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ON THE CENTRAL PATH FOR NONLINEAR SEMIDEFINITE PROGRAMMING (*)

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Abstract. – *In this paper we study the welldefinedness of the central path associated to a given nonlinear (convex) semidefinite programming problem. Under standard assumptions, we establish that the existence of the central path is equivalent to the nonemptiness and boundedness of the optimal set. Other equivalent conditions are given, such as the existence of a strictly dual feasible point or the existence of a single central point. The monotonic behavior of the logarithmic barrier and the objective function along the trajectory is also discussed. Finally, the existence and optimality of cluster points are established.*

Keywords: Nonlinear semidefinite programming, semidefinite programming, central path, logarithmic barrier function, convex programming.

1. INTRODUCTION

The notion of central path can be traced up to the 60's with the works of Parissot [21] and Fiacco and McCormick [3]. Parissot studied the barrier function method in connection with linear programming, while Fiacco and McCormick did it in the nonlinear (constrained) case; they developed the S.U.M.T. (Sequential Unconstrained Minimization Technique). Roughly speaking, the method consists of finding the exact minimizers of auxiliary functions for decreasing values of the barrier parameter. These auxiliary functions are defined in terms of the problem functions in such a way that

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they have a singularity at each boundary point of the feasible set, forcing their minima to remain strictly feasible. The central path associated to the problem is the set of those minimizers together with their Karush-Kuhn-Tucker multipliers, that is to say that the trajectory lies in the primal-dual space (actually, in the relative interior of the primal-dual feasible set).

With the fast computer developments in the late seventies, researchers became interested in efficient implementations of interior point methods and particularly of the so-called path following methods, *i.e.* methods which trace the central path up to an optimal solution of the problem. An adequate notion of complexity for linear programming algorithms was given by Khachiyan in 1979, when he proposed the first polynomial algorithm for linear programming problems (see [11]). In 1984 Karmarkar in his seminal paper [10] proposed the first competitive polynomial algorithm for linear programming, with lower complexity than Khachiyan's. His method performs in the relative interior of the feasible set far away from the boundary. In 1986 Renegar [22] developed the first polynomial path-following algorithm (in a maximization framework) for linear programming. Ever since, this field has been extremely active and many polynomial algorithms were proposed. These facts renewed the interest on the theoretical issues concerning central paths.

The existence and convergence of central paths associated to special classes of convex problems has been widely studied in the last decade. In the context of linear programming Megiddo [14] provides conditions which guarantee welldefinedness of the central path, and Megiddo and Schub [15] characterize the end point of the primal curve as the analytic center of the primal optimal set. In the same context these issues were also considered by Bayer and Lagarias [2], Adler and Monteiro [16] and Sonnevend [26].

For linearly constrained convex problems satisfying the Slater condition, it was proved in [8] that the existence of the central path is equivalent to any of the following three conditions: nonemptiness and boundedness of the optimal set, existence of a single central point and existence of a strictly feasible (Wolfe) dual point; under additional hypotheses on the objective function it is also shown that the primal curve converges to the analytic center of the optimal set. For weighted central paths associated to convex problems, this fact was also proved by Monteiro and Zhou [18] for the case when the constraint and objective functions are analytic. Still in the context of convex programming, the same results on existence and convergence of the central path were also obtained in [6] under less restrictive assumptions. Similar results on the limiting behavior of weighted central paths related to monotone complementarity problems can be found in [12] and [17].

Central paths which arise from more general problems, like variational inequalities, were considered in [9]; in this paper, welldefinedness and convergence of paths derived from a large class of penalty barriers were studied. For linear programming – viewed as a particular case of variational inequality problem – these issues are studied in [5] for a primal-infeasible, dual-feasible path; the results on existence and convergence are similar to those of [8].

Welldefinedness of the central path associated to a semidefinite programming problem was already studied in [4] and [7]; roughly speaking, the minimal conditions under which the central path exists are the same required on the (nonlinear) convex programming case. In [13] it is also established the existence of the end point of the central path assotiated to a semidefinite linear complementarity programming problem, and in [4] this point is completely characterized in the semidefinite programming case.

