

RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

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Rendiconti del Seminario Matematico della Università di Padova,
tome 21 (1952), p. 316-334

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NOTE ON AUTOPOLAR PLANE CUBICS

Memoria (*) *di* HARI DAS BAGCHI,
e di MANINDRA CHANDRA CHAKI (*a Calcutta*)

INTRODUCTION

The present paper deals with the general type of « autopolar » (or « self-polar ») cubic, *i.e.*, a plane cubic, which is its own polar reciprocal *w.r.t.* one or more « auxiliary » conics. As a matter of convenience, this paper has been divided into two sections and an addendum. Sec. I concerns itself mainly with certain characteristic traits of an « autopolar » cubic (Γ) with special reference to its one-parameter family of « auxiliary » conics; besides, there are incidental references to three familiar varieties of autopolar cubics, *viz.*, the semi-cubical parabola, the cubical parabola and the Cissoid of Diocles. Then Sec. II reckons with the effect of certain types of Cremona transformations (particularly of the *automorphic* type) on an autopolar cubic.

Lastly there is an Addendum, dealing with the *general* type of « bicursal » (or « elliptic ») cubic. An apology is needed for this digression upon the *general* cubic. The *main* reason is that the transformations, whose effect on the *general* cubic has been studied in the Addendum, are precisely those, whose effect on the *autopolar* cubic has been discussed in Sec. II. A subsidiary reason is that the subject-matter of the Addendum could not be fittingly inserted anywhere else in this paper.

On the whole this paper is believed to contain some amount of *original* matter, although there are casual references to *known* results.

*) Pervenuta in Redazione il 25 aprile 1952.

SECTION I.
CERTAIN NOVEL FEATURES
OF AN AUTOPOLAR CUBIC

1. - The *degree* and *class* of any algebraic plane curve Γ being interchanged by polar reciprocation, it follows that, for Γ to be « autopolar », a necessary — but by no means sufficient — condition is that its *degree* and *class* should be equal. Consequently, if the curve Γ (supposed autopolar) be of degree n and has δ nodes and k cusps, we have by a formula of Plücker's

$$2\delta + 3k = n(n - 2).$$

Setting $n = 3$, we infer that the discrimination of the type of autopolar cubic (assumed to exist) depends on the positive integral solution of the *indeterminate* equation:

$$2\delta + 3k = 3.$$

The only possible (positive) integral solution being

$$\delta = 0 \quad \text{and} \quad k = 1,$$

we arrive at the proposition:

PROP. I. — *If a cubic be autopolar, it must be cuspidal.*

The converse of this result will be examined in the next article, where it will be found expedient to proceed in an *indirect* manner.

2. - We know that every (non-degenerate) cuspidal cubic Γ has only one inflexion (say, B), and that, if the tangent at B cuts that at the cusp (C) at the point A , then the equation of Γ , — expressed in terms of homogeneous or projective coordinates (x, y, z) , referred to ABC as the « fundamental triangle » (Fig. 1) — admits of the canonical form ¹⁾:

$$(1) \quad x^3 = \lambda y^2 z,$$

where λ is a parameter.

¹⁾ Vide Hilton: *Plane Algebraic Curves*, (1920), pp. 204-207.

The presence of a single parameter, *viz.* λ , in (1) could be foreseen from general considerations. For, to be given a cusp of a cubic (together

Evidently when we interchange the meanings of any two of the three coordinates (x, y, z) , — so that the « fundamental » triangle ABC remains practically the same as before, — the equation (1) can at pleasure be turned into any one of the five forms:

$$\left. \begin{array}{l} (2), (3) \\ (4), (5) \\ (6) \end{array} \right\} \begin{array}{l} x^3 = \lambda_1 y z^2, \quad y^3 = \lambda_2 z^2 x, \quad y^3 = \lambda_3 z x^2, \quad z^3 = \lambda_4 x^2 y, \\ z^3 = \lambda_5 x y^2, \end{array}$$

where the λ 's are parameters.

