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HARI DAS BAGCHI
BISWARUP MUKHERJI

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NOTE ON CERTAIN REMARKABLE TYPES OF CURVES, SURFACES AND HYPER-SURFACES

Memoria (*) *di* HARI DAS BAGCHI *e di* BISWARUP MUKHERJI
(*a Calcutta*).

ABSTRACT

The present paper is divided into three sections, dealing respectively with plane curves, surfaces and hyper-surfaces of *specialised* categories. *Firstly*, Sec. I begins with an extension of R. A. ROBERT'S [1]) famous theorem on the intersections of an *arbitrary* ellipse with a plane curve (of even degree $2m$), having each of the two *circular points* at infinity (I, J) for a multiple point of order m. Indeed a much wider class of plane curves, for which the above theorem holds, has been discussed and characterised geometrically with reference to the points I, J; particular varieties of such curves have also been noted in this connection. *Secondly*, Sec. II concerns itself with a generalisation of Robert's theorem to an interesting type of surfaces (of *even* degree), specially related to the (imaginary) *circle at infinity*; incidentally a remarkable class of surfaces, — including *anallagmatic* surfaces as a « sub-class » — has also been taken into consideration. *Finally*, Sec. III reckons with a further generalisation of Robert's theorem to a comprehensive class of n-surfaces, lying in a Euclidean space (R_{n+1}) of $(n+1)$ dimensions and

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(*) Pervenuta in Redazione il 22 dicembre 1951.

[1]) R. A. ROBERT, *Examples and Problems on Conics and Cubics*, (1882), [Ex. 122, P. 62 and Ex. 410, P. 186].

bearing special geometrical relations with the *fixed* (imaginary) $(n-1)$-sphere (Ω), along which an *arbitrary* n-sphere (in R_{n+1}) intersects the n-flat at infinity. Special varieties of n-surfaces of this description, which contain within their fold the species of « anallagmatic » n-surfaces, have also been touched upon in the sequel.

INTRODUCTION

Among the special notations and conventions used in this paper, the following are note-worthy:

(i) that an arbitrary point in R_{n+1} is defined by means of $(n+1)$ Cartesian coordinates $(x_0, x_1, x_2, \ldots, x_n)$, referred to a set of $(n+1)$ rectangular axes $(OX_0, OX_1, OX_2, \ldots, OX_n)$;

(ii) that in the Cartesian equation of an n-surface in R_{n+1}, the symbol u_p or v_p stands for a homogeneous quantic of degree p in $(x_0, x_1, \ldots, x_n)$;

and (iii) that particular forms of the conventions of (i) and (ii), answering to the special cases ($n=1$ and $n=2$), are adopted in Sections I and II.

SECTION I.

SPECIAL TYPES OF CURVES

1. - If Γ denotes a plane algebraic curve of *even* degree $2m$, its Cartesian equation, referred to an arbitrary origin O, can evidently be put in the form:

$$(1) \qquad \sum_{p=0}^{p=2m} u_p = 0 \,,$$

where u_p stands for a homogeneous function of degree p in x, y. For an *arbitrary* right line L, drawn through O to cut Γ at the points $(P_1, P_2, \ldots, P_{2m})$, the product

$$OP_1 \,.\, OP_2 \,. \ldots .\, OP_{2m}$$

will be *constant* (*i.e.*, *independent* of the direction of L), if and only if

$$(2) \qquad u_{2m} \equiv (\text{const.}) \times r^{m}, \qquad r^2 \equiv x^2 + y^2).$$

This being the case, the constant can be made equal to unity (without loss of generality), so that, subject to (2), (1) becomes:

$$(3) \qquad r^{2m} + \sum_{p=0}^{p=2m-1} u_p = 0 .$$

Obviously (3) retains the *same* characteristic form, *viz.*

$$r'^{2m} + \sum_{p=0}^{p=2m-1} u'_p = 0 ,$$

when any other point O' is chosen as the origin. Consequently the contingency (2) must connote some inherent geometrical property of the curve Γ.