In this work we are concerned with the central path associated to a primal-dual pair of particular nonlinear semidefinite programming problems: the convex linearly constrained ones, *i.e.* those in which the primal constraint is a matrix valued affine function. (For the general case, in which the constraint is a positive semidefinite convex function, see [24], [20] and [25].) For simplicity, we will keep on calling it the nonlinear semidefinite programming problem. We recall that a nonlinear semidefinite programming problem consists of the minimization of a nonlinear (convex) function of a symmetric matrix variable constrained to belong to the intersection of a linear manifold and the cone of positive-semidefinite matrices. Since we can always describe such matrices in terms of its coordinates in a maximal affinely independent system, a nonlinear semidefinite programming problem can, therefore, be regarded as the minimization of a nonlinear function of a vector variable subject to a linear matrix inequality (see Sect. 2). Nonlinear semidefinite programming extends semidefinite programming, which, in turn, extends linear programming (the nonnegative orthant is replaced by the cone of symmetric semidefinite matrices), so it is not surprising that many algorithms for linear programming admit extensions to semidefinite programming [19, 27], in particular the path following ones. Moreover, the central path for a linear programming problem can be viewed as the central path of a suitable semidefinite programming problem [27].

Our existence results are a direct generalization to the nonlinear semidefinite programming case of those that can be found in [13] and [7]. The outline of this paper is as follows. In Section 2 we define the primal-dual pair of nonlinear semidefinite programming problems, as well as its central path; we introduce some notation and the assumptions that will be used throughout the work; we also recall some well known results on convex analysis, and finally we state and prove our main theorem, which provides some necessary and sufficient conditions for the existence of the central path associated to a nonlinear semidefinite programming pair of problems. In Section 3 we study some properties of the central path and we establish the existence and optimality of its cluster points. In Section 4 we make some final remarks.

2. THE NONLINEAR SEMIDEFINITE PROGRAMMING PROBLEM AND THE CENTRAL PATH

Consider the problem of minimizing a nonlinear function of $x \in \Re^n$, where x is the vector of coordinates of the positive semidefinite matrices of a certain affine manifold in $\Re^{m \times m}$:

$$\text{(NP)} \qquad \begin{cases} \min & f(x) \\ \text{s.t.} & F(x) \geq 0, \end{cases}$$

where $F(x) := F_0 + \sum_{i=1}^{n} x_i F_i$, with $F_0, \ldots, F_m$ symmetric real matrices of order m and $f : \Re^n \to \Re \cup \{+\infty\}$ is a proper closed convex function differentiable in its effective domain, $ed(f)$, assumed to be open. The inequality $F(x) \geq 0$ stands for $z^\top F(x) z \geq 0$ for all $z \in \Re^m$.

Due to the symmetry of $F(x)$, we can replace the infinite set of linear constraints on x, $z^\top F(x) z \geq 0$ for all $z \in \Re^m$, by a finite one, namely the one consisting of the inequalities $\det F(x)_{JJ} \geq 0$, where $F(x)_{JJ}$ stands for the leading principal submatrix of $F(x)$ corresponding to $J \subset \{1, 2, \ldots, n\}$. The price we pay is that we may lose convexity, since $\det F(x)_{JJ}$ is a polynomial in x, which, in general, is not convex. Here we will keep the constraint $F(x) \geq 0$ in its convex form.

Since both the objective and the constraint of (NP) are convex functions, (NP) is a convex optimization problem. It is a well known fact that it includes as particular cases many important optimization problems, such as linear programming problems, quadratically constrained quadratic programming problems and the problem of minimizing the maximum singular value of a rectangular matrix, among others (see [27]).

Recall that the (Wolfe) dual problem associated with (NP) is (see [27])

$$
\text{(DNP)} \qquad \begin{cases} \max f(x) - F(x) \bullet Z \\ \text{s.t. } \dfrac{\partial f}{\partial x_i}(x) = F_i \bullet Z, \quad i = 1, \ldots, n, \\ \qquad Z \geq 0, \end{cases}
$$

where $F_i \bullet Z$ stands for the usual inner product of F_i and $Z \in S^m = \{X \in \Re^{m \times m} \mid X^\top = X\}$, *i.e.* $F_i \bullet Z := \mathrm{Tr}(F_i Z) = \sum_{j=1}^{m} (F_i Z)_{jj}$ (see [1]). We recall that the Frobenius norm of $A \in S^m$ is given by $||A||_F^2 = \mathrm{Tr}(A^2) = \sum_{i,j=1}^{m} A_{ij}^2$. Here the variable is the pair (x, Z), where $x \in \Re^n$ and Z is an $m \times m$ symetric matrix, which is constrained to satisfy n linear equalities and the nonnegativity condition.

