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## A Monte Carlo simulation of the flow network reliability using importance and stratified sampling

# A MONTE CARLO SIMULATION OF THE FLOW NETWORK RELIABILITY USING IMPORTANCE AND STRATIFIED SAMPLING

by S. Bulteau ([1]) and M. El Khadiri ([2])

Abstract. – *We consider the evaluation of the flow network reliability parameter. Because its exact computation has exponential time complexity, simulation methods are alternatives used to evaluate large networks. In this paper, we use the state space decomposition methodology of Doulliez and Jamoulle in order to construct a new simulation method which combines the importance and the stratified Monte Carlo principles. We show that the related estimator belongs to the variance reduction family and it offers more accurate estimates than those obtained by a previous stratified sampling technique based on the same decomposition. By experimental results, we show the interest of the proposed method when compared to previous methods.* © Elsevier, Paris

Keywords: Flow network, maximum st-flow, Monte Carlo simulation, reliability, variance reduction.

Résumé. – *Nous considérons le problème de l'évaluation de la fiabilité d'un réseau de transport stochastique. Le fait que le calcul analytique de ce paramètre soit de complexité temporelle exponentielle, conduit à utiliser des simulations de Monte Carlo pour l'analyse des réseaux de grande taille. Dans ce papier, nous exploitons la procédure de décomposition de l'espace d'états de Doulliez et Jamoulle pour construire une nouvelle méthode de simulation. Elle est basée sur les échantillonnages stratifié et préférentiel. Nous démontrons théoriquement que cette méthode appartient à la famille des méthodes de réduction de la variance et que l'estimateur associé est de variance plus réduite que celle d'un estimateur stratifié basé sur la même décomposition. Par des résultats expérimentaux, nous montrons l'intérêt de la méthode proposée en la comparant à d'autres méthodes.* © Elsevier, Paris

Mots clés : Réseau de transport, st-flot maximal, simulation de Monte Carlo, fiabilité, réduction de la variance.

## 1. INTRODUCTION

A basic mission of a flow network is the establishment of a flow between a source node $s$ and a sink node $t$ that meets or exceeds a fixed demand $d$. In the stochastic case, the network components have random capacities and the success of this mission is a random event. Systems that can

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be regarded as flow networks are electric-power transmission, distribution systems, transportation networks and computer networks. The probability of successful mission is a performance measure, often called the *flow network reliability measure*.

Several papers have been devoted to its evaluation when nodes do not limit flow transmission and when arcs capacities are discrete, multi-valued and statistically independent random variables [1, 7, 11, 15, 16]. When all arcs have only two possible capacities 1 or 0 and the demand is $d = 1$, the problem becomes the source-terminal reliability problem [3, 6, 10] which is NP-hard [2]. Consequently, the general case considered here is also an NP-hard problem. This implies that the computational time will be prohibitive when the network size is large [8]. Then Monte Carlo approaches are alternatives allowing, in a reasonable time, the evaluation of large networks.

For a specified sample size $K$, the estimation of the flow network reliability parameter by the standard estimator is the frequency of operating capacity vectors in the set of $K$ independent trials drawn from the whole state space $\Omega$ of the random capacity vector. The well known drawback of the standard Monte Carlo sampling is the large size $K$ required to obtain sufficiently small variance and reasonable relative error [10]. The objective of variance reduction methods is to offer more accurate estimates than those obtained by using the standard estimator, for the same sample size $K$.

The Doulliez and Jamoulle decomposition (DJD) procedure [1, 7] partitions an input rectangular subset $R \subseteq \Omega$ that contains operating and failed vectors into an operating rectangular set of operating vectors, failed rectangular sets of failed vectors and undetermined rectangular sets of vectors not yet classified as failed or operating. Based on this procedure, Fishman and Shaw proposed a stratified Monte Carlo technique [13]. Bounds on the reliability parameter and other informations produced by calls to the DJD procedure, are exploited to reduce the sampling to a subset $U$ of $\Omega$. Moreover, the $K$ samples are distributed on strata of $U$. Unfortunately, the trials within each stratum is accomplished by the standard sampling technique. The aim of our work is to show that if we estimate the contribution of each stratum by the importance sampling method proposed in [5], we induce additional accuracy in the evaluation of the measure of interest.