To go into the question more fully, we observe that, if I and J denote the two *circular points at infinity,* the equation of the two *isotropic* lines through O (*viz.*, OI, OJ) is

$$r^2 \equiv x^2 + y^2 = 0 .$$

As a result the geometrical interpretation of (2) is that the *$2m$ points at infinity* on Γ consist simply of the points I, J, *each counted m times.*

The converse of this property can be readily substantiated.

Now let us consider the intersections of a curve Γ of the type (3) with an arbitrary ellipse, *viz.*,

$$(4) \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 .$$

If Φ denotes the eccentric angle of any of the $4m$ points of intersection, say (x, y), we may write:

$$(5) \qquad x = \frac{a}{2}\left(t + \frac{1}{t}\right), \qquad y = \frac{b}{2i}\left(t - \frac{1}{t}\right), \qquad (i \equiv \sqrt{-1}),$$

where $t = e^{i\Phi}$. So

$$(6) \quad r^2 = \lambda\left(t^2 + \frac{1}{t^2}\right) + \mu, \text{ where } \lambda \equiv \frac{a^2 - b^2}{4} \text{ and } \mu = \frac{a^2 + b^2}{2} .$$

Substitution of (5) in (4) and subsequent simplification ultimately lead to an algebraic equation of degree $4m$ in t, which can be put in the symbolic form:

$$(7) \qquad \lambda^m \cdot t^{4m} + c_1 t^{4m-1} + c_2 t^{4m-2} + \dots + c_{4m-1} t + \lambda^m = 0 ,$$

where $c_1, c_2, \dots, c_{4m-1}$ are certain constants and λ is the constant, already defined by (6).

If the $4m$ roots of (7) be $(t_1, t_2, \dots, t_{4m})$, we have:

$$t_1 t_2 t_3 \dots t_{4m} = \lambda^m \div \lambda^m = 1 ,$$

so that

$$(8) \qquad e^{i(\Phi_1 + \Phi_2 + , \dots , + \Phi_{4m})} = 1 ,$$

where $\varphi_1, \varphi_2, \dots, \varphi_{4m}$ are the eccentric angles of the $4m$ points of intersection of (3) with (4).

Inasmuch as (8) is equivalent to

$$\sum_{p=1}^{4m} \Phi_p = 2k\pi ,$$

(where k is zero or any integer), we may sum up our results in the form of a theorem:

THEOREM **A** — *If a plane curve Γ of even degree $2m$ partakes of any one of the following attributes, viz.:*

(i) *that the $2m$ points at infinity shall consist of the two circular points at infinity, each counted m times,*

(ii) *that the product of the distances from any fixed point P of the $2m$ points, where an arbitrary line drawn through P cuts Γ, is constant,*

and (iii) *that the Cartesian equation, referred to an arbitrary origin, shall have its terms of the highest order $2m$ in the form (const.) r^{2m},*

then it (Γ) *must partake of the other two properties and at the same time it must enjoy the two additional properties, viz.*

(IV) *that the algebraic sum of the eccentric angles of the $4m$ points of intersection of Γ with an arbitrary ellipse must be either zero or an even multiple of π,* *and* (v) ***that the algebraic sum of the cotangents of the angles of intersection of Γ with an arbitrary transversal must be nil*** [2]).

In the next article we shall take into consideration certain particular phases of Theorem A.

2. - For obvious reasons the *essential* geometrical condition to be satisfied by the type of curve Γ of even degree $2m$, contemplated in Theorem A, *viz.,* that its $2m$ points of intersection with the line at infinity shall consist of the two circular points I, J (*each counted m times*), can be fulfilled in various ways. Thus if Γ has the line at infinity for a *multiple tangent*, the two (distinct) points of m-pointic contact being I, J, the afore-mentioned condition is automatically fulfilled.