When, however, we have to deal with a *single* determinate cuspidal cubic Γ , we may keep to the form (1) and merge the parameter λ in one of the coordinates (say, z), so that the equation assumes the simpler form:

$$(7) \quad x^3 = y^2 z.$$

Plainly an *arbitrary* point on (7) can be taken as

$$(t, 1, t^3),$$

(where t is a parameter) and then its polar *w.r.t.* a conic S , — as yet *undetermined* —, *viz.*

$$(8) \quad ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$

can be written in the symbolic form:

$$(9) \quad Wt^3 + Ut + V = 0,$$

where U, V, W stand for three linear functions of (x, y, z) , defined by:

$$(10), (11) \quad U \equiv ax + hy + gz, \quad V \equiv hx + by + fz,$$

$$(12) \quad W \equiv gx + fy + cz.$$

with its associated tangent) amounts to *five* conditions, whereas to be given a point of inflexion (together with its associated tangent) amounts to *three* conditions. Inasmuch as the algebraic structure of an unrestricted cubic Γ involves *nine* parameters, it is but meet and proper that, when the « fundamental triangle » ABC (defined as above) is *assigned*, the number of independent parameters in the equation of Γ should be $(9 - 5 - 3)$, *i. e.*, 1.

It is easy to see that the polar reciprocal (Σ) of Γ *w.r.t.* S (chosen as the « auxiliary » conic) is to be obtained by equating to zero the t -discriminant of the equation (9). Thus the equation to Σ becomes

$$(13) \quad 4 U^3 = -27 V^2 W.$$

Recognising that the equation (13) is of the *same* symbolic form as any of the equations of the set (1)-(6) and interpreting (13) geometrically *w.r.t.* the $\Delta A'B'C'$, (formed by the

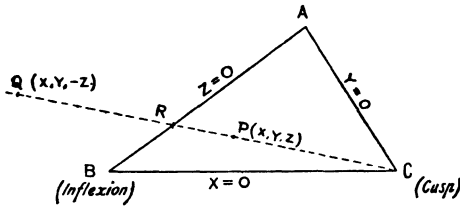


Fig. 1 (Cubic Γ)

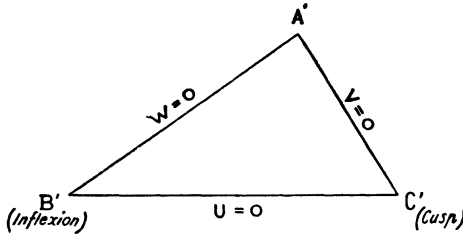


Fig. 2 (Reciprocal cubic Σ)

three lines $U = 0, V = 0, W = 0$, as shewn in Fig. 2), we conclude that the reciprocal cubic Σ has C' and B' respectively for a cusps and an inflexion and has $C'A'$ and $B'A'$ for the attached tangents at the two points.

If, then, the cuspidal cubic Γ is to be « autopolar », it must be possible to adjust the constants of S , *viz.*

$$a : b : c : f : g : h,$$

(which are hitherto undefined), so that the cubics Σ and Γ may be *identical*. If we now attend to the elementary lemmas, *viz.*:

- (a) that a (proper) cubic cannot have more than one cusp,
- (b) that a (proper) cuspidal cubic cannot have more than one inflexion

and (c) that a (proper) cuspidal cubic can have only one cusp, one inflexion with determinate tangents thereat,

it is palpably plain that, if Γ is to be autopolar, a *necessary* (but not sufficient) condition is that the points (A' , B' , C') of Fig. 2 must coincide respectively with the points (A , B , C) of Fig. 1. This implies that the three equations ($U = 0$, $V = 0$, $W = 0$), must coincide respectively with the three equations ($x = 0$, $y = 0$, $z = 0$), leading ultimately to the equalities

$$(14), (15), (16) \quad f = 0, \quad g = 0, \quad h = 0.$$

When (14), (15), (16) are substituted in (13) and the result of substitution is identified with (7), we derive one additional relation, *viz.*,

$$(17) \quad 4a^3 + 27b^2c = 0.$$

Remarking that the cubic Γ reciprocates into itself *w.r.t.* the « auxiliary » conic S , whose coefficients conform to the four relations (14), (15), (16) and (17), we may summarise our conclusions in the form of a proposition:

PROP. II. — *Every cuspidal cubic Γ , representable as always in the normal form (7), is « autopolar », there being a system of ∞^1 « auxiliary conics » of the form:*

$$(18) \quad ax^2 + by^2 + cz^2 = 0,$$

where the ratios $a : b : c$ are subject to the single condition (17).

For obvious reasons Prop. II is the « converse » of Prop. I of § 1.