We point out that we have a weak duality result, since for x and $(\bar{x}, \bar{Z})$ primal- and dual-feasible, respectively, it holds $f(x) - (f(\bar{x}) - F(\bar{x}) \bullet \bar{Z}) = F(x) \bullet \bar{Z} + f(x) - f(\bar{x}) - \nabla f(\bar{x})^\top (x - \bar{x}) \geq 0$, in view of the convexity of f and the fact that the trace of two positive semidefinite matrices is positive. So, whenever we have a zero gap, we have a primal-dual pair of optimal solutions.

Note also that when the primal objective function is linear, *i.e.* when we consider a semidefinite programming problem, we can put the dual problem in the same form as the primal, since the dual objective function is also linear; that is to say that the dual of a semidefinite programming problem is also a semidefinite programming problem. (For a primal-dual formulation much similar to a primal-dual pair of linear problems see [1].)

We can now define the central path. For this, we first need to introduce the *logarithmic barrier*:

$$
\Phi(x) := \begin{cases} \log\det F(x)^{-1}, & \text{if } \quad F(x) > 0, \\ +\infty, & \text{otherwise}, \end{cases}
$$

where $F(x) > 0$ means that $F(x)$ is a positive definite matrix. So Φ is a barrier function for the primal feasible set or, in other words, Φ tends to infinite as x approaches to the boundary of $\{x \mid F(x) \geq 0\}$. It can be seen that Φ is strictly convex if $F_1, \ldots, F_n$ are linearly independent (see [27]).

We will need the following assumptions.

Assumptions

A.1. The symmetric matrices $F_1, \ldots, F_n$ are linearly independent.

A.2. There exists a primal strictly feasible point $\tilde{x}$ in $ed(f)$, *i.e.* $F(\tilde{x}) > 0$ and $f(\tilde{x}) < +\infty$.

Observe that Assumption A.2 is essential since, without it, it makes no sense to consider the logarithmic barrier. As we already mentioned, Assumption A.1 guarantees that the barrier Φ is strictly convex.

The logarithmic barrier function method [3] applied to (NP) generates the family of unconstrained problems

$$(\mathrm{NP}_\mu) \qquad \begin{cases} \min f(x) - \mu \log\det F(x), \\ \text{s.t. } F(x) > 0, \end{cases}$$

where $\mu > 0$ is the barrier penalty parameter. Observe that the minimand in (NP$_\mu$) is a strictly convex function, and so the problem (NP$_\mu$) has no more than one (global) minimizer, which is characterized by

$$\begin{cases} 0 = \dfrac{\partial f}{\partial x_i}(x) - \mu\, F(x)^{-1} \bullet F_i, \quad i = 1, \ldots, n, \\ F(x) > 0, \end{cases}$$

where $F(X)^{-1} \bullet F_i = \dfrac{\partial \Phi}{\partial x_i}(x)$ (see [27]). Calling $Z = \mu\, F(x)^{-1}$, we can rewrite the above conditions as

$$\begin{cases} \dfrac{\partial f}{\partial x_i}(x) = F_i \bullet Z, \quad i = 1, \ldots, n, \\ ZF(x) = \mu I, \\ F(x) > 0, \\ Z > 0. \end{cases} \tag{1}$$

Therefore, if $(x(\mu), Z(\mu))$ satisfies (1), it is strictly dual-feasible and $x(\mu)$ is strictly primal-feasible. The *central path* associated to the pair of problems (NP) and (DNP) is the set

$$\{(x(\mu), Z(\mu)) \text{ solution of } (1), \text{ for } \mu > 0\}.$$

For each $\mu > 0$, welldefinedness of the corresponding central point depends on the existence and uniqueness of the solution of (1). In Theorem 1 we will give a set of necessary and sufficient conditions for the existence of the central path.