Another remarkable case [3]) arises, when Γ has each of the points I, J for a multiple point of order m. If this curve Γ (of degree $2m$) be further restricted to have a third m-tuple point, then, referred to this point as origin (O), the Cartesian equation takes the symbolic form:

$$r^{2m} + r^{2m-2}v_1 + r^{2m-4}v_2 + , \dots , + r^2 v_{m-1} + u_m = 0 . \tag{9}$$

The deficiency (or genus) being equal to $\dfrac{(m-1)(m-2)}{2}$, it is crystal-clear that a curve Γ of the type (9) will be *bicursal,* when and only when $m = 3$. That is to say, the *only* bicursal curve, belonging to the category (9), is a sextic, which has three triple points, *viz.* I, J, O and whose Cartesian equation, referred to the finite point O as origin, is accordingly

$$r^6 + r^4 v_1 + r^2 v_2 + u_3 = 0 . \tag{10}$$

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[2]) The property (v) follows by logarithmic differentiation from the proved result $\prod_{p=1}^{2m} r_p = \text{const.}$, where $r_1, r_2, r_3, \dots, r_{2m}$ denote the distances (from any fixed point O) of the $2m$ points where an arbitrary transversal (through O) cuts Γ.

[3]) R. A. Robert (*loc. cit.*) has studied the afore-mentioned specialised curve (along with its particular variety, *viz.*, a bicircular quartic) and has also discussed certain other properties, which have been proved as above to hold for a *much wider class of curves.*

Needless to say, this sextic can be converted at pleasure into a (bicursal) bicircular quartic or into a (bicursal) circular cubic by an appropriate Cremona transformation.

SECTION II.

SPECIAL TYPES OF SURFACES

3. - Starting with an algebraic surface Γ (of *even* degree $2m$) in the symbolic Cartesian form:

$$\sum_{p=0}^{p=2m} u_p = 0\,, \tag{11}$$

(where u_p denotes the set of terms, homogeneous and of degree p in x, y, z), we readily perceive that, if Σ be the section of Γ made by the plane at infinity, the equation to the cone, having the origin O for vertex and Σ for base, is

$$u_{2m} = 0\,. \tag{12}$$

Consequently, the necessary and sufficient condition for the (plane) curve Σ to consist of the « circle at infinity (Ω) », counted m times, is that the following identity in x, y, z shall hold:

$$u_{2m} \equiv (\text{const.}) \times r^{2m}, \qquad (r^2 \equiv x^2 + y^2 + z^2)\,. \tag{13}$$

There is no difficulty in verifying (as in § 1) that the condition (13) may also be interpreted as the condition that the product of the distances (measured from any origin K) of the $2m$ points, at which Γ is cut by an *arbitrary* straight line (L), drawn through K, shall be *independent* of the direction of L.

4. - When it is required to take account of the points of intersection of an arbitrary ellipse π with a surface of the special type, contemplated in § 3, we may represent π in the Cartesian form:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,, \qquad z = 0\,.$$

So if $t \equiv e^{i\Phi}$, where φ denotes the eccentric angle of any of the $4m$ points of intersection of Γ and π, we may, as in § 1, write:

$$(14) \qquad x = \frac{a}{2}\left(t + \frac{1}{t}\right), \quad y = \frac{b}{2i}\left(t - \frac{1}{t}\right) \quad \text{and} \quad z = 0,$$

so that

$$r^2 = x^2 + y^2 + z^2 = \lambda\left(t^2 + \frac{1}{t^2}\right) + \mu,$$

where λ, μ are certain constants, independent of t.

The equation of Γ being (by § 3) of the symbolic form:

$$(15) \qquad r^{2m} + u_{2m-1} + u_{2m-2} + , \dots , + u_2 + u_1 + u_0 = 0,$$

a moment's reflection shows that, when (14) is substituted in (15), the latter equation ultimately takes the form:

$$(16) \qquad \lambda^m t^{4m} + c_1 t^{4m-1} + c_2 t^{4m-2} + , \dots , + c_{4m-1} t + \lambda^m = 0,$$

where $c_1, c_2, \dots, c_{4m-1}$ are all constants.

Hence if $(t_1, t_2, \dots, t_{4m})$ be the $4m$ roots of (16), we find, as in § 1,

$$t_1 t_2 t_3 \dots t_{4m} = 1,$$

whence $\sum_{q=1}^{q=4m} \Phi_q = 0$ or an even multiple of π.