If we now introduce a single variable parameter u according to the relation

$$u = \frac{a}{b},$$

and take note of the conditional relation (17), we can re-write the equation (18) in the symbolic form:

$$(19) \quad S_u \equiv F(x, y, z, u) = 0,$$

where the function F is defined by

$$(20) \quad F(x, y, z, u) \equiv 27ux^2 + 27y^2 - 4u^3z^2.$$

Propositions I and II may now be amalgamated together and presented as a single proposition as follows:

PROP. III. — *The necessary and sufficient condition for a plane cubic Γ to be autopolar is that it be cuspidal. This condition being fulfilled by Γ and its (homogeneous) equation being taken in the canonical form (7), there exists a family of ∞^1 « auxiliary » conics $\{S_u\}$ given by (19), (20), with respect to any one of which Γ is « autopolar ».*

Further characterisation of the family of « auxiliary » conics $\{S_u\}$ with special reference to the autopolar cubic Γ being deferred to § 4, we propose to corroborate Propositions II and III (in § 3) by an *alternative* method, which, although more lengthy, is yet *more algebraic* than the foregoing method.

3. The polar line (9) of an *arbitrary* point $(t, 1, t^3)$ on the cubic (7) can be easily put in the symbolic form:

$$(21) \quad x \cdot l(t) + y \cdot m(t) + z \cdot n(t) = 0,$$

wherein the coefficients are given by

$$(22) \quad l(t) = gt^3 + at + h,$$

$$(23) \quad m(t) = ft^3 + ht + b,$$

$$(24) \quad n(t) = ct^3 + gt + f.$$

Manifestly Γ will be its own polar reciprocal *w.r.t.* (8), provided that the line (21) touches Γ for *all* values of t ; for this to be possible, the necessary and sufficient condition is that the « tangential equation » of Γ , *viz.*,

$$(25) \quad 4\lambda^3 + 27\mu^2\nu = 0$$

shall be satisfied *independently* of t , as soon as λ, μ, ν are replaced respectively by the three « line-coordinates » of (21), *viz.*,

$$l(t), \quad m(t), \quad n(t).$$

That is to say, Γ will be autopolar, when and only when the equality

$$(26) \quad 4\{l(t)\}^3 = -27\{m(t)\}^2n(t)$$

holds *identically*. Supposing α, β, γ to be the three *zeros* of

the cubic function $l(t)$, it is plain that the nine *zeros* of the polynomial (of degree 9) on the L.S. of (26) consists of

$$\alpha, \beta, \gamma, \text{ (each counted } \textit{thrice}),$$

whereas the nine *zeros* of the polynomial (of degree 9) on the R.S. of (26) consist of

$$\begin{array}{ll} \text{the 3 zeros of the cubic } m(t), & \text{(each counted } \textit{twice}) \\ \text{and the 3 zeros of the cubic } n(t), & \text{(each counted } \textit{once}). \end{array}$$

Inasmuch as the L.S. and the R.S. of (26) denote *essentially the same* polynomial (of degree 9), whenever (26) is an *identity*, it is palpably plain that their zeros must also be the same, so that α, β, γ must be the three zeros of each of the other two cubics $m(t), n(t)$. Thus the identity (26) *ordinarily* implies that the three cubics $l(t), m(t), n(t)$ are essentially identical, differing from one another at most by a numerical multiplier. Thus under normal circumstances when

$$(27) \quad f, g, h \neq 0,$$

the identity (26) apparently gives rise to the relations of proportionality:

$$g : a : h = f : h : b = c : g : f,$$

leading ultimately to

$$A, B, C, F, G, H \quad \text{each} \quad = 0,^2)$$

and signifying geometrically that the conic (8) consists of *two coincident* right lines. The contingency (27) must, however, be ruled out, for the « auxiliary » conic is impliedly *non-degenerate*. Clearly, then, *if Γ is to be autopolar w.r.t. the conic S , viz. (8), a necessary (but not sufficient) condition*

²⁾ As usual, the capital letters denote the cofactors of the corresponding small letters in the determinant

$$\begin{vmatrix} g & f & c \\ a & h & g \\ h & b & f \end{vmatrix}$$

to be fulfilled by S is that at least one of its three coefficients (f, g, h) must be nil.

Thus there are three cases to consider:

$$\text{CASE i} \quad f = 0.$$

$$\text{CASE ii} \quad g = 0.$$

$$\text{CASE iii} \quad h = 0.$$

Before we deal with the three cases separately, we note the two subsidiary relations:

$$(28), (29) \quad 4h^3 = -27b^2f \quad \text{and} \quad 4g^3 = -27cf^2,$$

which are deducible from (26) by equating the absolute terms, and the coefficients of t^3 of the L.S. and the R.S. of (26).

