RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

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Rendiconti del Seminario Matematico della Università di Padova, tome 20 (1951), p. 365-380
<http://www.numdam.org/item?id=RSMUP_1951__20__365_0>

NOTE ON A CIRCULAR CUBIC HAVING ONE OR MORE SEXTACTIC POINTS AT INFINITY

Nota I (*) *di* HARIDAS BAGCHI *e di* BISWARUP MUKHERJI (*a Calcutta*).

Introduction. - The present investigation relates *primarily* to the special category of circular cubics, possessing sextactic points at infinity and *secondarily* to other types of cubics. As a matter of convenience, the entire paper has been divided into four sections. Sec I deals in the first instance with the *separate* conditions for a circular cubic Γ to have its real point (K) at infinity and its double focus (O) to be sextactic, then follows the connecting link between the two conditions. Then Sec II outlines an *alternative* method of ascertaining the conditions for K to be a sextactic point, which has the advantage of finding the actual equation of the osculating conic at K, (if and when sextactic). Thirdly, Sec III deals (in the manner of Sec II) with the conditional determination of the osculating conics at the two circular points I, J, *when known to be sextactic*: the latter portion of this section classifies Γ into two categories, according as the three points K, I, J (supposed sextactic) are «cognate» or «non-cognate» [1]. Special prominence is given to the first category of cubics, for which K, I, J are «cognate» sextactic points; it is shewn further that this type of cubic is also definable as a «central» circular cubic, endowed, as a matter of course, with a characteristic property expressible in terms

(*) Pervenuta in Redazione il 2 settembre 1950.

(1) Two sextactic points (of a cubic) have here been termed «cognate» when they possess a *common* tangential.

of the triad of «cognate» sextactic conics at infinity. Lastly, Sec IV deals with the projective generalisation of the property (just mentioned) to the *most general* type of bicursal cubic (circular or otherwise).

Although the subject-matter is classical in origin, and there are casual references to *known* results, still the paper is believed to embody a decent amount of original matter.

Section I.

1. - Condition for the real point K at infinity on a circular cubic Γ to be sextactic.

2. - Condition for the double focus O to be a sextactic point.

1. - With the double focus taken as the origin and the real asymptote taken parallel to the axis of y, the equation of a circular cubic Γ assumes the form:

$$(x - \lambda)(x^2 + y^2) + ax + by + c = 0 \,, \tag{1}$$

where $ax + by + c = 0$ represents the satellite to the line at infinity.

Let K be the real point at infinity on Γ and K' its tangential. Then K is sextactic, if and only if K' is an inflexion. But K' is none other than the intersection of $x - \lambda = 0$ and $ax + by + c = 0$ and has, therefore, for its co-ordinates:

$$\left(\lambda, \; -\frac{a\lambda + c}{b}\right).$$

An *arbitrary* transversal through K', which may be taken in the form:

$$ax + by + c = \mu\,(x - \lambda) \,, \tag{2}$$

(where μ is a *disposable* parameter) will cut (1) at two other points, whose co ordinates will satisfy the equation :

$$(3) \qquad x^2 + y^2 + \mu = 0 .$$

and, therefore, also the y — eliminant of (2) and (3), *viz*

$$(4) \quad x^2 \{ b^2 + (\mu - a)^2 \} - 2x(\mu - a)(\mu\lambda + c) + b^2\mu + (\mu\lambda + c)^2 = 0 .$$

Consequently K' is an inflexion, provided that there exists a 'μ', for which (4) is identical with :

$$(5) \qquad (x - \lambda)^2 = 0 .$$

Identification of (4) and (5) gives rise to two equations, whose μ — eliminant, *viz*

$$(6) \quad b^2 \{ (a^2 + b^2)\lambda + ac \} + (\lambda a + c) \{ (\lambda a + c)^2 + b^2\lambda^2 \} = 0$$

must then be the necessary and sufficient condition for the cubic Γ to have K for a sextactic point.

2. - The (circular) cubic Γ being supposed to be given by the equation (1) of Art. 1, we shall now investigate the condition for the double focus (*i. e.*, the origin O) to be a sextactic point on Γ.