The result may then be summarised in the form of a theorem:

THEOREM **B** — *If a surface Γ of even degree $2m$ possesses any one of the following properties,* viz.,

(i) *that the associated « curve at infinity » shall consist of the circle at infinity (counted m times),*

(ii) *that, for an arbitrary transversal L, drawn through any given point O, the product of the distances (from O) of the 2m points of intersection of Γ and L shall be a constant, i.e., independent of the direction of L,*

and (iii) *that the set of terms, homogeneous and of degree 2m in x, y, z, occurring in the Cartesian equation of Γ, shall be a numerical multiple of r^{2m},*

***it* (Γ) *must possess the other two properties and at the same time it must enjoy yet another property, viz. that the sum of the eccentric angles of the points of intersection of* Γ *with an arbitrary ellipse (in the 3-space) is either zero or an even multiple of* π.**

5. - We shall now consider special varieties of surfaces, considered in § 3 and 4.

Plain reasoning shews that, upon a surface Γ (of the aforesaid description) the *circle at infinity* (Ω) will be a multiple curve of multiplicities 2, 3, 4, ..., *m*, according as the Cartesian equation of Γ (referred to an arbitrary origin) is of the form:

$$r^{2m} + r^{2m-2}v_1 + u_{2m-2} + u_{2m-3} + , \dots , + u_2 + u_1 + u_0 = 0$$

or $r^{2m}+r^{2m-2}v_1+r^{2m-4}v_2+u_{2m-3}+u_{2m-4}+ ,\dots, +u_2+u_1+u_0=0.$

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or $r^{2m} + r^{2m-2}v_1 + r^{2m-4}v_2 + , \dots , + r^2v_{m-1} + u_m + u_{m-1} +$
$+ , \dots , +u_2 + u_1 + u_0 = 0.$

It may be remarked in passing that the surface Γ, which is defined by the equation last written, and has the circle at infinity for an *m*-tuple curve, will behave like Darboux's *anallagmatic surface,* provided that the involved coefficients conform to certain special relations.

When $m = 2$, we obtain a highly specialised surface of this kind, *viz.,* a *cyclide,* which has the *circle at infinity* for a double curve and which plays the same rôle in a 3-space as the ordinary bicircular quartic does in a 2-space.

SECTION III.

SPECIAL TYPES OF HYPER-SURFACES

6. - In order to generalise the results of Sections I and II to a multi-space, let us start with an *arbitrary* point (x_0, x_1, x_2, ..., x_n) in an $(n+1)$-space (R_{n+1}), so that the Cartesian equation of an algebraic n-surface (of even degree $2m$) may be written in the symbolic form:

$$\sum_{p=0}^{p=2m} u_p = 0\,, \tag{17}$$

where u_p is a homogeneous polynomial of degree p in (x_0, x_1, x_2, ..., x_n). It is easy to see that, if π denote the $(n-1)$-surface along which Γ is cut by the *n-flat at infinity* (say, K), the equation to the n-cone, having O for vertex and π for base, is

$$u_{2m} = 0\,. \tag{18}$$

As is well-known, an *arbitrary* n-sphere in R_{n+1} intersects the *n-flat at infinity* (K) along a *fixed* (*imaginary*) $(n-1)$-*sphere at infinity* [4]), which will be symbolised as Ω. Further, the equation to the quadric n-cone, having O for vertex and Ω for base, is

$$r^2 = 0\,, \qquad (\text{where} \quad r^2 \equiv \sum_{p=0}^{p=n} x_p{}^2)\,. \tag{19}$$

Comparing (18) and (19), we readily see that the necessary and sufficient condition for the $(n-1)$-surface π to consist of Ω, *reckoned m times,* is that the relation

$$u_{2m} \equiv (\text{const.}) \times r^{2m} \tag{20}$$

shall hold *identically*. Other geometrical interpretations, somewhat similar to those of §§ 1 and 3, can be put upon (20).

Subject to the condition (20), the equation to Γ assumes the form:

$$r^{2m} + u_{2m-1} + u_{2m-2} + \,, \ldots, + u_2 + u_1 + u_0 = 0\,. \tag{21}$$

In order to find its $4m$ points of intersection with an arbitrary ellipse Σ, *viz.,*

$$(x_0 = a\cos\Phi,\ x_1 = b\sin\Phi,\ x_2 = 0,\ x_3 = 0,\ \ldots,\ x_n = 0)\,,$$

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[4]) It is hardly necessary to remark