In order to study welldefinedness of the central path, we define the functions Φ_μ and $\tilde{f}$ by

$$\Phi_\mu(x) := \begin{cases} f(x) - \mu \log\det F(x), & \text{if} \quad F(x) > 0 \quad \text{and} \quad x \in ed(f), \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$\tilde{f}(x) := \begin{cases} f(x), & \text{if} \quad F(x) \geq 0 \quad \text{and} \quad x \in ed(f), \\ +\infty, & \text{otherwise.} \end{cases}$$

It is easy to see that, under our assumptions, Φ_μ and $\tilde{f}$ are both proper closed convex functions and, moreover, that Φ_μ is strictly convex in its effective domain. Note that $\nabla\Phi_\mu(x) = 0$ if and only if x solves (NP$_\mu$), *i.e.* if $x = x(\mu)$.

From now on, $\Gamma(\alpha, \mu)$ and Γ_α will stand for the level sets of Φ_μ and $\tilde{f}$ corresponding to $\alpha \in \mathfrak{R}$, respectively, *i.e.*

$$\Gamma(\alpha, \mu) := \{x \in \mathfrak{R}^n \mid \Phi_\mu(x) \leq \alpha\}$$

and

$$\Gamma_\alpha := \{x \in \mathfrak{R}^n \mid \tilde{f}(x) \leq \alpha\}.$$

Both $\Gamma(\alpha, \mu)$ and Γ_α are closed convex subsets of $\mathfrak{R}^n$ because Φ_μ and $\tilde{f}$ are closed convex functions.

We recall that the *recession cone of a convex set* $C \subset \mathfrak{R}^n$ is given by

$$O^+C := \{v \in \mathfrak{R}^n \mid C + tv \subset C, \text{ for all } t \geq 0\}.$$

We also recall some very well known results on convex analysis (see [23], Ths. 8.4 and 8.7) which we will use on the sequel. The first one states that a nonempty closed convex set C in $\mathfrak{R}^n$ is bounded if and only if its recession cone O^+C consists of the zero vector alone. The second one states that the nonempty level sets of a closed proper convex function are either all bounded or all unbounded. From this fact (using compactness arguments) it follows that a closed proper convex function has nonempty bounded level sets if and only if the set of its (unconstrained) minimizers is nonempty and bounded.

Now we will state and prove our main theorem which gives necessary and sufficient conditions for the welldefinedness of the central path associated to the pair of problems (NP) and (DNP).

THEOREM 1: Under Assumptions A.1 and A.2, the following conditions are equivalent:

1. the solution set of problem (NP), Sol(NP), is nonempty and bounded,
2. the central path $\{(x(\mu), Z(\mu)) \mid \mu > 0\}$ is well defined,
3. for some $\mu_0 > 0$ the central point $(x(\mu_0), Z(\mu_0))$ is well defined,
4. there exists a strictly dual-feasible point $(\bar{x}, \overline{Z}) \in \Re^n \times S^m$, *i.e.* such that $F_i \bullet \overline{Z} = \dfrac{\partial f}{\partial x_i}(\bar{x})$ for $i = 1, 2, \ldots, n$ and $\overline{Z} > 0$.

Proof: Suppose that Condition 1 holds. We will show that Condition 2 is true.

Take $\mu > 0$. Using $\tilde{x}$ as in Assumption A.2, define $\tilde{\alpha} := \Phi_\mu(\tilde{x})$. We claim that $O^+\Gamma(\tilde{\alpha}, \mu) = \{0\}$. Let $v \in \Re^n$ be an element of $O^+\Gamma(\tilde{\alpha}, \mu)$. For all $x \in \Gamma(\tilde{\alpha}, \mu)$ and $t \geq 0$ we have that

$$\tilde{\alpha} \geq \Phi_\mu(x + tv) = f(x + tv) - \mu \log \det F(x + tv). \tag{2}$$

Using the gradient inequality for f in (2), we obtain that for all $x \in \Gamma(\tilde{\alpha}, \mu)$ and $t \geq 0$,

$$\tilde{\alpha} \geq f(x) + t\nabla f(x)^\top v - \mu \log \det F(x + tv).$$

Therefore, since $t \mapsto \det F(x + tv)$ is a polynomial, we have

$$\nabla f(x)^\top v \leq 0, \quad \text{for all} \quad x \in \Gamma(\tilde{\alpha}, \mu). \tag{3}$$