CASE i — If we now start with the hypothesis $f = 0$, we at once deduce from (28), (29),

$$h = 0 \quad \text{and} \quad g = 0,$$

so that (26) gives finally

$$(17) \quad 4a^3 = -27b^2c.$$

In other words, the vanishing of f implies that of both g and h and at the same time leads to the relation (17).

Similar results can be easily verified in Cases ii and iii.

So unifying these results, we infer that, when the conic (8) is *non-degenerate*, the identity (25) or (26) can be valid, if and only if all the three coefficients f, g, h vanish simultaneously and at the same time a, b, c conform to the relation (17). Thus once again we are led to the conclusion that *the cubic (7) is autopolar w.r.t. any conic S of the form (18), the coefficients $a:b:c$ of which conform to the relation (17).*

This completes the *algebraic* confirmation of the truth of Propositions II and III.

4. - As a preliminary to the consideration of the envelope of the set of auxiliary conics $\{S_u\}$ of § 2, it is necessary to associate, with the cubic Γ given by (7), a second cubic Γ' ,

$$(30) \quad x^3 = -y^2z.$$

A little reflection shews that, if in Fig. 1 (§ 2) we take an arbitrary point $P(x, y, z)$ and mark the point R , where the line CP (produced, if necessary) cuts the line AB , then the point $Q(x, y, -z)$ is no other than the *fourth* conjugate of the three collinear points C, P, R ³). Evidently the correspondence between the points P, Q is involutory and when P moves on the curve

$$\varphi(x, y, z) = 0,$$

Q must move on the curve

$$\varphi(x, y, -z) = 0.$$

In particular, when P moves on the cubic (7), Q moves on the cubic (30). For felicity of expression, the two cubics Γ, Γ' will be said to be « conjugate » to each other. Plainly the two « conjugate » (cuspidal) cubics possess the *same cusp*, the *same inflexion* and the *same tangents* thereat and the *same « fundamental » triangle*, and further admit of the *same set of* ∞^1 of « auxiliary » conics of the type $\{S_u\}$.

Recognising that the improper sextic equation, *viz.*

$$(x^3 + y^2z)(x^3 - y^2z) = 0,$$

obtained by multiplying the two equations (7) and (30), can be algebraically interpreted as the u -eliminant of the two equations

$$F(x, y, z, u) = 0 \quad \text{and} \quad \frac{\partial F}{\partial u} = 0,$$

where F is defined by (20), we readily derive a geometrical proposition, which reads as follows:

PROP. IV. — *The one-parameter family of « auxiliary » conics $\{S_u\}$, with respect to any one of which any given cuspidal cubic Γ and therefore also its « conjugate » Γ' are « autopolar », have for the complete envelope an improper sextic curve, composed of the two cubics. Any particular « auxiliary*

³) Obviously, the range of points C, P, R, Q is harmonic.

ry » conic S_u has double contact ⁴⁾ with each of the two conjugate cubics Γ, Γ' . Furthermore the quadrangle, formed by the two pairs of points of contact of S_u with Γ, Γ' , has for its three « centres » the vertices of the « fundamental » triangle.

5. - We shall now make a few observations on three special categories of cuspidal cubics ⁵⁾, viz.,

(i) a semi-cubical parabola, given in the standard form

$$(31) \quad X^3 = aY^2,$$

(ii) a cubical parabola, given in the standard form

$$(32) \quad X^3 = a^2Y,$$

and (iii) a cissoid of Diocles, given in the standard form

$$(33) \quad X(X^2 + Y^2) = aY^2, \quad \text{i.e.,} \quad X^3 = Y^2(a - X),$$

into each of which the general type of « autopolar » or cuspidal cubic can be (conically) projected ⁶⁾.

By general reasoning the reader can readily substantiate the following statements:

⁴⁾ An interesting feature of the system of conics $\{S_u\}$ may be noted here. In general, *three* conditions have to be fulfilled by an *undefined* conic (T), when a self-conjugate triangle (Δ) is given and two conditions have to be fulfilled, when T is required to touch a *given* curve Σ twice; so when T is to have Δ for a self-conjugate triangle and to have, besides, double contact with Σ , it has ordinarily to satisfy $(3 + 2)$, or, 5 conditions; and consequently the total number of such conics is, in general, *finite*. But there are *exceptional* cases, where the number of conics, having an *assigned* self-conjugate triangle (Δ) and touching an *assigned* curve (Σ) twice, is *not* finite but infinite. Obviously such a contingency can arise only when, owing to some peculiar relation between Δ and Σ the number of independent conditions to be satisfied is not five but four. As an instance of this exceptional case we may point to the set of ∞^1 auxiliary conics of the type S_u , each of which has ΔABC for a self-conjugate Δ and has double contact with the cuspidal cubic Γ . That every such conic has double contact with the conjugate cubic Γ' is to be looked upon as a *subsidiary* property.

