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TRIVIAL CASES FOR THE KANTOROVITCH PROBLEM (*)

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Abstract. – *Let X and Y be two compact spaces endowed with respective measures μ and ν satisfying the condition $\mu(X) = \nu(Y)$. Let c be a continuous function on the product space $X \times Y$. The mass transfer problem consists in determining a measure ξ on $X \times Y$ whose marginals coincide with μ and ν, and such that the total cost $\iint c(x,y)\, d\xi(x,y)$ be minimized. We first show that if the cost function c is decomposable, i.e., can be represented as the sum of two continuous functions defined on X and Y, respectively, then every feasible measure is optimal. Conversely, when X is the support of μ and Y the support of ν and when every feasible measure is optimal, we prove that the cost function is decomposable.*

Keywords: Continuous programming, transportation.

1. INTRODUCTION

The mass transfer problem was initially studied by Monge [6] in 1781. His work was pursued by Appel [2] in 1887 and Appel [3] in 1928. In 1942, Kantorovitch [1] formulated the problem in a functional space, and this approach has been adopted since. This problem, a typical problem of continuous programming, has been considered by many authors, among which Anderson and Nash [1].

Specifically, let X and Y be two compact spaces endowed with respective measures μ and ν satisfying the condition $\mu(X) = \nu(Y)$. Let c be a continuous function on the product space $X \times Y$. The Kantorovitch problem

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consists in determining a measure ξ on $X \times Y$ whose marginals coincide with μ and ν, and such that the total cost

$$\iint_{X\times Y} c(x,y)\,d\xi(x,y) \tag{1}$$

be minimized. The dual of the above linear program consists in determining two real continuous functions r and s defined on X and on Y, respectively, such that

$$r(x)+s(y)\le c(x,y) \qquad \forall(x,y)\in X\times Y \tag{2}$$

and that the objective

$$\int_X r(x)\,d\mu(x)+\int_Y s(y)\,d\nu(y) \tag{3}$$

be maximized. The primal problem (1) and the dual problem (2, 3) are equivalent in the sense that they admit extremal optimal solutions whose objective values are equal. In this paper we show that, under restrictive assumptions on the cost function c, the objective

$$\iint_{X\times Y} c(x,y)\,d\xi(x,y) \tag{4}$$

is constant over the set of measures ξ defined over the product space $X \times Y$ having respective marginals μ and ν. Next we prove that a slightly modified converse statement holds as well.

2. PRELIMINARY CONSIDERATIONS

Throughout the paper, we will denote by $P_C\xi$ the projection of a measure ξ over a set C and by S_ξ the support of a measure ξ, *i.e.*, the set of points p of its domain such that ξ is not the zero measure on any neighborhood of the point p.

The Kantorovitch problem, or primal problem, is defined as

$$\begin{aligned} \gamma = \quad & \inf_{\xi\ge 0} && \iint_{X\times Y} c(x,y)\,d\xi(x,y) \\ & \text{subject to} && P_X\xi=\mu \\ & && P_Y\xi=\nu. \end{aligned} \tag{5}$$

We say that a measure on the space $X \times Y$ is primal-feasible if, for any compact subset $K \subseteq X$ and $L \subseteq Y$ we have that $\xi(K\times Y)=\mu(K)$ and

$\xi(X \times L) = \nu(L)$. This condition holds if and only if, for each pair of continuous functions ϕ and ψ defined on X and Y, respectively, we have:

$$\iint_{X\times Y} \phi(x)\, d\xi(x,y) = \int_X \phi(x)\, d\mu(x)$$

and

$$\iint_{X\times Y} \psi(y)\, d\xi(x,y) = \int_Y \psi(y)\, d\nu(y).$$

In particular, the marginals of the measure ξ are μ and ν.

Whenever the transportation cost

$$\iint_{X\times Y} c(x,y) d\xi_0(x,y)$$

of a feasible measure ξ_0 is equal to γ we say that ξ_0 is optimal. Kantorovitch [5] has shown that the set of optimal measures is nonempty.