Thus, since $\tilde{x} + tv \in \Gamma(\tilde{\alpha}, \mu)$ for all $t \geq 0$, we get from (3),

$$\nabla f(\tilde{x} + tv)^\top v \leq 0, \quad \text{for all} \quad t \geq 0.$$

Hence $f(\tilde{x} + tv)$ is a nonincreasing function of $t \geq 0$, so

$$f(\tilde{x} + tv) \leq f(\tilde{x}), \quad \text{for all} \quad t \geq 0. \tag{4}$$

From the feasibility of $\{\tilde{x} + tv \mid t \geq 0\}$ and (4) we conclude that

$$\{\tilde{x} + tv \mid t \geq 0\} \subseteq \Gamma_{\tilde{\beta}}, \quad \text{where} \quad \tilde{\beta} := f(\tilde{x}).$$

On the other hand, by Condition 1 (and our comments before this theorem), the nonempty set $\Gamma_{\tilde{\beta}}$ is bounded, therefore $v = 0$ and so

$$O^+\Gamma(\tilde{\alpha}, \mu) = \{0\}.$$

It follows that $\Gamma(\tilde{\alpha}, \mu)$ is a (nonempty) bounded set. We conclude that Φ_μ attains its minimum $x(\mu)$, which is unique due to the strict convexity of Φ_μ

in its effective domain. Whence $x(\mu)$ and $Z(\mu) = \mu\, F(x(\mu))^{-1}$ satisfy the optimality conditions for problem (NP$_\mu$), so Condition 2 holds.

Condition 3 is an obvious consequence of Condition 2.

If Condition 3 holds, then $(x(\mu_0), Z(\mu_0))$ satisfies system (1). So Condition 4 is true with $(\bar{x}, \overline{Z}) = (x(\mu_0), Z(\mu_0))$.

Now assume that Condition 4 holds and let us verify Condition 1. We know that, in view of the symmetry of $\overline{Z}$, there exists an orthogonal matrix Q such that $\overline{Z} = Q^\top \operatorname{diag}(\lambda_1(\overline{Z}), \ldots, \lambda_m(\overline{Z}))Q$, where $\lambda_i(\overline{Z})$ is the i-th eigenvalue of $\overline{Z}$. We also know that for three arbitrary square matrices A, B and C it holds that $\operatorname{Tr}(ABC) = \operatorname{Tr}(B^\top A^\top C^\top)$; so, for $x \in ed(f)$ such that $F(x) \geq 0$ we have that

$$\begin{aligned}
\overline{Z} \bullet F(x) &= \operatorname{Tr}(\overline{Z}F(x)) \\
&= \operatorname{Tr}(\operatorname{diag}(\lambda_1(\overline{Z}), \ldots, \lambda_m(\overline{Z}))QF(x)Q^\top) \\
&= \sum_{i=1}^{m} \lambda_i(\overline{Z})(QF(x)Q^\top)_{ii} \\
&\geq \lambda_{\min}(\overline{Z}) \sum_{i=1}^{m} (QF(x)Q^\top)_{ii} \\
&= \lambda_{\min}(\overline{Z})\operatorname{Tr}(QF(x)Q^\top) \\
&= \lambda_{\min}(\overline{Z})\operatorname{Tr}(F(x)),
\end{aligned}$$

where $\lambda_{\min}(\overline{Z})$ is the smallest eigenvalue of $\overline{Z}$ and the inequality holds in view of the fact that $F(x)$ is a positive semidefinite matrix. Using once again that $F(x)$ is symetric and positive semidefinite, we can write $\operatorname{Tr}(F(x)) = \operatorname{Tr}(F(x)^{1/2}F(x)^{1/2}) = ||F(x)^{1/2}||_F^2$, where $||\ ||_F$ stands for the Frobenius norm; so we can rewrite the above inequality as

$$\overline{Z} \bullet F(x) \geq \lambda_{\min}(\overline{Z})||F(x)^{1/2}||_F^2, \tag{5}$$