The paper is organized as follows. The following section introduces general notation and definitions. In section 3, we briefly explain the use of the DJD procedure in the exact computation context and its use to transform the considered problem to the evaluation of networks with state spaces smaller

than $\Omega$. In section 4, we present the stratified technique of Fishman and Shaw [13]. In section 5, we explain our stratified Monte Carlo technique. Section 6 is devoted to numerical comparisons and Section 7 to some conclusions.

## 2. NOTATION AND MODEL DEFINITION

We will use the following notation and definitions. For ease of explanation, additional definitions and notation will be given in later sections.

- $G = (V, A, \vec{C}, s, t, d)$: the flow network
- $V$: the set of nodes
- $n$: the number of nodes
- $A = \{e_1, \ldots, e_a\}$: the set of arcs
- $a$: the number of arcs
- $s$: the source node
- $t$: the sink node
- $d$: the demand required at node $t$
- For each arc $e_j \in A$,
    - $C_j$: the random discrete capacity of arc $e_j$
    - $n_j$: the cardinality of the state space of the random capacity of arc $e_j$
    - $0 \leq c_{j1} < c_{j2} < \ldots < c_{jn_j} < +\infty$: the $n_j$ possible values of the random variable $C_j$
    - $\Omega_j = \{c_{j1}, \ldots, c_{jn_j}\}$: the state space of the random variable $C_j$
    - $p_{jn}$: the probability that $C_j$ has capacity $c_{jn}$ in $\Omega_j$
- $\vec{C} = (C_1, \ldots, C_a)$: the random state vector of the network $G$
- $\Omega = \bigotimes\limits_{j=1}^{a} \Omega_j$: the network state space (the state space of random variable $\vec{C}$)
- A set $R \subseteq \Omega$ is *rectangular* if and ony if there is a lower and an upper vector of capacities $\alpha(R) = (\alpha_1(R), \ldots, \alpha_a(R))$ and $\beta(R) = (\beta_1(R), \ldots, \beta_a(R))$, respectively, in $\Omega$ such that every vector of capacities $\vec{C} = (c_1, \ldots, c_a)$ with $\alpha_j(R) \leq c_j \leq \beta_j(R)$ for all arcs $e_j$, belongs to $R$ and $R$ contains only those vectors [7]
- $v(c_1, \ldots, c_a)$: the value of maximum $st$-flow when the arc $e_j$ has capacity $c_j$, $e_j \in A$
- $\Phi$ the non-decreasing structure function of the network $G$ defined for all state vectors $\vec{C}$ by

$$\Phi(\vec{C}) = \begin{cases} 1 & \text{if } v(\vec{C}) \geq d \\ 0 & \text{otherwise} \end{cases}$$

- For any rectangle $R \subseteq \Omega$,
    - $\vec{C}(R)$: the random vector having the same distribution as the conditional distribution of $\vec{C}$ given that $\vec{C} \in R$
    - $\mathcal{B}(R) = \Phi(\vec{C}(R))$
    - $g(R) = E\{\mathcal{B}(R)\} = E\{\Phi(\vec{C}(R))\} = E\{\Phi(\vec{C})|\vec{C} \in R\}$
- $g(\Omega) = E\{\Phi(\vec{C})\} = E\{\mathcal{B}(\Omega)\}$: the flow network performance measure of interest.

## 3. THE EXACT COMPUTATION AND THE NETWORK RELIABILITY TRANSFOR-MATION BASED ON THE DJD PROCEDURE

The Doulliez and Jamoulle decomposition (DJD) procedure [7] partitions an input rectangular subset $R \subseteq \Omega$ that contains operating and failed vectors into an operating rectangular set, failed rectangular sets and undetermined rectangular sets of vectors not yet classified as failed or operating. In this section, we recall the use of this decomposition in the exact computation context and its use to transform this problem to the evaluation of networks with state spaces smaller than $\Omega$.