Now let r and s be continuous real functions over the sets X and Y, respectively. The dual problem of (5) is defined as

$$\begin{array}{rcl} \gamma^* = & \sup\limits_{r,s} & \int_X r(x)\, d\mu(x) + \int_Y s(y)\, d\nu(y) \\ & \text{subject to} & r(x) + s(y) \leq c(x,y) \\ & & (x,y) \in X \times Y. \end{array} \tag{6}$$

If ξ is primal-feasible and the pair (r,s) is dual-feasible, then we have the weak duality inequality:

$$\int_X r(x)\, d\mu(x) + \int_Y s(y)\, d\nu(y) \leq \iint_{X\times Y} c(x,y)\, d\xi(x,y).$$

Feasible solutions to the above dual problem (6) are said to form a (continuous) **cost partition**. A cost partition (r_0, s_0) is dual-optimal if it achieves the optimal dual objective γ^*, *i.e.*,

$$\int_X r_0(x)\, d\mu(x) + \int_Y s_0(y)\, d\nu(y) = \gamma^*.$$

Dubuc and Tanguay [4] have proved that the set of optimal dual solutions is nonempty.

3. CHARACTERIZING AN OPTIMAL PARTITION

The following theorem, due to Dubuc and Tanguay [4], provides a criterion for checking the optimality of a cost partition.

THEOREM 1: *Let J be a compact subset of Y and let us define the set $I(J) = \{x \in X : \exists y \in J : r(x) + s(y) = c(x, y)\}$. A cost partition (r, s) is dual-optimal if and only if $\mu(I) \geq \nu(J)$ for every compact subset J of Y.*

Let us illustrate this result by an example where we wish to determine a measure ξ defined on the square $[0, 1] \times [0, 1]$, whose marginals are the Lebesgue measures restricted to the interval $[0, 1]$, and that minimizes the objective

$$\int_0^1 \int_0^1 |x - y| \, d\xi(x, y).$$

This problem can be rewritten as:

$$\begin{array}{ll} \inf\limits_{\xi} & \displaystyle\int_0^1 \int_0^1 |x - y| \, d\xi(x, y) \\ \text{subject to} & P_{[0,1]}\xi = \mu \quad \text{(Lebesgue measure).} \end{array}$$

Its dual problem is:

$$\begin{array}{ll} \sup\limits_{(r,s)} & \displaystyle\int_0^1 r(x) \, dx + \int_0^1 s(y) \, dy \\ \text{subject to} & r(x) + s(y) \leq |x - y| \\ & (x, y) \in [0, 1] \times [0, 1]. \end{array}$$

Let us show that setting $r(x) = x$ and $s(y) = -y$ yields an optimal partition. To this effect, let J be a compact subset of $[0, 1]$. We have:

$$I(J) = \{x \in [0, 1] : \exists y \in J : \ r(x) + s(y) = |x - y|\}.$$

From the theorem, it is sufficient to show that $J \subseteq I$. Indeed, $y \in I(J)$ whenever $y \in J$ since

$$r(y) + s(y) = y - y = |y - y|.$$

The optimal value for this problem is

$$\gamma = \gamma^* = \int_0^1 x \, dx + \int_0^1 -y \, dy = 0.$$

We notice that setting $r(x) = 0$ and $s(y) = 0$ for all $(x, y) \in [0,1] \times [0,1]$ also yields an optimal partition.

Let us now consider the measure:

$$\xi_0(A) = \frac{\lambda(A \cap D)}{\sqrt{2}},$$

where D denotes the main diagonal of the unit square and λ the Lebesgue measure. Since

$$\int_0^1 \int_0^1 |x - y| \, d\xi_0(x, y) = 0,$$

the measure ξ_0 is primal-optimal.

Notice that dual solutions are, at best, unique up to an additive constant. Indeed, whenever $(r(x), s(y))$ is an optimal partition, so is $(r(x)+c, s(y)-c)$, for any number c. In the previous example, we exhibited two dual solutions, $(x, -y)$ and $(0, 0)$, whose difference is nonconstant.

4. THE CASE OF DECOMPOSABLE COSTS

This section includes the paper's main results. We first show that if the cost function c is **decomposable**, *i.e.*, can be represented as the sum of two conti-nuous functions defined on X and Y, respectively:

$$c(x, y) = f(x) + g(y) \qquad \forall (x, y) \in X \times Y,$$

then every feasible measure is optimal. Next we prove that a modified form of the reverse statement holds.