for all primal–feasible x. On the other hand, from the convexity of f, it follows that

$$\begin{aligned}
f(x) &\geq f(\bar{x}) + \nabla f(\bar{x})^\top (x - \bar{x}) \\
&= f(\bar{x}) + \sum_{i}^{m} (F_i \bullet \overline{Z})(x_i - \bar{x}_i) \\
&= f(\bar{x}) - F(\bar{x}) \bullet \overline{Z} + F(x) \bullet \bar{Z} \\
&= K + F(x) \bullet \bar{Z},
\end{aligned} \tag{6}$$

where the first equality is a consequence of the dual feasibility of $(\bar{x}, \overline{Z})$ and $K = f(\bar{x}) - F(\bar{x}) \bullet \bar{Z}$. So, combining (5) and (6), we get that

$$f(x) \geq K + \lambda_{\min}(\overline{Z}) ||F(x)^{1/2}||_F^2, \quad \text{for all feasible } x. \tag{7}$$

Hence, since $\lambda_{\min}(\overline{Z}) > 0$, it follows from (7) that $F^{1/2}$ is bounded on all the primal objective level sets Γ_α. Using now the fact that for the Euclidean operator norm it holds that $||F(x)|| \leq ||F(x)^{1/2}||^2$, we conclude that F is bounded on all the primal objective level sets Γ_α.

Now suppose that, for some α, Γ_α is a nonempty and unbounded level set for the primal objective function. Then there exists a sequence $\{x^k\} \subset \Gamma_\alpha$ such that $\lim\limits_{k\to\infty} ||x^k|| = \infty$. So $\lim\limits_{k\to\infty} ||F(x^k)|| \geq \lim\limits_{k\to\infty} ||\sum\limits_{i=1}^{m} x_i^k F_i|| - ||F_0|| = +\infty$, in view of the linear independence of $\{F_1, \ldots, F_m\}$ (Assumpt. A.1). Since, in particular $\{x^k\}$ is a primal-feasible sequence, our last inequality clearly contradicts what was established about the boundedness of F on primal objective level sets. Hence all level sets of the objective function are bounded and, in particular, its set of minimizers Sol(NP) is nonempty and bounded. □

3. BEHAVIOR OF THE CENTRAL PATH

In this section we discuss some properties of the central path. From now on, in addition to Assumptions A.1 and A.2, we will also suppose that Sol(NP) is a nonempty and bounded subset of $\Re^n$. It follows from Theorem 1 that the central path is well defined.

In the next proposition we establish the monotonic behavior of the logarithmic barrier and of the primal objective along the central path.

PROPOSITION 2: *Let Φ be the logarithmic barrier, i.e. $\Phi(x) = -\log\det F(x)$. If $0 < \mu_1 < \mu_2$, then*

$$\Phi(x(\mu_2)) \leq \Phi(x(\mu_1)),$$

and

$$f(x(\mu_1)) \leq f(x(\mu_2)).$$

Proof: Call $x^i := x(\mu_i)$, for $i = 1, 2$. Then

$$x^i = \operatorname*{argmin}_{F(x)>0} \; f(x) + \mu_i \Phi(x), \text{ for } i = 1, 2.$$

So

$$f(x^1) + \mu_1 \, \Phi(x^1) \leq f(x^2) + \mu_1 \, \Phi(x^2), \tag{8}$$

and

$$f(x^2) + \mu_2 \, \Phi(x^2) \leq f(x^1) + \mu_2 \, \Phi(x^1). \tag{9}$$

Adding up (8) and (9) we obtain

$$0 \leq (\mu_2 - \mu_1)(\Phi(x^1) - \Phi(x^2)).$$

Therefore, since $\mu_1 < \mu_2$,

$$\Phi(x^2) \leq \Phi(x^1). \tag{10}$$

Now combining (8) and (10) we see that

$$f(x^1) \leq f(x^2).$$

We will need the following lemma in order to establish the existence of accumulation points of the central path.