In the first place we observe that Γ will have O for an «*ordinary*» point, if and only if

$$(1) \qquad c = 0$$

and that, subject to this condition, the equation to Γ reduces to

$$(2) \qquad (x - \lambda)(x^2 + y^2) + ax + by = 0 .$$

A glance at (2) shows that the tangential O' of O, which is, by the way, the intersection of (2) with the tangent at O *viz*.

$$(3) \qquad ax + by = 0$$

is designable also as the intersection of (3) and

$$x - \lambda = 0 .$$

Noticing that O will be sextactic when and only when O' is an inflexion and employing a mode of reasoning, similar to that of Art 1, we find without much difficulty that the point O will be sextactic provided that, over and above the condition (1), the special relation:

$$(4) \qquad \lambda^2 a + b^2 = 0$$

is satisfied.

Recognising that the condition (6) of Art. 1 is automatically satisfied as soon as the two conditions (1) and (4) of the present article are assumed to hold, we are squarely led to the following theorem:

THEOREM A. - *If a circular cubic (bicursal or unicursal) has its double focus for a sextactic point, it must have its real point at infinity for a sextactic point.*

For obvious reasons, the *direct* converse of the above theorem is **not** generally true. Appeal to á priori reasoning, however, shews that a *modified* form of converse theorem is valid. In point of fact, if K be a sextactic point and O lies on the cubic, it is easy to see that O also will be of the sextactic type. Hence the following.

THEOREM B. - *If the double focus of any circular cubic lies on its periphery and the real point at infinity is a sextactic point, then the double focus must also be of the sextactic type.*

3. - In this article we propose to verify Theorems A and B by the Principle of Residuation.

I, J, K being, as before the two circular points and the real point at infinity, and the double focus being supposed to lie on the curve, we have the following equations of residuation:

$$(1), (2), (3) \quad [I + J + K] = 0, \; [2I + O] = 0, \; [2J + O] = 0.$$

Manifestly (1), (2), (3), when associated with the additional relation $[6\,O] = 0$ lead to $[6\,K] = 0$, thus confirming Theorem A.

If, on the other hand, (1), (2) and (3) be associated with $[6\,K] = 0$, then $[6\,O] = 0$ follows automatically, thereby confirming Theorem B.

Section II.

1. - Alternative investigation of the condition for K to be a sextactic point on Γ.

2. - Determination of the osculating conic at K, if and when sextactic.

4. We shall now outline an *independent* method, which determines not only the criterion for K to be a sextactic point but also the actual equation of the associated sextactic conic (when existent).

Keeping to the notations and conventions of the foregoing Section, we start with the equation of a circular cubic Γ in the following form:

$$(1) \qquad (x - \lambda)(x^2 + y^2) + a\,x + b\,y + c = 0.$$

Let us put:

$$(2) \qquad U = x - \lambda, \qquad V = a\,x + b\,y + c - n\,(x - \lambda),$$

where n is an *undefined* parameter.

Solving (2) for x and y, we get:

(3) $$x = U + \lambda, \qquad y = pU + qV + r,$$

where $p = \dfrac{n - a}{b}$, $\quad q = \dfrac{1}{b}$, $\quad r = -\dfrac{a\lambda + c}{b}$.

Substitution of (3) in (1) reduces it to the form;

$$(1 + p^2)\,U^3 + 2\,(pr + \lambda)\,U^2 + (\lambda^2 + r^2 + n)\,U +$$
$$+ (2\,pq\,U^2 + q^2\,UV + 2\,qr\,U + 1)\,V = 0,$$

which can be reduced to the form:

$$(1 + p^2)\,U^3 + (2\,pq\,U^2 + q^2\,UV + 2\,qr\,U + 1)\,V = 0,$$

if and only if it is feasible to choose the *single* disposable parameter n consistently with the pair of conditions:

(5) $$\begin{cases} pr + \lambda = 0, \\ \lambda^2 + r^2 + n = 0. \end{cases}$$

Evidently the whole thing hinges on the evanescence of the n — eliminant, *i.e.*, on the condition:

(6) $$b^2\{(a^2 + b^2)\lambda + ca\} + (\lambda a + c)\{(\lambda a + c)^2 + \lambda^2 b^2\} = 0.$$

That is to say, when (6) holds, Γ can be accommodated to the symbolic form:

(7) $$k\,U^3 + SV = 0,$$

where the constant k and the conic S are defined by:

(8) $$k = \frac{b^2}{(\lambda a + c)^2}\{(\lambda a + c)^2 + \lambda^2 b^2\},$$

and

$$(9) \qquad S \equiv \frac{2\lambda b^2}{\lambda a + c} U^2 + UV - 2(\lambda a + c) U + b^2 = 0 .$$

Now since the right line U touches the conic S at a *real* point at infinity (say, K) we conclude, on the strength of the lemma (2) quoted below, that the equation (7) represents a cubic, having a six-pointic contact wih S at K, provided that the relation (6) holds. To be more explicit, *when the condition* (6) *is fulfilled, the circular cubic* Γ, *defined by* (1), *will have its real point at infinity, viz.* K *for a sextactic point, the osculating (i.e., six-pointic) conic thereat being the conic* S, *defined by* (9).

Section III.

1. - Conditions for the circular points (at infinity) I, J to be sextactic points on Γ.

2. - Equations to the associated sextactic conics (when existent).

5. A circular cubic Γ being, as before, taken in the Cartesian form:

$$(1) \qquad (x - \lambda)(x^2 + y^2) + ax + by + c = 0 ,$$

we now propose to find the condition that the circular point I (at which the tangent is $x + iy = 0$) may be a sextactic point. If we put:

(2) Apply the lemma that if a right line ξ, touches a conic T at a point P, the cubic

$$T - \mu \xi^3 \qquad \text{(where } \mu \text{ is a constant)}$$

must have P for a sextactic point, the osculating (*i.e.*, six-pointic) conic whereat is T.

$$(2) \qquad \begin{cases} U_1 \equiv x + iy\,, \\ V_1 \equiv ax + by + c - n_1(x + iy)\,, \end{cases}$$

(where n_1 is an *undetermined* parameter), we have, on solving (2) for x, y,

$$x = \frac{(n_1 + ib)\, U_1 + V_1 - c}{a + ib} \quad \text{and} \quad y = \frac{(a - n_1)\, U_1 - V_1 + c}{i\,(a + ib)}\,.$$

The equation (1), when expressed in terms of U_1, V_1, evidently takes the form:

$$\begin{aligned} & U_1^3 (n_1 + ib)(2 n_1 - a + ib) + U_1^2 [(n_1 + ib)(-2c) - \\ & \qquad - (2 n_1 - a + ib) \{c + \lambda (a + ib)\}] \\ & + U_1 \cdot [2c \{c + \lambda (a + ib)\} + n_1 (a + ib)^2] \\ & + V_1 [U_1^2 (4 n_1 - a + 3ib) + 2 U_1 V_1 \quad 2 U_1 \{2c + \\ (3) \qquad & \qquad + \lambda (a + ib)\} + (a + ib)^2] = 0\,. \end{aligned}$$

This will admit of the *symbolic* form:

$$(4) \qquad k_1 U_1^3 + S_1 V_1 = 0\,,$$

(where k_1 is a constant and S_1 a conic), when and only when it is possible to adjust the single parameter n_1 (so long undetermined) in such a way that the coefficients of both U_1^2 and U_1 in (3) may vanish simultaneously.

For this to be possible, the necessary and sufficient condition is to be derived by equating to zero the n_1 - eliminant of the two equations:

$$\begin{cases} 2c\,(n_1 + ib) + (2n_1 - a + ib)\,\{ c + \lambda\,(a + ib) \} = 0\,, \\ 2c\,\{ c + \lambda\,(a + ib) \} + n_1\,(a + ib)^2 = 0\,, \end{cases}$$

and is accordingly expressible in the form:

$$\text{(5)} \qquad \begin{aligned} &(a + ib)\,(a^2 + b^2)\,\{ c + \lambda\,(a + ib) \} + 4c\,\{ 2c^2 + \\ &+ 3c\lambda\,(a + ib) + \lambda^2\,(a + ib)^2 \} - 2ibc\,(a + ib)^2 = 0\,. \end{aligned}$$

If, then, we interpret the equation (4) in the light of the lemma, quoted in the foot note to Art 4, we reach the following conclusions:

(i) that (5) represents the condition for Γ to have the circular point I as a sextactic point,

and (ii) that, subject to the condition (5), the sextactic conic (S_1) at I is given by:

$$\text{(6)} \qquad \begin{aligned} S_1 \equiv\; & U_1^2\,(4n_1 - a + 3ib) + 2\,U_1\,V_1 \\ & - 2\,U_1\,\{ 2c + \lambda\,(a + ib) \} + (a + ib)^2 = 0\,, \end{aligned}$$

where

$$n_1 = -\frac{2c\,\{ c + \lambda\,(a + ib) \}}{(a + ib)^2}\,.$$

6. - Judging from considerations of symmetry, we readily realise that the necessary and sufficient condition for Γ to have the other circular point at infinity (J) for a sextactic point is deducible from (4) of Art 5 by changing i into $-i$ and is accordingly expressible as:

$$\text{(1)} \qquad \begin{aligned} &(a - ib)\,(a^2 + b^2)\,\{ c + \lambda\,(a - ib) \} + \\ &+ 4c\,\{ 2c^2 + 3c\lambda\,(a - ib) + \lambda^2\,(a - ib)^2 \} \\ &+ 2ibc\,(a - ib)^2 = 0\,. \end{aligned}$$

This condition being complied with, the sextactic conic at J can be presented in the form:

$$(2) \qquad S_2 \equiv U_2^2 (4 n_2 - a - 3 ib) + 2 U_2 V_2 - 2 U_2 \{ 2c + \lambda (a - ib) \} + (a - ib)^2 = 0 ,$$

where

and

$$\begin{cases} U_2 \equiv x - i y , \\ V_2 \equiv (a x + b y + c) - n_2 (x - i y) , \\ n_2 = - \dfrac{2 c \{ c + \lambda (a - ib) \}}{(a - ib)^2} . \end{cases}$$

7. - Amalgamating the results of Arts 4-6, we can readily obtain, in the following form, the typical equation of a circular cubic Γ, for which the three points at infinity (K, I, J) are sextactic:

$$(1) \qquad (x - \lambda)(x^2 + y^2) + a x + b y + c = 0 ,$$

it being tacitly understood that the coefficients λ, a, b, c satisfy the three relations (6) of Art 4, (5) of Art 5 and (1) of Art 6.

Furthermore, subject to the afore-said conditions, the sextactic conics (S, S_1, S_2) at K, I, J are respectively:

$$\begin{cases} S \equiv \dfrac{2 \lambda b^2}{\lambda a + c} \cdot U^2 + U V - 2 (\lambda a + c) U + b^2 = 0 , \\ S_1 \equiv U_1^2 (4 n_1 - a + 3 ib) + 2 U_1 V_1 - 2 U_1 \{ 2c + \lambda (a + ib) \} + (a + ib)^2 = 0 , \end{cases}$$

$$S_2 \equiv U_2^2 (4 n_2 - a - 3 ib) + 2 U_2 V_2 - 2 U_2 \{ 2c + \lambda (a - ib) \} + (a - ib)^2 = 0 ,$$

where

$$U \equiv x - \lambda, \quad V \equiv (ax + by + c) - n\,(x - \lambda),$$

$$U_1 \equiv x + iy, \quad V_1 \equiv (ax + by + c) - n_1\,(x + iy),$$

$$U_2 \equiv x - iy, \quad V_2 \equiv (ax + by + c) - n_2\,(x - iy),$$

and $n = \dfrac{\lambda(a^2 + b^2) + ac}{a\lambda + c}, \quad n_1 = -\dfrac{2c\{c + \lambda(a + ib)\}}{(a + ib)^2},$

$$n_2 = -\frac{2c\{c + \lambda(a - ib)\}}{(a - ib)^2}.$$

Now it is a tame affair to verify that the centres of aberrancy (³) (say, C, A, B), answering to the points K, I, J, are respectively given by:

$$\left(\lambda, \frac{\lambda a + c}{b}\right), \quad \left\{\frac{c + \lambda(a + ib)}{a + ib}, -\frac{c + \lambda(a + ib)}{i(a + ib)}\right\},$$

and $$\left\{\frac{c + \lambda(a - ib)}{a - ib}, \frac{c + \lambda(a - ib)}{i(a - ib)}\right\}.$$

Before we proceed further, it is easy to see that if two sextactic points P, Q of a (bicursal) cubic be cognate (*i. e.*, co-tangential) the third sextactic point R, collinear with them must also be «cognate» with them. If, on the other hand, P, Q be «non-cognate» (sextactic) points, R also is «non-cognate»