PROPOSITION 2: *If the cost function c is decomposable, then the primal objective function is constant over the set of primal-feasible measures.*

Proof: Let ξ be a measure with respective marginals μ and ν. We write:

$$\iint_{X \times Y} f(x) \, d\xi(x, y) = \int_X f(x) \, d\mu(x)$$

and

$$\iint_{X \times Y} g(y) \, d\xi(x, y) = \int_Y g(y) \, d\nu(y).$$

Hence:

$$\begin{aligned} F(\xi) &= \iint_{X\times Y} (f(x)+g(y))\, d\xi(x,y) \\ &= \iint_{X\times Y} f(x)\, d\xi(x,y) + \iint_{X\times Y} g(y)\, d\xi(x,y) \\ &= \int_X f(x)\, d\mu(x) + \int_Y g(y)\, d\nu(y), \end{aligned}$$

and the result follows from the observation that this last term is constant. □

THEOREM 3: *Let the primal objective function be constant over the set of primal feasible measures. Then there exist continuous functions f and g defined over the sets X and Y, respectively, such that c is decomposable over $S_\mu \times S_\nu$.*

Proof: The measure $\xi_0 = (\mu \otimes \nu)/\mu(X)$ is clearly primal-feasible, hence it is also primal-optimal. Let (r_0, s_0) be a dual-optimal solution; there exists as it was said at the end of Section 2. The following relation holds by the strong duality result (*i.e.* the primal and the dual problems have the same optimal objective value):

$$\int_X r_0(x)\, d\mu(x) + \int_Y s_0(y)\, d\nu(y) = \iint_{X\times Y} c(x,y)\, d\xi_0(x,y) = \gamma \tag{7}$$

and

$$r_0(x) + s_0(y) \le c(x,y) \qquad \forall (x,y) \in X \times Y. \tag{8}$$

Now, since ξ_0 has respective marginals μ and ν, there follows:

$$\begin{aligned} \int_X r_0(x)\, d\mu(x) &= \iint_{X\times Y} r_0(x)\, d\xi_0(x,y) \\ \int_Y s_0(y)\, d\nu(y) &= \iint_{X\times Y} s_0(y)\, d\xi_0(x,y). \end{aligned}$$

Therefore (7) is equivalent to

$$\iint_{X\times Y} (r_0(x) + s_0(y))\, d\xi_0(x,y) = \iint_{X\times Y} c(x,y)\, d\xi_0(x,y) = \gamma. \tag{9}$$

From the relationships (8) and (9) we conclude that

$$r_0(x) + s_0(y) = c(x,y) \qquad \forall (x,y) \in S_{\xi_0}.$$

For any points x_0 and y_0 in S_μ and S_ν, respectively, we have that $(x_0, y_0) \in S_{\xi_0}$ and there follows:

$$r_0(x_0) + s_0(y_0) = c(x_0, y_0),$$

from which we infer that c is decomposable over $S_\mu \times S_\nu$, as required. □

We conclude this section with two simple results, whose proofs are straightforward.

Remark 1: If c is decomposable then, for any four-tuple $(x' \in X, y' \in Y, x \in X, y \in Y)$ there holds:

$$c(x,y) - c(x,y') + c(x',y') - c(x',y) = 0. \tag{10}$$

Remark 2: If there exists a pair $(x', y') \in X \times Y$ such that

$$c(x,y) - c(x,y') + c(x',y') - c(x',y) = 0 \qquad \forall (x,y) \in X \times Y, \tag{11}$$

then the cost function c is decomposable.

Proof: It suffices to consider the functions $f(x) = c(x,y') - c(x',y')$ and $g(y) = c(x', y)$. □

5. EXAMPLES

In this section we denote by $F(\xi)$ the primal objective function of the Kantorovitch problem. We consider three examples in which X and Y are subsets of the reals. The usefulness of Theorem 1 is illustrated in Example 3.

Example 1: Let $c(x,y) = |y - x|$ and the measures μ and ν have equal mass. We want to find conditions under which the functional F is constant over the set of measures on $X \times Y$ with respective marginals equal to μ and ν. If $S_\mu \subseteq (-\infty, a]$ and $S_\nu \subseteq [a, +\infty)$ for some number a, then $c(x,y) = y - x$, and Proposition 2 ensures that F is constant. This conclusion also holds if $S_\mu \subseteq [a, +\infty)$ and $S_\nu \subseteq (-\infty, a]$, in which case $c(x,y) = x - y$.