### 3.1. The DJD procedure in the exact evaluation context

The exact algorithm of Doulliez and Jamoulle [1, 7] starts by calling the DJD procedure to partition the state space $\Omega$. Each undetermined set subsequently undergoes the same procedure for its decomposition. This process terminates when all generated sets are classified operating or failed. The probability that the random vector state belongs to any generated operating set is easy to compute and the flow network reliability parameter is the sum of these probabilities [1, 7]. More formally, we have

$$g(\Omega) = \sum_{i=1}^{T(\Omega)} \Pr\{\vec{C} \in W_1(\Omega)\} \tag{1}$$

where $T(\Omega)$ is the number of generated operating rectangular subsets and $W_i(\Omega)$, $1 \leq i \leq T(\Omega)$, are these sets. This number corresponds to the number of calls to the DJD procedure. A detailed example of this process is given in [9].

As the DJD procedure has polynomial time complexity [13], the exponential time complexity of the exact evaluation of $g(\Omega)$ by the formula (1) results from the exponential growth of $T(\Omega)$, the number of calls to the DJD procedure. Consequently, we will interest in later sections to Monte

Carlo methods that exploit bounds deduced from a number $M$ of calls to the DJD procedure, with $M$ smaller than $T(\Omega)$.

## 3.2. The network reliability transformation based on the DJD procedure

Some additional notation and definitions follow, which are needed to define the transformation in lemma 3.1 and all random variables that serve to construct the estimators in later sections. They concern all undetermined rectangular sets $R \subseteq \Omega$ resulting from the DJD procedure when it is called to compute exactly $g(\Omega)$ by formula (1),

- $T(R)$: the number of decompositions that allows to compute exactly the parameter $g(R)$ as in formula (1),

$$g(R) = \sum_{i=1}^{T(R)} \Pr\{\vec{C}(R) \in W_i(R)\} \tag{2}$$

where $W_i(R)$, $1 \leq i \leq T(R)$, are the generated operating rectangular subsets of $R$.

- For a fixed integer $M$ such that $M \leq T(R) (M \leq T(R) \leq T(\Omega))$,

  - $W(R)$: the subset of vectors classified as operating in first $M$ calls to the DJD in order to compute $g(R)$ by formula (2)
  - $F(R)$: the subset of vectors classified as failed in first $M$ calls to the DJD in order to compute $g(R)$ by formula (2)
  - $h(R)$: the number of undetermined rectangular subsets not yet decomposed
  - $S(R) = \{U_1(R), \ldots, U_{h(R)}(R)\}$: the set of undetermined rectangular subsets not yet decomposed
  - $U(R) = \bigcup\limits_{j=1}^{h(R)} U_j(R)$
  - $g_l(R) = \Pr\{\vec{C}(R) \in W(R)\} = \sum\limits_{i=1}^{M} \Pr\{\vec{C}(R) \in W_i(R)\}$: a lower bound on $g(R)$
  - $\pi_j(R) = \Pr\{\vec{C}(R) \in U_j(R)\}$, for $j \in \{1, \ldots, h(R)\}$
  - $\pi(R) = \Pr\{\vec{C}(R) \in U(R)\} = \sum\limits_{j=1}^{h(R)} \pi_j(R)$
  - $g_u(R) = g_l(R) + \pi(R)$: an upper bound on $g(R)$
  - If $S(R) \neq \phi$, $H(R)$ denotes the random variable defined by

$$\Pr\{H(R) = h\} = \Pr\{\vec{C}(R) \in U_h(R) \,|\, \vec{C}(R) \in U(R)\}$$
$$= \pi_h(R)/\pi(R), \qquad \text{for } h \in \{1, \ldots, h(R)\}.$$

The following lemma expresses the parameter $g(\Omega)$ as a function of bounds on this parameter and the exact reliabilities of networks with state spaces smaller than $\Omega$. Both stratified simulation methods presented here use this transformation.