(³) As is well-known, the centre β of the osculating conic of any given analytic curve (Ω) at a point α is often designated as the *centre of aberrancy* of Ω (at α) and the line $\alpha\beta$ as the associated *axis of aberrancy*. When α moves on Ω, β will trace out a curve (say, Ξ), called the *curve of aberrancy*, which is often definable as the envelope of the *variable* axis of aberrancy. The line $\alpha\beta$ touches Ξ at β, which becomes a cusp (on Ξ), as soon as α becomes a sextactic point on Ω.

with them. This basic principle being borne in mind, (bicursal) circular cubics (endowed with three sextactic points at infinity, *viz* K, I, J) may, broadly speaking, be divided into two distinct species, according as the three sextactic points are « cognate » or « non-cognate ».

These two categories of circular cubics will now be taken up one by one.

Species I. - When K, I, J are « non-cognate », the cubic is given by equation (1) of Art 7, it being implied that the coefficients λ, a, b, c are subject to three relations, *viz.* (6) of Art 4, (5) of Art 5 and (1) of Art 6. Plain reasoning shows that the three centres of aberrancy C, A and B are all distinct and are further collinear [1], the actual line of collinearity being:

$$(3) \quad x \{ \lambda (a^2 + b^2) + ca \} + bc \cdot y - \{ (\lambda a + c)^2 + \lambda^2 b^2 \} = 0 .$$

It may be noted that this line is « conjugate » to the line at infinity with respect to each of the sextactic conic S, S_1 and S_2 at K, I and J.

Furthermore, the axes of aberrancy at I and J (which are none other than the lines: $x + iy = 0$ and $x - iy - 0$) intersect each other at the double focus. Interested readers may pursue this investigation still further.

Species II. - When, however, K, I, J are « cognate » the line at infinity is a harmonic polar of Γ. Consequently, the common tangential, (which is an inflexion), is the double focus and at the same time $\lambda = c = 0$. Accordingly the cubic Γ becomes *central*, its centre being no other than the double focus

[1] This result is, in a sense, a confirmation of G. D. Bhar's observation that the three centres of aberrancy — called limiting centres by him — answering to the three points at infinity of a plane cubic-lie on a right line. (Vide *Bull. Cal. Math. Soc.*, Vols. XII - XIII). What is remarkable is that this lemma, proved by Bhar only for cubics with three real points at infinity, holds even when one of the points is real and the other two are imaginary circular points at infinity.

(0, 0). Moreover the equation of this central circular cubic, referred to the centre as origin, assumes the simple form:

(4) $$x(x^2 + y^2) + ax + by = 0.$$

The three centres of aberrancy being now obviously coincident with the double focus (0, 0) by virtue of (2) of Art 7, we arrive at the following theorem:

THEOREM C. – *If a* (bicursal) *circular cubic has the three points at infinity for «cognate» sextactic points, it must reduce to a central circular cubic having the double focus for its centre* ([5]). Furthermore, *the three sextactic conics at infinity must be concentric, having the centre of the cubic for its common centre.*

The converse also is true. For, from the equation (4) of a central circular cubic it is manifest that the three points at infinity K, I, J are co-tangential, having the double focus (0, 0) for the common tangential; and since the double focus is an inflexion, K, I, J are «cognate» sextactic points

We can now designate a «*central circular cubic*» as a (circular) cubic, possessing three «cognate» sextactic points at infinity. In the succeeding articles we shall be exclusively interested in this variety of circular cubics.

8. – At the very outset we shall state an elementary lemma of Projective Geometry, *viz.* that the n (≥ 3) pairs of straight lines of the type

$$a_r x^2 + 2h_r xy + b_r y^2 = 0 \qquad (r = 1, 2, 3, .., n)$$

([5]) Apropos of Theorem C, one may refer to CHASLES's general proposition, which being paraphrased, reads thus:

The necessary and sufficient condition for a (bicursal) cubic to be central is that the line at infinity be a harmonic polar [Vide S. M. GANGULY's «*Theory of Plane Curves*», Vol. II. p. 4].

are in involution, if and only if, the *rank* of the rectangular matrix

$$\begin{pmatrix} a_1, & a_2, \ldots, & a_n \\ h_1, & h_2, \ldots, & h_n \\ b_1, & b_2, \ldots, & b_n \end{pmatrix}$$

be equal to 2.