LEMMA 3: *Assume that E is a finite dimensional linear space, $\langle \cdot, \cdot \rangle$ is an inner product defined on E and that $\| \cdot \|$ is the corresponding norm. Assume also that K is a closed convex cone and that K^*, the polar cone, given by $K^* = \{y \in E \mid \langle y, x \rangle \geq 0$ for all $x \in K\}$, has nonempty topological interior. Then, for all $w \in$ int(K^*) there exists $\sigma > 0$ such that $\langle w, x \rangle \geq \sigma \|x\|$ for all $x \in K$.*

Proof: First of all, we will establish that $\text{int}(K^*) = \{w \in V : \langle w, v \rangle > 0 \ \forall \ 0 \neq v \in K\}$. If $w \in \text{int}(K^*)$ then $\langle w, v \rangle \geq 0$ for all $v \in K$ by definition of K^*, and we cannot have $\langle w, v \rangle = 0$ for some nonnull $v \in K$ because in such a case $w' = w - \alpha v$ belongs to K^* for $\alpha \in \Re_{++}$ small enough, and then

$$\langle w', v \rangle = \langle w, v \rangle - \alpha \|v\|^2 = -\alpha \|v\|^2 < 0,$$

contradicting the definition of K^*. If $\langle w, v \rangle > 0$ for all $0 \neq v \in K$, then $\langle w, \cdot \rangle$ attains its minimum on the compact intersection of K with the unit sphere of V, with a positive value, say β. Is is easy to check that $\langle u, v \rangle \geq 0$ for all $v \in K$ and all u in the ball with center at w and radius β. Thus, this ball is contained in K^* and therefore $w \in \text{int}(K^*)$. Now it suffices to take $\sigma = \min\{\langle w, x \rangle \mid x \in K, \|x\| = 1\}$. $\square$

Our next result shows that the set of central points is bounded when the parameter μ is bounded above.

PROPOSITION 4: *For all $\bar{\mu} > 0$ the set $\{(x(\mu), Z(\mu)) \mid 0 < \mu \leq \bar{\mu}\}$ is bounded.*

Proof: Take $\bar{\mu} > 0$. Define $\bar{\alpha} := f(x(\bar{\mu}))$. From Proposition 2

$$\{x(\mu) \mid 0 < \mu \leq \bar{\mu}\} \subset \Gamma_{\bar{\alpha}}\,.$$

So $\{x(\mu) \mid 0 < \mu \leq \bar{\mu}\}$ is a bounded subset of $\mathfrak{R}^n$.

Now we will prove that $\{Z(\mu) \mid 0 < \mu \leq \bar{\mu}\}$ is a bounded set too. Let $\tilde{x} \in \mathfrak{R}^n$ be a strictly primal-feasible point, whose existence is guaranteed by Assumption A.2. Using the convexity of f

$$\begin{aligned} f(\tilde{x}) &\geq f(x(\mu)) + \nabla f(x(\mu))^{\mathsf{T}}(\tilde{x} - x(\mu)) \\ &= f(x(\mu)) + \sum_{i=1}^{m}(F_i \bullet Z(\mu))(\tilde{x}_i - x_i(\mu)) \\ &= f(x(\mu)) + \sum_{i=1}^{m}\tilde{x}_i F_i \bullet Z(\mu) - \sum_{i=1}^{m} x_i(\mu) F_i \bullet Z(\mu) \\ &= f(x(\mu)) + F(\tilde{x}) \bullet Z(\mu) - F(x(\mu)) \bullet Z(\mu), \end{aligned} \tag{11}$$

where the first equality is a consequence of the dual feasibility of $(x(\mu), Z(\mu))$ and the last one follows from the definition of F.

Now taking $E = \mathfrak{R}^{m\times m}$, $\langle X, Y\rangle = X \bullet Y = \mathrm{Tr}(XY)$ for all $X, Y \in S^m$, $K = S^m$ and $w = F(\tilde{x})$ in Lemma 3, we conclude that

$$F(\tilde{x}) \bullet Z(\mu) \geq \sigma ||Z(\mu)||_F, \text{ for all } \mu > 0, \tag{12}$$

where σ is positive (and doesn't depend on μ). Using the fact that along the central path the gap is given by $F(x(\mu)) \bullet Z(\mu) = m\mu$, we have from (11) and (12) that

$$\frac{f(\tilde{x}) + m\mu - f(x(\mu))}{\sigma} \geq ||Z(\mu)||_F, \qquad \text{for all } \mu > 0. \tag{13}$$