LEMMA 3.1: *For the fixed integer $M \leq T(\Omega)$, we have the following transformation*

$$g(\Omega) = g_l(\Omega) + \sum_{h=1}^{h(\Omega)} \pi_h(\Omega) \, g(U_h(\Omega)). \qquad (3)$$

*Proof:* As the sets $W(\Omega)$, $F(\Omega)$ and the undetermined sets $U_h(\Omega)$, $h \in \{1, \ldots, h(\Omega)\}$, form a partition of $\Omega$, the total expectation theorem gives

$$\begin{aligned}
g(\Omega) = E\{B(\Omega)\} = {} & E\{B(\Omega) \mid (\vec{C} \in W(\Omega))\} \Pr\{\vec{C} \in W(\Omega)\} \\
& + E\{B(\Omega) \mid (\vec{C} \in F(\Omega))\} \Pr\{\vec{C} \in F(\Omega)\} \\
& + \sum_{h=1}^{h(\Omega)} E\{B(\Omega) \mid (\vec{C} \in U_h(\Omega))\} \Pr\{\vec{C} \in U_h(\Omega)\}.
\end{aligned}$$

Since $W(\Omega)$ is an operating set and $F(\Omega)$ is a failed set, we have

$$E\{B(\Omega) \mid (\vec{C} \in W(\Omega))\} = 1$$

and

$$E\{B(\Omega) \mid (\vec{C} \in F(\Omega))\} = 0.$$

We then obtain

$$g(\Omega) = E\{B(\Omega)\} = g_l(\Omega) + \sum_{h=1}^{h(\Omega)} g(U_h(\Omega)) \, \pi_h(\Omega). \qquad (4)$$

∎

## 4. THE STRATIFIED SAMPLING METHOD OF FISHMAN AND SHAW [13]

The stratified sampling technique of Fishman and Shaw is based on the transformation given in the previous lemma and the standard sampling technique that we recall below.

## 4.1. The standard Monte Carlo method

For a sample size $K$ and any rectangular set $R \subseteq \Omega$, the standard Monte Carlo estimator of the parameter $g(R)$ is a sample mean based on $K$ independent trials $\Phi(\vec{\mathcal{C}}^{(1)}(R)), \ldots, \Phi(\vec{\mathcal{C}}^{(K)}(R))$ of the random variable $\mathcal{B}(R) = \Phi(\vec{\mathcal{C}}(R))$. More formally, this estimator is

$$\hat{\Phi}(K, R) = \frac{1}{K} \sum_{i=1}^{K} \Phi(\vec{\mathcal{C}}^{(i)}(R)). \tag{5}$$

Its variance is

$$\text{Var}\{\hat{\Phi}(\vec{\mathcal{C}}(R))\} = \frac{\text{Var}\{\Phi(\vec{\mathcal{C}}(R))\}}{K} = \frac{g(R)(1 - g(R))}{K}. \tag{6}$$

## 4.2. The Monte Carlo method of Fishman and Shaw [13]

By replacing $R$ by $\Omega$ in definition (5), we obtain the standard estimator of $g(\Omega)$ with the sample size $K$. The well known drawback of the standard estimator is the large sample size $K$ required to obtain sufficiently small variance and reasonable relative error [10]. Variance reduction methods is based on estimators having smaller variance than the standard one with the same sample size. Consequently, they get more accurate estimates.

Instead of drawing the $K$ trials from $\Omega$ using the distribution function of $\vec{\mathcal{C}} = \vec{\mathcal{C}}(\Omega)$, as in the standard Monte Carlo method, the transformation (3) leads Fishman and Shaw [13] to concentrate the $K$ samples in the set

$$U(\Omega) = \bigcup_{h=1}^{h(\Omega)} U_h(\Omega).$$

Moreover, they propose to distribute the $K$ trials on the undetermined sets. The number $K_h$ of trials drawn from each $U_h(\Omega)$ is fixed proportional to $\pi_h(\Omega)/\pi(\Omega)$, that is the probability that $\vec{\mathcal{C}}$ belongs to $U_h(\Omega)$ given that $\vec{\mathcal{C}}$ belongs to $U(\Omega)$. The reader can see [13] for a discussion about the rules to distribute the $K$ trials.