⁵⁾ It is presumed that the symbols X, Y , used in (31)-(33), denote rectangular Cartesian coordinates.

⁶⁾ Vide BASSER, *Cubics and Quartics*, (1901), Art 360, p. 239; and HILTON, *Plane Algebraic Curves*, (1920), Ex 1, p. 205.

- (a) A semi-cubical parabola, given by (31), is « autopolar » and admits of the ∞^1 « auxiliary » conics

$$\frac{X^2}{4u^2a^2} + \frac{Y^2}{4u^3a^2} = 1, \quad (u \text{ being a parameter}),$$

whose complete envelope consists of the cubic (31) and its conjugate ($X^2 = -aY^2$);

- (b) A cubical parabola, given by (32), is « autopolar » and admits of the family of ∞^1 « auxiliary » conics, *viz.*,

$$\frac{Y^2}{27a^2} - \frac{X^2}{a^2} = 1, \quad (u \text{ being a parameter}),$$

whose complete envelope consists of the cubic (32) and its conjugate, *viz.*, $X^3 = -a^2Y$;

- (c) A cissoid of Diocles, given by (33), is « autopolar » and admits of the family of ∞^1 « auxiliary » conics, *viz.*,

$$27uX^2 + 27Y^2 - 4u^3(X - a)^2 = 0, \quad (u \text{ being a parameter}),$$

whose complete envelope consists of the cubic (33) and its conjugate, *viz.*, $X^3 = -Y^2(a - X)$.

SECTION II.

EFFECT OF CERTAIN SPECIAL VARIETIES OF CREMONA TRANSFORMATIONS ON AN AUTOPOLAR CUBIC

6. - Plain reasoning shows that a plane collineation (\mathbb{E}) of the *unrestricted* type, when operating on any given plane curve Γ (not necessarily a cubic), must conserve, among other entities,

(a) its *degree* and *class*,

and (b) the *geometrical character* — as distinguished from the absolute position — of:

(i) a point of inflexion and its associated tangent, and (ii) a multiple point and each of its associated tangents.

In particular, \mathbb{E} converts a cuspidal cubic Γ into another cuspidal cubic Γ_1 , such that the « fundamental » triangle ABC of Γ , — as contemplated in § 2 (Fig. 1) — is carried over into the « fundamental » triangle $A_1B_1C_1$ of Γ_1 . Because the points A_1, B_1, C_1 correspond respectively to the points A, B, C , it follows that if Γ_1 is to coincide with Γ so that this curve remains *invariant* under the transformation \mathbb{E} , a necessary — but not sufficient — condition is that A_1, B_1, C_1 coincide respectively with A, B, C . In other words, the *invariance* of the cubic Γ necessitates the *invariance* of the three points A, B, C . So an « automorphic » projective transformation (\mathbb{E}) of the cuspidal cubic Γ must belong to the *two-parameter*⁷⁾ family of collineations, which leave the three points A, B, C *invariant* and are accordingly representable in the analytic form:

$$(34) \quad \rho x' = ax, \quad \rho y' = by, \quad \rho z' = z,$$

it being implied that ABC is the « triangle of reference » and that ρ is a factor of proportionality and that a, b are two *arbitrary* (or *disposable*) parameters.

The condition for (34) to leave the equation (7) of Γ *unaltered* being easily seen to be

$$(35) \quad a^3 = b^3,$$

⁷⁾ Vide GRAUSTEIN, *Introduction to Higher Geometry*, (1946), Ex 4, p. 173. The existence of *two* independent parameters a, b in (34) could be foreseen from general considerations, based on the following lemmas:

- (i) that the number of collineations, which convert *four* points of a given tetrad into *four* points of another given tetrad, is *only one*, provided that no three points of either tetrad are collinear;
 - (ii) that the totality of collineations, which convert three given (non-collinear) points into three other (non-collinear) points, is ∞^2 ;
- and (iii) that, as a particular case of (ii), the totality of collineations, which leave three given (non-collinear) points *invariant*, is also ∞^2 .

we conclude that *the general type of « automorphic » collineations of a cuspidal cubic Γ , given by (7), is representable in the form:*

$$(36) \quad \rho x' = \mu^2 x, \quad \rho y' = \mu^2 y, \quad \rho z' = z,$$

where μ is a variable parameter.