Example 2 (nonconstant F)**:** Let, as before, $c(x,y) = |y - x|$ and consider the measures:

$$\mu(A) = \lambda(A \cap [0, 2/3])$$
$$\nu(B) = \lambda(B \cap [1/3, 1]),$$

where λ denotes Lebesgue's measure. Now $S_\mu = [0, 2/3]$ and $S_\nu = [1/3, 1]$. Since

$$c(0, 1/3) - c(0, 1) + c(2/3, 1) - c(2/3, 1/3) = -2/3 \neq 0.$$

Remark 2 ensures that c is not decomposable on $[0, 2/3] \times [1/3, 1]$. Hence, by Theorem 3, F is not constant.

Example 3: The dual program of the above example takes the form

$$\begin{array}{ll} \sup\limits_{r,s} & \displaystyle\int_{[0,2/3]} r(x)\, d\mu(x) + \int_{[\frac{1}{3},1]} s(y)\, d\nu(y) \\ \text{subject to} & r(x) + s(y) \leq |y - x| \\ & x \in [0, 2/3],\ y \in [1/3, 1]. \end{array}$$

Based on dual optimality, let us determine primal-optimal solutions. From Theorem 1, a cost partition is optimal if and only if $\mu(I) \geq \nu(J)$, where J is a compact subset of the interval $[1/3, 1]$ and

$$I(J) = \{x \in [0, 2/3] : \exists y \in J : \ r(x) + s(y) = |y - x|\}\cdot$$

The partition $(r(x) = -x, s(y) = y)$ is optimal. Indeed, for any compact subset J of $[1/3, 1]$,

$$I(J) = \left\{x \in \left[0, \frac{2}{3}\right] : \exists y \in J \quad y - x = |y - x|\right\} = \left[0, b\right] \cap \left[0, \frac{2}{3}\right]$$

where b is the least upper bound of J. we obtain:

- $b \geq 2/3 \Rightarrow I(J) = [0, 2/3]$ and $\mu(I(J)) \geq \nu(J)$.
- $b \leq 2/3 \Rightarrow I(J) = [0, b]$. Since $J \subseteq [1/3, b]$, we have that

$$\nu(J) \leq \nu([1/3, b]) = b - 1/3.$$

Now: $\mu(I(J)) = \mu([0, b]) = b \geq b - 1/3 \geq \nu(J)$.

The problem's optiml value is

$$-\int_{[0,2/3]} x\, d\mu(x) + \int_{[1/3,1]} y\, d\nu(y) = -\int_0^{2/3} x\, dx + \int_{1/3}^1 y\, dy = \frac{2}{9}.$$

Let us consider the measure ξ_1 defined as

$$\xi_1(A) = \frac{\sqrt{2}}{2} \lambda(A \cap D),$$

where $D = \{(x,y) : y = x + 1/3, x \in [0,2/3]\}$, and λ is Lebesgue's measure defined on the segment D. The measure ξ_1 is feasible, since its marginals are μ and ν. Furthermore it is optimal since

$$\iint_{X\times Y} |y-x|\, d\xi_1(x,y) = 1/3 \iint_{X\times Y} d\xi_1(x,y) = \frac{1}{3}\cdot\frac{2}{3} = \frac{2}{9}\cdot$$

Let us now consider the measure ξ_2 defined as

$$\xi_2(A) = \frac{\sqrt{2}}{2}\,[\lambda(A\cap D_1) + \lambda(A\cap D_2)]$$

where $D_1 = \{(x,y) : y = x + 2/3, x \in [0,1/3]\}$, $D_2 = \{(x,y) : y = x, x \in [1/3,2/3]\}$ and λ is the one dimensional Lebesgue measure. The marginals of ξ_2 are μ and ν. We set

$$P_1 = \{(x,y) : 0 \le x \le 1/3,\ 1/3 \le y \le 1\}$$
$$P_2 = \{(x,y) : 1/3 \le x \le 2/3,\ 1/3 \le y \le 1\}\cdot$$