More formally, they suggest to use the following unbiased estimator

$$\tilde{\mathcal{F}} = g_l(\Omega) + \sum_{h=1}^{h(\Omega)} \pi_h(\Omega) \hat{\Phi}(K_h, U_h(\Omega)) \tag{7}$$

where

$$K_h = K \Pr \{H(\Omega) = h\}$$
$$= K \pi_h(\Omega)/\pi(\Omega), \qquad \text{for } h = 1, \dots, h(\Omega).$$

and $\hat{\Phi}(K_h, U_h(\Omega))$ is the standard unbiased estimator of $g(U_h(\Omega))$ based on $K_h$ independent trials of $\vec{C}(U_h(\Omega))$. In [13], the authors show that the variance of this estimator is

$$\mathrm{Var}\{\tilde{\mathcal{F}}\} = (g(\Omega) - g_l(\Omega))(g_u(\Omega) - g(\Omega))/K$$
$$- \sum_{h=1}^{h(\Omega)} \pi_h(\Omega)(g(U_h(\Omega)) - g(U(\Omega))^2/K. \qquad (8)$$

Then

$$\mathrm{Var}\{\tilde{\mathcal{F}}\} \le (g(\Omega) - g_l(\Omega))(g_u(\Omega) - g(\Omega))/K$$
$$\le g(\Omega)(1 - g(\Omega))/K.$$

Because $g(\Omega)(1 - g(\Omega))/K$ corresponds to the variance of the standard estimator of $g(\Omega)$, the last inequality implies that $\tilde{\mathcal{F}}$ offers more accurate estimates than the standard one.

## 4.3. Algorithmic description of the method of Fishman and Shaw

For a given sample size $K$, an estimate of $g(\Omega)$ by this sampling strategy can be obtained from the following function with the parameter $R$ equal to $\Omega$.

Function *StratifiedSampling*1$(K, R)$

0. Preliminary step: *Bounds* $(R, M, g_l(R), g_u(R), h(R), S(R), \{\pi_1(R), \dots, \pi_{h(R)}(R)\})$
1. Initialization: *Est* := 0
2. For $h := 1$ to $h(R)$ do
   2.1. Evaluate the size $K_h$ : $K_h := K \times \pi_h(R)/\pi(R)$
   2.2. Compute the standard estimation of $g(U_h(R))$ : $Est_h$ := *StandardSampling* $(U_h(R), K_h)$
   2.3. Add the contribution of the stratum $h$ to *Est* : $Est := Est + \pi_h(R) \times Est_h$
3. Return the estimate of $g(R)$ : $return (g_l(R) + Est)$

In step 0, we use the procedure *Bounds* $(R, M, g_l(R), g_u(R), h(R), S(R), \{\pi_1(R), \dots, \pi_{h(R)}(R)\}))$ that computes, for a given rectangular subset $R$ of $\Omega$ and a fixed integer $M$ of decompositions, the bounds $g_l(R), g_u(R)$, the number $h(R)$ of undetermined rectangular subsets, the

set of undetermined sets $S(R) = \{U_1(R), \ldots, U_{h(R)}(R)\}$, the set of associated weights $\{\pi_1(R), \ldots, \pi_{h(R)}(R)\}$. If $T(R) \leq M$, this procedure gives $g_l(R) = g_u(R) = g(R)$ and $h(R) = 0$. The reader can see [7, 13], for details about the computing process of the procedure *Bounds* ( ), even if this is not necessary to understand our work.