A little reflexion shows that the *two-parameter* family of collineations, defined by (34), constitute a « group », of which a « sub-group » is composed of the *one-parameter* family of « automorphic » collineations, defined by (36).

In this connection it is worth while to reckon with the *bigger* group (G) of all possible collineations, which permute *among themselves* the three vertices A, B, C of the « fundamental » triangle. Clearly G can be divided into *six* distinct (two-parameter) subsets (G_1, G_2, \dots, G_6), whose *symbolic* and *analytic* representations are recorded in the following table. As is to be expected from other considerations, each of the six subsets involves *two* effective parameters in its *analytic* representation. In recording the *symbolic* representation the current notation of the *Theory of Substitutions* has been adopted. Thus the notation (ABC) means that the points A, B, C are being converted respectively into B, C, A , so that *the cyclic order remains the same*. Also the « transposition » (AB) or (BA) implies that A, B simply change places. In particular, the transposition (AA) means that the point A remains *fixed*. Subject to these conventions, the six subsets are tabulated as under:

$G \equiv$ Collineation - group, connected with the $\triangle ABC$.

Subsets	Symbolic representations	Analytic representations
G_1	$(AA)(BC)$	$\rho x' = x, \rho y' = bz, \rho z' = cy$
G_2	$(BB)(CA)$	$\rho x' = az, \rho y' = y, \rho z' = cx$
G_3	$(CC)(AB)$	$\rho x' = ay, \rho y' = bx, \rho z' = z$
G_4	$(AA)(BB)(CC)$	$\rho x' = ax, \rho y' = by, \rho z' = z$
G_5	(ABC)	$\rho x' = az, \rho y' = bx, \rho z' = y$
G_6	(ACB)	$\rho x' = ay, \rho y' = bz, \rho z' = x$

It is superfluous to add that, of all the six subsets ($G_1 - G_6$) only one, *viz.*, G_4 , counts as a *sub-group*. For, the «unity» or «identity» element being absent in the remaining five G 's, *none* of them is a sub-group. Plainly the sub-group G_4 , defined heretofore by (34), is Abelian, and can further be characterised as the *largest* sub-group, which possesses the property that, when any one of its constituent transformations is multiplied by *any* transformation of the compound group G , the «commutative law» holds good. Accordingly the sub-group G_4 must be designated as the «centre» or «centrum» of the larger group G .

7. - In this article we shall touch briefly upon another type of Cremona transformations, which are quadratic in character but are nevertheless *automorphic* in relation to any given «autopolar» cubic Γ , supposed to be given as before in the canonical form (7).

It is well-known that every «quadratic» transformation⁸⁾ can be analytically represented as

$$(37) \quad \rho x x' = a, \quad \rho y y' = b, \quad \rho z z' = 1,$$

where a and b are two independent parameters. The two-

⁸⁾ It is scarcely necessary to remark that, for any *fixed* values of the parameters a, b , (37) admits of a simple geometrical interpretation. In fact, (37) being written in the symmetric form

$$(i) \quad p x x' = a, \quad p y y' = b, \quad p z z' = c,$$

and the constants ($a : b : c$) being kept *fixed*, it is easy to verify geometrically that an *arbitrary* pair of points (x, y, z) and (x', y', z') , connected by the relations (i), are «conjugate» to each other in relation to the «pencil» of conics, passing through the four *fixed* points, *viz.*

$$(\sqrt{a}, \sqrt{b}, \sqrt{c}), \quad (-\sqrt{a}, \sqrt{b}, \sqrt{c}), \quad (\sqrt{a}, -\sqrt{b}, \sqrt{c}) \quad \text{and} \quad (\sqrt{a}, \sqrt{b}, -\sqrt{c}).$$

For obvious reasons, the four *common* (or «base») points of the «pencil» of conics will be *real*, when the parameters a, b, c (supposed real) have the *same* algebraic sign.

[Vide: (i) GRAUSTEIN, *loc. cit.*, pp. 320-21;

(ii) COOLIDGE, *Algebraic Plane Curves*, (1931), Book II, (Ch. I), Pp. 196-200;

(iii) HUDSON, *Cremona Transformations*, (1927), Ch. III].

parameter set of *involutory* transformations of the type (37) cannot be a « group » in the *technical* sense of the term, seeing that the « identity » (*i.e.*, *identical* transformation) is absent. However it is of some interest to note that an *arbitrary* transformation of the type (37) converts an equation of the form (1) into another equation of a *similar* form (with, of course, a *different* values of λ), showing thereby that *an autopolar (or cuspidal) cubic is carried over into an autopolar cubic by (37)*. It is easy to see that the set of transformations (37) contains a *subset*, which is « automorphic » in relation to the autopolar cubic, given in the normal form (7), (*viz.*, $\omega^3 = y^2z$). The actual analytical representation of the subset is

$$(38) \quad \rho x x' = \mu^2, \quad \rho y y' = \mu^3 \quad \rho z z' = 1,$$

where μ is an *arbitrary* parameter.