(i) that in space R_2: $(n = 1)$, Ω consists simply of two points. *viz., the two circular points at infinity,*

and (ii) that in space R_3: $(n = 2)$, Ω becomes the *circle at infinity.*

we proceed precisely as in §§ 1 and 4 and eventually arrive at the result:

$$\Phi_1 + \Phi_2 + , \ldots , + \Phi_{4m} = 0 \text{ or an even multiple of } \pi.$$

Putting this and that together, we may finalise our conclusions in the form of a theorem:

TEOREM **C** — *If an algebraic n-surface of degree 2m, (lying in R_{n+1}), possesses any one of the following properties:*

(i) *that the associated $(n-1)$-surface at infinity shall consist of the fixed (imaginary) $(n-1)$-sphere at infinity Ω, reckoned m times,*

(ii) *that for an arbitrary transversal L, drawn through any fixed point O, the product of the distances (from O) of the 2m points of intersection of Γ with L shall be constant* (i.e., *independent of the direction of L*),

and (iii) *that the terms of the highest order, occurring in the Cartesian equation (of Γ), referred to an arbitrary set of axes, shall be a numerical multiple of r^{2m},*

then it (Γ) must possess the other two properties and at the same time it must have another concomitant property, viz., that the algebraic sum of the eccentric angles of the 4m points of intersection of Γ with an arbitrary *ellipse must be either zero or an even multiple of π.*

7. - We shall now close this topic with a laconic reference to certain varieties of *n*-surfaces, which have the $(n-1)$-sphere at infinity (Ω) for a multiple $(n-1)$-surface [5]). Without going

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[5]) In space R_{n+1}, a point P, lying on an n-surface Γ, is said to be a *multiple point* (*of multiplicity k*), provided that an *arbitrary* right line (drawn through P and lying in R_{n+1}) meets Γ at a number of points, of which k coincide with P. Further, an $(n-1)$-surface Ξ, lying on Γ, is called a *multiple* $(n-1)$-*surface* (*of order k*), provided that *every* point, lying on Ξ (and therefore also on Γ) is a multiple point *of order k* on Γ.

General reasoning shows that, although the existence of a *multiple* $(n-1)$-surface, — or for the matter of that, of even a single *multiple*, into details one can easily perceive that the groups of homogeneous terms, *viz.,* u_{2m-1}, u_{2m-2}, ..., occurring in (21), have to be particularised to a certain extent in order that Ω may be a *multiple* $(n-1)$-surface on Γ. There is no difficulty in shewing that Ω will be a multiple $(n-1)$-surface of respective multiplicities 2, 3, 4, ..., m, on the following $(m-1)$-species of n-surfaces:

Species (*i*) $r^{2m}+r^{2m-2}v_1+u_{2m-2}+, \dots, +u_2+u_1+u_0=0;$

Species (*ii*) $r^{2m}+r^{2m-2}v_1+r^{2m-4}v_2+u_{2m-3}+, \dots, +$
$+u_2+u_1+u_0=0;$

Species (*iii*) $r^{2m}+r^{2m-2}v_1+r^{2m-4}v_2+r^{2m-6}v_3+u_{2m-4}+$
$+, \dots, +u_2+u_1+u_0=0;$

. . . .

. . . .

Species $(m-1)$ $r^{2m}+r^{2m-2}v_1+r^{2m-4}v_2+, \dots, +r^2v_{m-1}+$
$+u_m+u_{m-1}+, \dots, +u_2+u_1+u_0=0.$

It is worth mentioning that the $(m-1)$th. species of n-surface, which is represented by the last equation of the above set and which has Ω for a multiple $(n-1)$-surface of multiplicity m, comprises as a sub-class the aggregate of generalised *anallagmatic* n-surfaces.

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point — on an n-surface Γ (given in the Cartesian form) is to be regarded as *accidental* or *exceptional* and is contingent upon certain conditions to be fulfilled by the attached coefficients, still *under favourable conditions* an n-surface Γ (supposed to be of degree $2m$) may possess a multiple $(n-1)$-surface, whose multiplicity ranges from 2 to m. This readily explains why it is necessary in the above enumeration to reckon with $(m-1)$-species of n-surfaces (like Γ), having the $(m-1)$-sphere (at infinity) Ω for a multiple $(m-1)$-surface of multiplicities varying form 2 to m.