The standard sampling function used in step 2.2 can be written as follows:

<div align="center">Function <em>StandardSampling</em>$(R, K)$</div>

1. Initialization: $Est := 0$
2. For each experiment $k := 1, \ldots, K$ do
   2.1. For each link $j \in A$ accomplish a trial $c_j$ of $C_j(R)$
   2.2. Compute $v(c_1, c_2, \ldots, c_a)$
   2.3. If $v(c_1, c_2, \ldots, c_a) \geq d$ then $Est := Est + 1$
3. Return $(Est/K)$

The Ford-Fulkerson flow-augmenting method can be used to compute $v(c_1, c_2, \ldots, c_a)$, that is the maximum *st*-flow corresponding to the capacity vector $(c_1, c_2, \ldots, c_a)$. A method to generate a trial of a discrete random variable can be found for instance in [12].

## 5. A NEW STRATIFIED SAMPLING ESTIMATOR

In the above algorithm, the estimation of each reliability parameter $g(U_h(\Omega))$ associated to the stratum $U_h(\Omega)$ is accomplished by the standard sampling. If these reliabilities are evaluated by estimators that belong to the variance reduction family, we will obtain more accurate estimator than $\tilde{\mathcal{F}}$ (7). As a consequence, we propose to replace the standard estimator by the recursive importance sampling one presented in [4, 5]. This estimator is briefly recalled in the next subsection.

### 5.1. The recursive importance sampling estimator [4, 5]

In [4, 5], we proposed an efficient variance reduction method. It transforms the sampling in a given rectangular set $R \subseteq \Omega$ of state vectors into the sampling is one subset among the undetermined subsets resulting from the DJD procedure when it is called to partition the rectangle $R$. This subset subsequently undergoes a similar step until the transformation is called on a

rectangular set $R$ with $T(R)$ smaller or equal to $M$. The related estimator is a sample mean

$$\hat{Z}(K, R) = \frac{1}{K} \sum_{i=1}^{K} Z^{(i)}(R), \qquad (9)$$

based on the random variable $Z(R)$ given in the following lemma.

LEMMA 5.1: *The recursive random variable $\mathcal{Z}(R)$ defined by*

$$\mathcal{Z}(R) = \begin{cases} g(R) & \text{if } T(R) \leq M \\ g_l(R) + \pi(R) \mathcal{Z}(U_{H(R)}(R)) & \text{otherwise,} \end{cases} \qquad (10)$$

*where all parameters are defined as in section 3.2, satisfies*

$$E\{\mathcal{Z}(R)\} = E\{\mathcal{B}(R)\} = g(R) \qquad (11)$$

*and*

$$\begin{aligned} \text{Var}\{\mathcal{Z}(R)\} &\leq (g(R) - g_l(R))(g_u(R) - g(R)) \\ &\leq g(R)(1 - g(R)) = \text{Var}\{\mathcal{B}(R)\} \end{aligned} \qquad (12)$$

The proof is given in [4, 5].

Results (11) and (12) implie that the estimator defined by (9) belongs to the variance-reduction family.

## 5.2. The proposed stratified estimator

PROPOSITION 5.2: *The estimator defined by*

$$\tilde{\mathcal{Z}} = g_l(\Omega) + \sum_{h=1}^{h(\Omega)} \hat{Z}(K_h, U_h(\Omega)) \pi_h(\Omega) \qquad (13)$$

*verifies*

$$E\{\tilde{\mathcal{Z}}\} = g(\Omega) = E\{\tilde{\mathcal{F}}\} \qquad (14)$$

*and*

$$\text{Var}\left\{\tilde{\mathcal{Z}}\right\} \leq \text{Var}\left\{\tilde{\mathcal{F}}\right\} \leq \text{Var}\left\{\hat{\Phi}\left(\vec{\mathcal{C}}\right)\right\} \tag{15}$$

*where*

$$\tilde{\mathcal{Z}}\left(K_h, U_h\left(\Omega\right)\right) = \frac{1}{K_h} \sum_{i=1}^{K_h} \mathcal{Z}^{(i)}\left(U_h\left(\Omega\right)\right)$$

