx^2}+\frac{1}{3\,!}\frac{d^3}{dx^3}\right)u_x$$

Putting $n=2$ and 3, two similar equations may be obtained. From these three equations, eliminating $\dfrac{d^2u_x}{dx^2}$ and $\dfrac{d^3u_x}{dx^3}$, we have

$$\frac{du_x}{dx}=\left\{\sum_{p=1}^{3}(-)^{p-1}\,{}^3C_p\frac{{}^pO_1^-}{p}\right\}A^1u_x-\left\{\sum_{p=2}^{3}(-)^{p-2}\,{}^3C_p\frac{{}^pO_2^-}{p}\right\}A^2u_x$$
$$+\left\{\sum_{p=3}^{3}(-)^{p-3}\,{}^3C_p\frac{{}^pO_3^-}{p}\right\}A^3u_x$$

ie

$$\frac{du_x}{dx}=\sum_{r=1}^{3}\left\{\sum_{p=r}^{3}(-)^{p-1}\,{}^3C_p\frac{{}^pO_r^-}{p}A^ru_x\right\}$$

The general case may be similarly treated

COR.

$$\frac{du_x}{dx}=\Delta u_x-\frac{\Delta^2u_x}{2}+\frac{\Delta^3u_x}{3}-\dots \tag{10}$$

which is a well-known theorem in F. D.

[If $a \to 1$, the coefficient of $(-)^{r-1}A^r u_x$ in (8) becomes

$$\sum_{p=r}^{n} (-)^{p-r}\, {}^nC_p\, {}^pC_r/p$$

which may be written

$$\sum_{p=0}^{n-r} (-)^p \frac{1}{r+p}\, {}^nC_{r+p}\, {}^{r+p}C_r$$

$$= {}^nC_r \sum_{p=0}^{n-r} (-)^p \frac{1}{r+p}\, {}^{n-r}C_p$$

$$= {}^nC_r \frac{(n-r)!\,(r-1)!}{n!} \quad \text{by Finite Differences}$$

$$= \frac{1}{r}$$

Hence (10) follows from (8)].