ADDENDUM

EFFECT OF CERTAIN CREMONA TRANSFORMATIONS ON A « BICURSAL » CUBIC

8. - In this Addendum we shall trace briefly the effect of the Cremona transformations of §§ 6 and 7 on the *general* type of a « bicursal » cubic (*i.e.*, a cubic of *unit* genus or deficiency). To that end we feel it incumbent to quote the following common-places of Higer Plane Curves, (it being presumed that P is an *arbitrary* point on an unrestricted « bicursal » cubic Γ):

- (i) that the system of ∞^1 cubics, having eight-pointic contacts with Γ at P , cut this curve again at a certain fixed point (P'), often called the « Halphen » point ⁹⁾ of P ;
- (ii) that the total number of *united* points of the correspondence (P, P') — often called « coincidence » points, or simply « c -points » — is, in general, 72;

⁹⁾ Vide WILCZYNSKI, *Projective Differential Geometry of Curves and Ruled Surfaces*, (1906), pp. 67-82.

- (iii) that P being any of the c -points, its first tangential (Q) is also a c -point and its second tangential (R) — which is after all the first tangential of Q — is also a c -point; and the third tangential of P — which is but the first tangential of R — coincides with P itself ¹⁰);
- (iv) that a triangle PQR of the type (iii) is often called a « Hart-triangle » (or an « H-triangle ») of Γ ;
- (v) that the 72 c -points (real or imaginary) of Γ can be distributed into 24 distinct triplets, such that the 3 points of each triplet make up an « H-triangle »;
- and (vi) that the equation of the cubic Γ , expressed in terms of « homogeneous » or « projective » coordinates (x, y, z), referred to any of the H -triangles (say, ABC), must belong to one or other of the following forms ¹¹):

$$(39) \quad px^2y + qy^2z + rz^2x + sxyz = 0, \quad \text{or}$$

$$(40) \quad p'xy^2 + q'yz^2 + r'zx^2 + s'xyz = 0,$$

according as the tangentials of (A, B, C) are either (C, B, A) or (B, C, A) .

The presence of 3 ($= 9 - 6$) effective parameters, *viz.*, the ratios $(p : q : r : s)$ in (39), or the ratios $(p' : q' : r' : s')$ in (40), is a foregone conclusion, considering that the pre-assignment of an H -triangle amounts to 6 conditions and that the typical equation of a bicursal cubic involves 9 constants.

The lemma (vi) can evidently be enunciated in the following alternative form:

With an assigned triangle ABC as an H -triangle, there can be constructed two distinct « three-parameter » families of cubics, whose homogeneous equations, referred to ABC as the

¹⁰) A c -point is alternatively definable as a point (on Γ), which coincides with its own *third* tangential. [Vide HALPHEN, *Journal de Mathématiques*, 3^e series, t. II (p. 376)].

¹¹) Vide HILTON, *loc. cit.*, Ex 2 (p. 236), and Ex 15 (p. 260).

« standard triangle », are respectively of the forms (39) and (40), it being implied that the two triads of ratios $(p:q:r:s)$ and $(p':q':r':s')$ are independent parameters.

Appealing to geometrical intuition, one can immediately substantiate the following statements, regarding the conversion of one bicursal cubic Γ into another such cubic Γ' by means of an unrestricted (plane) collineation:

- (a) that the « Halphen »-point of any point on Γ is converted into the « Halpen-point » of the corresponding point on Γ' ,
- (b) that a c -point on Γ is converted into a c -point on Γ' ,
- and (c) that an « H -triangle » of Γ is converted into an « H -triangle » of Γ' .

As regards (c), it is manifest that the two corresponding H -triangles are, in general, *distinct*. Let us now propose to examine if it is possible to find a collineation Ξ , which shall transform a given (bicursal) cubic Γ into another (bicursal) cubic Γ' in such a way that a certain specified H -triangle (say, ABC) of Γ shall be an H -triangle of Γ' also. For this to be possible, the essential condition to be fulfilled is that the three correlated c -points of Γ , *viz.*, the vertices of the ΔABC , shall also be (correlated) c -points of Γ' . This does not necessarily mean that the points A, B, C shall remain invariant. Rather it is enough that these three points shall be permuted among themselves by Ξ . The logical conclusion is, then, that Ξ must belong to the collineation-group G , (considered in § 6), and must as such be a member of one or other of the six possible subsets G_1 - G_6 , mentioned in that article.