*is the sample mean based on $K_h$ independent trials using the distribution function of $\mathcal{Z}\left(U_h\left(\Omega\right)\right)$.*

*Proof:* By replacing $R$ by $U_h\left(\Omega\right)$ in equality (11), we obtain that $E\left\{\hat{\mathcal{Z}}\left(K_h, U_h\left(\Omega\right)\right)\right\} = g\left(U_h\left(\Omega\right)\right)$ and then,

$$E\left\{\tilde{\mathcal{Z}}\right\} = g_l\left(\Omega\right) + \sum_{h=1}^{h\left(\Omega\right)} \pi_h\left(\Omega\right) E\left\{\hat{\mathcal{Z}}\left(K_h, U_h\left(\Omega\right)\right)\right\}$$
$$= g_l\left(\Omega\right) + \sum_{h=1}^{h\left(\Omega\right)} \pi_h\left(\Omega\right) g\left(U_h\left(\Omega\right)\right) = g\left(\Omega\right)$$

For the variance, by applying the result (12) with $R$ equal to $U_h\left(\Omega\right)$, we obtain

$$\text{Var}\left\{\hat{\mathcal{Z}}\left(K_h, U_h\left(\Omega\right)\right)\right\} = \frac{\text{Var}\left\{\mathcal{Z}\left(U_h\left(\Omega\right)\right)\right\}}{K_h}$$
$$\leq \frac{\text{Var}\left\{\mathcal{B}\left(U_h\left(\Omega\right)\right)\right\}}{K_h} = \text{Var}\left\{\hat{\Phi}\left(K_h, U_h\left(\Omega\right)\right)\right\}.$$

As

$$\text{Var}\left\{\tilde{\mathcal{Z}}\right\} = \sum_{h=1}^{h\left(\Omega\right)} \pi_h^2\left(\Omega\right) \text{Var}\left\{\hat{\mathcal{Z}}\left(K_h, U_h\left(\Omega\right)\right)\right\}$$

and

$$\text{Var}\left\{\tilde{\mathcal{F}}\right\} = \sum_{h=1}^{h\left(\Omega\right)} \pi_h^2\left(\Omega\right) \text{Var}\left\{\hat{\Phi}\left(K_h, U_h\left(\Omega\right)\right)\right\}$$

We deduce that

$$\text{Var}\left\{\tilde{\mathcal{Z}}\right\} \leq \text{Var}\left\{\tilde{\mathcal{F}}\right\}.$$

∎

Result (14) implies that $\tilde{\mathcal{Z}}$ is an unbiased estimator of $g(\Omega)$. Result (15) implies that the use of the recursive importance sampling to evaluate the contributions of the strata leads to a more accurate estimator than the stratified estimator proposed in [13].

## 5.3. Algorithmic description

For a given sample size $K$, an estimate of $g(\Omega)$ by the proposed strategy can be obtained from the function *StratifiedSampling2()* with the parameters $K$ and $\Omega$. As in the function *StratifiedSampling1()*, the new function uses the procedure *Bounds ()*.

Function *StratifiedSampling2*$(K, R)$

0. Preliminary step:
   *Bounds* $(R, M, g_l(R), g_u(R), h(R), S(R), \{\pi_1(R), \dots, \pi_{h(R)}(R)\})$
1. Initialization: $Est := 0$
2. For $h := 1$ to $h(R)$ do
   2.1. Evaluate the size $K_h$ : $K_h := K \times \pi_h(R)/\pi(R)$
   2.2. Compute the standard estimation of $g(U_h(R))$ by a sample mean based on $K_h$ trials of $\mathcal{Z}(U_h(R))$ :
       2.2.1. $Est_h := 0$
       2.2.2. For $k := 1$ to $K_h$ do $Est_h := Est_h + ImportanceSampling (U_h(R))$
       2.2.3. $Est_h := Est_h/K_h$
   2.3. Add the contribution of the stratum $h$ to $Est$ : $Est := Est + \pi_h(R) \times Est_h$
3. Return the estimate of $g(R)$ : $return (g_l(R) + Est)$