Since we have already proved that $\{x(\mu) \mid 0 < \mu \leq \bar{\mu}\}$ is bounded, it follows from (13) that there exists $K > 0$ such that $||Z(\mu)||_F \leq K$, for all $0 < \mu \leq \bar{\mu}$. Whence $\{(x(\mu), Z(\mu)) \mid 0 < \mu \leq \bar{\mu}\}$ is a bounded set. $\square$

We say that $(\bar{x}, \overline{Z}) \in \mathfrak{R}^n \times \mathfrak{R}^{m\times m}$ is a *cluster point* of the central path if there exists a sequence $\{\mu_k\} \subset \mathfrak{R}_{++}$ such that $\lim_{k\to\infty} \mu_k = 0$ and $\lim_{k\to\infty} (x(\mu_k), Z(\mu_k)) = (\bar{x}, \overline{Z})$.

We know, from Proposition 4, that the set of cluster points of the central path is nonempty. Now we will see that the cluster points are optimal solutions of the primal-dual pair of problems (NP) and (DNP), in the sense that if $(\bar{x}, \overline{Z})$ is a cluster point, then $(\bar{x}, \overline{Z}) \in$ Sol(DNP) and $\bar{x} \in$ Sol(NP).

PROPOSITION 5: *All cluster points of the central path are optimal solutions of the primal-dual pair of problems (NP) and (DNP).*

Proof: Assume that $(\bar{x}, \overline{Z}) \in \mathfrak{R}^n \times \mathfrak{R}^{m\times m}$ is a cluster point of the central path and that $\{\mu_k\} \subset \mathfrak{R}_{++}$ is such that $\lim_{k\to\infty} \mu_k = 0$ and $\lim_{k\to\infty} (x^k, Z^k) = (\bar{x}, \overline{Z})$, with $(x^k, Z^k) = (x(\mu_k), Z(\mu_k))$. Since the primal and the dual feasible sets are both closed because f is continuously differentiable on $ed(f)$, we conclude that $\bar{x}$ and $(\bar{x}, \overline{Z})$ are feasible for (NP) and (DNP) respectively.

In order to prove optimality we just need to check that the gap function g vanishes at $(\bar{x}, (\bar{x}, \overline{Z}))$. We have that

$$g(x^k, (x^k, Z^k)) = F(x^k) \bullet Z^k = m\,\mu_k\,, \quad \text{for} \quad k = 1, 2, \ldots \tag{14}$$

Letting $k \to \infty$ in (14) we see that

$$g(\bar{x}, (\bar{x}, \overline{Z})) = F(\bar{x}) \bullet \overline{Z} = 0.$$

4. FINAL REMARKS

1. As mentioned in the introduction, for the case of standard convex optimization, our assumptions seem to be insufficient for establishing convergence of the central path, *i.e.* existence of a unique cluster point. In [18] such a result is proved under the assumption that the data are analytic; in [6] such assumption is weakened to the following one: if any of the data functions is constant on a segment, then it is constant over the whole line containing it. We have not been able to extend such results to the convex semidefinite programming case. The issue of assumptions enough for such a result is left as an open problem.

2. The following result has been stated in [27] (Th. 1):
In semidefinite programming, *i.e.* when $f(x) = c^\top x$ in (NP), the primal optimal value is equal to the dual optimal value if any of the following conditions holds:

a) there exists a strictly primal-feasible point;
b) there exists a strictly dual-feasible point;
c) the primal optimal set is nonempty and bounded;
d) the dual optimal set is nonempty and bounded.

We observe that, as a consequence of Theorem 1, particularly from the equivalence between Conditions 1 and 4, it follows that some pairs of alternative conditions of the result just stated imply some others. More especifically, (a) and (b) imply (c), and (a) and (c) imply (b). In fact, they are immediate consequences of Theorem 1 because (a) is just Assumption 2, (b) and (c) are Conditions 4 and 1 of Theorem 1, respectively. Since the dual of a semidefinite programming problem is also a semidefinite programming problem, we have the dual version of Theorem 1 and so we can prove, in a similar fashion, that (a) and (b) imply (d), and (b) and (d) imply (a).

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