We obtain:

$$\iint_{X\times Y} |y-x|\, d\xi_2(x,y) = \iint_{P_1} |y-x|\, d\xi_2(x,y) + \iint_{P_2} |y-x|\, d\xi_2(x,y).$$

Now, since $y = x$ over the set $S_{\xi_2} \cap P_2$, there follows:

$$\iint_{P_2} |y-x|\, d\xi_2(x,y) = 0$$

and ξ_2 is optimal, since its objective is equal to 2/9:

$$\begin{aligned}\iint_{X\times Y} |y-x|\, d\xi_2(x,y) &= \iint_{P_1} |y-x|\, d\xi_2(x,y)\\ &= \frac{2}{3}\iint_{P_1} d\xi_2(x,y)\\ &= \frac{2}{3}\mu([0,1/3]) \quad = \quad \frac{2}{9}\cdot\end{aligned}$$

Actually, a solution is primal-optimal if it is a convex combination of ξ_1 and ξ_2.

If the support of ξ is contained in $\{(x,y) : y \ge x\}$, then $|y-x| = y - x$ almost everywhere. Furthermore:

$$\iint_{X\times Y} |y-x|\, d\xi(x,y) = \int_Y y\, d\nu(y) - \int_X x\, d\mu(x).$$

Any measure having Lebesgue measures as marginals and being uniformly distributed over the $k+2$ diagonals given in Table 1 is optimal (see also Fig. 1).

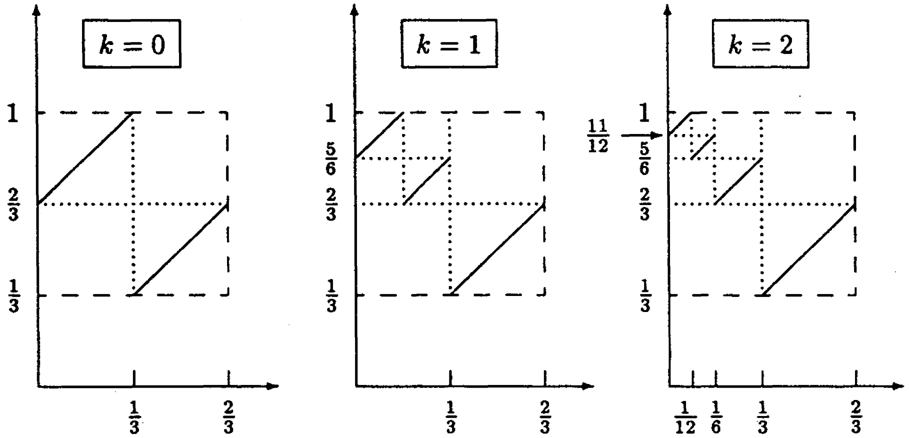

Figure 1. – Three optimal measures.

TABLE 1
Diagonal segments supporting the optimal measure.

square	diagonal segment
0	$\left[\left(\frac{1}{3},\frac{1}{3}\right),\left(\frac{2}{3},\frac{2}{3}\right)\right]$
1	$\left[\left(\frac{1}{6},\frac{2}{3}\right),\left(\frac{1}{3},\frac{5}{6}\right)\right]$
2	$\left[\left(\frac{1}{12},\frac{5}{6}\right),\left(\frac{1}{6},\frac{11}{12}\right)\right]$
$\vdots$	$\vdots$
k	$\left[\left(\frac{1}{3\times 2^k},1-\frac{1}{3\times 2^{k-1}}\right),\left(\frac{1}{3\times 2^{k-1}},1-\frac{1}{3\times 2^k}\right)\right]$
$k+1$	$\left[\left(0,1-\frac{1}{3\times 2^k}\right),\left(\frac{1}{3\times 2^k},1\right)\right]$

6. CONCLUSION

The decomposability of a cost function is a very restrictive property. Note however that nondecomposable functions such as $c(x,y)=|x-y|$ become decomposable when they are restricted to specific subrectangles of their domain of definition. This helps to understand the geometrical structure of solution sets for some Kantorovitch problems. For instance, in Example 3, the partial decomposability of the cost function enabled us to obtain infinitely many extremal optimal solutions

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