If it be now desired to investigate whether there can exist a collineation, which shall leave a specified H -triangle (ABC) of Γ *invariant* and shall at the same time transform Γ into itself, it is necessary to attend to the following basic lemmas (or riders):

- (d) that an arbitrary collineation, belonging to any of the subsets (G_4, G_5, G_6) converts any cubic of either of the two families (39), (40) into another cubic of the *self-same* family;

and (e) that an arbitrary collineation, belonging to any of the three remaining subsets (G_1, G_2, G_3) converts a cubic of the family (39) into one of the family (40) and *vice versa*.

Suppose now (for the sake of precision) that the original cubic Γ , referred as before to an H -triangle (ABC), is given initially in the form (39), where, of course, p, q, r are *known* constants. Then in view of the lemmas (d) and (e), it is crystal-clear that any *automorphic* collineation Ξ , which leaves the H -triangle (ABC) *invariant*, must, *if at all*, belong to one of the three subsets G_4, G_5, G_6 . That is to say, a *necessary* — but by no means sufficient — condition to be fulfilled by Ξ is that it shall be a member of one of the three subsets G_4, G_5, G_6 .

In order to discuss *sufficient* conditions, we have therefore to examine the three cases separately.

CASE I. - *Firstly*, supposing Ξ to belong to the subset G_4 , whose defining equations are (§ 6):

$$\rho x' = ax \quad , \quad \rho y' = by \quad , \quad \rho z' = z \quad ,$$

we can easily see that, if the equation (39) is to remain *intact*, the conditions are

$$a = b = 1 \quad ,$$

showing that Ξ is none other than the identity.

CASE II. - *Secondly*, supposing that Ξ belongs to the subset G_5 , whose defining equations are (§ 6)

$$\rho x' = az \quad , \quad \rho y' = bx \quad , \quad \rho z' = y \quad ,$$

we observe that, if (39) is to remain *intact*, the conditions are

$$a = \frac{r}{p} \quad \text{and} \quad b = \frac{r}{q} \quad .$$

So the only automorphic collineation of the type G_5 is that given by

$$(i) \quad \sigma x' = qrz \quad , \quad \sigma y' = rpx \quad , \quad \sigma z' = pqy \quad ,$$

where σ is a factor of proportionality and p, q, r are the (known) numerical coefficients in the equation (39) of the given cubic Γ .

CASE III. - *Thirdly*, supposing Ξ to belong to the subset G_6 , whose defining equations are (§ 6)

$$\rho x' = ay \quad , \quad \rho y' = bz \quad , \quad \rho z' = x \quad ,$$

we remark that, if (39) is to remain *unaltered*, the conditions are

$$a = \frac{q}{p} \quad \text{and} \quad b = \frac{r}{p} .$$

Hence the only automorphic collineation of the type G_6 is that given by

$$(ii) \quad \sigma x' = qy \quad , \quad \sigma y' = rz \quad , \quad \sigma z' = px \quad ,$$

where σ is a factor of proportionality and p, q, r are the *known* coefficients of (39).

Summarising the different cases, we can assert:

(A) *that, when a bicursal cubic Γ is represented analytically in the « homogeneous » form (39) with reference to one of its H-triangles (say, ABC), chosen as the « standard triangle », there exist two and only two automorphic collineations (other than the identity), which leave not only Γ but also the (H-triangle) ABC absolutely unaffected;*

and (B) *that the analytical representations of the two aforesaid collineations are no other than (i) and (ii).*

From considerations of symmetry the reader can easily write down the corresponding result, when the original bicursal cubic Γ is given in the form (40) instead of in the form (39).

We shall now give a finishing touch to the present investigation by talking a little about the effect of the « quadratic » transformation (37) upon a bicursal cubic, given in the form (39) or (40). Straightforward substitution clearly shows that any substitution of the type (37) converts a bicursal cubic of the category (39) into a (bicursal) cubic of the category (40) and *vice-versa*. An immediate inference is that *no « quadratic » transformation of the type (37), attaching to a given triangle ABC, can be automorphic in relation to a bicursal cubic, having ABC for an H-triangle.*