As an immediate application, let us take the general type of central circular cubic in the Cartesian form:

(1) $$x(x^2 + y^2) + a x + b y = 0,$$

it being understood that the centre (which is an inflexion) has been chosen as the origin and the real asymptote as the y- axis.

The equations to the three (concentric) sextactic conics at infinity, deducible from (5) of Art 4, (6) of Art 5 and (2) of Art 6 (by setting $\lambda = c = 0$) now assume the simple forms:

(2) $$a x^2 - b x y = b^2,$$

(3) $$x^2 (a + 3 ib) - 4 b x y + y^2 (a - ib) = -(a + ib)^2,$$

(4) $$x^2 (a - 3 ib) - 4 b x y + y^2 (a + ib) = -(a - ib)^2.$$

In view of the obvious identity:

$$\begin{vmatrix} a, & -\frac{b}{2}, & 0 \\ a + 3 ib, & -2b, & a - ib \\ a - 3 ib, & -2b, & a + ib \end{vmatrix} = 0,$$

the above mentioned lemma enables us to conclude that the three pairs of asymptotes of the three sextactic conics (2), (3), (4) are in involution. Consequently the three pairs of points of intersection of the above sextactic conics with the line at infinity are also in involution. Manifestly the foci (I', J') of this point-involution are the points, where the focal lines of the line-involution formed by the asymptotes meet the line at infinity.

For obvious reasons, the $\Delta O I' J'$ is self-conjugate with respect to the three (concentric) sextactic conics at K, I, J. We are, therefore, led to the following theorem:

THEOREM D. - *The points, where the line at infinity intersects the three (concentric) sextactic conics at infinity of a central circular cubic are in involution. Furthermore, these three (sextactic) conics at infinity possess a common self-conjugate triangle, one of whose vertices coincides with the double focus (i. e. the centre) of the cubic, the opposite side being located along the line at infinity.*

SECTION IV.

1. - Generalisation of Theorem *D* by projection.

2. - Triad of sextactic conics for any bicursal cubic (circular or otherwise).

10. - The above proposition on central circular cubics admits of easy generalisation through projective transformation, leading ultimately to interesting results about the *unrestricted* type of bicursal cubics. Suppose that Π is an unrestricted bicursal cubic, of which one of the (nine) inflexions is A and the associated harmonic polar is L. The three points, α, β, γ, where L cuts Π must be «cognate» sextactic points, having A for their common tangential. Let U, V and W be the sextactic conics of Π at α, β, γ respectively.

If we now project two of the sextactic points (say, β, γ) into the circular points at infinity (β', γ'), it is clear on all hands that Π will project into a central circular cubic Π', whose centre (*i. e.*, double-focus) will be the projection A' of A and whose real point at infinity - which is none other than a sextactic * point – will be the projection of α. Further the osculating conics U', V', W', answering to the three points (at infinity) α', β', γ' of Π' will be simply the projection of the sextactic conics U, V, W, belonging to Π.

Appealing to Theorem D on central circular cubics, we promptly perceive that the three sextactic conics U', V', W' of Π' possess a common self-conjugate triangle, one of whose vertices is at A', the opposite side being located along the line at infinity $(\beta' \gamma')$ and that the intersections of the line at infinity with U', V', W' are in involution. Now since polarity and involution are *projective* properties, we may safely conclude that the three sextactic conics, belonging to Π, possess a common self-conjugate triangle, one of whose vertices is located at A, the opposite side coinciding in position with the associated harmonic polar L and that L meets U, V, W in points of involution. It is needless to point out that though U', V', W' (of Π') are concentric, having A' for their common centre, U, V, W, (of Π) do not necessarily possess a common centre.

Remarking that the projective contrivance described in the previous paragraph is applicable to the triad of «cognate» sextactic conics, attaching to *any one* of the nine points of inflexion with an associated harmonic polar, we can finalise our results in the undermentioned form:

THEOREM E. – *Each of the nine triads of «cognate» sextactic conics of a bicursal cubic possesses a common self-conjugate triangle, which has the associated inflexion for one of its vertices, the opposite side being located along the corresponding harmonic polar. Furthermore, every such triad of sextactic conics intersects the attached harmonic polar in points in involution.*

* Manifestly, the inflexional or sextactic character of a point remains invariant during projective transformation.