The function *ImportanceSampling ()* used in step 2.2.2. gives a trial of the random variable $\mathcal{Z}(R)$ defined by the recursive formula (10). It can be described as follows:

Function *ImportanceSampling* $(R)$

1. Check and recursion condition:
   1.1. *Bounds* $(R, M, g_l(R), g_u(R), h(R), S(R), \{\pi_1(R), \dots, \pi_{h(R)}(R)\})$
   1.2. If $(h(R) = 0)$ return $(g_l(R))$
2. Accomplish a trial $h$ of $H(R)$
3. Recursive call: return $(g_l(R) + \pi(R) \times ImportanceSampling (U_h(R)))$

The reader can see [4, 5] for details about this procedure.

## 6. NUMERICAL ILLUSTRATIONS

For all simulation methods presented in this paper we fix the parameter $M$ to 1 and we use two topologies.

The first topology used in [13] and presented in Figure 1 has 10 nodes and 25 arcs. For each arc $j$ of this network, the random discrete capacity $C_j$ is a Bernoulli random variable that takes the value 0 with probability $q$ and the value on the arc $j$ with probability $p = 1 - q$. When the capacity of every arc is fixed at its upper value, the maximum $st$-flow is equal to 71. Consequently a demand greater than 71 at the sink node $t$ can not be satisfied. We consider five different demands $d = 20, 36, 50, 60$ and 71, and the probability $p = 0.9$. The corresponding exact flow network reliability values are tabulated in Column 2 of Table 1.
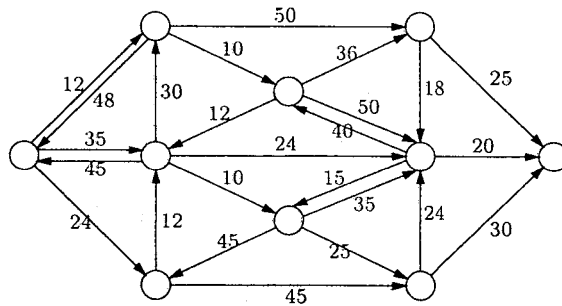


**Figure 1. – The network [13] used for numerical results in Tables 1 and 2.**

In Table 1, we present in Columns 2 and 3 the exact values of $g(\Omega)$ and their estimates by $\tilde{\mathcal{Z}}$ with $K = 2^{16}$. It can be observed that each estimate is close to the corresponding $g(\Omega)$ computed by the exact algorithm of Doulliez and Jamoulle. Also, this algorithm is used to deduce the parameters $g(U(\Omega))$, $g(U_h(\Omega))$ for $h = 1, \ldots, h(\Omega)$. These values serve to tabulate, in Column 4, the variance of $B(\Omega)$, which is equal to $g(\Omega)(1 - g(\Omega))$ and, in Column 5 the variance of $\tilde{\mathcal{F}}$ evaluated by formula (8). Since the variance of $\tilde{\mathcal{Z}}$ is unknown, its unbiased estimator [14]

$$\hat{V}_{\tilde{\mathcal{Z}}} = \sum_{h=1}^{h(\Omega)} \pi_h^2(\Omega) \frac{1}{K_h - 1} \sum_{i=1}^{K_h} (\hat{\mathcal{Z}}_h - \mathcal{Z}_h^{(i)})^2 \qquad (16)$$

is used to obtain estimates of $\mathrm{Var}\{\tilde{\mathcal{Z}}\}$ at Column 6.

TABLE 1

*Evolution of variances of $\hat{\Phi}(\vec{C})$, $\tilde{\mathcal{F}}$ and $\tilde{Z}$ as a function of the demand d on the network in Figure 1, with $p = 0.9$, $M = 1$ and $K = 2^{16}$.*

| $d$ | $g(\Omega)$ | $\tilde{Z}$ | Var $\{\hat{\Phi}(\vec{C})\}$ | Var $\{\tilde{\mathcal{F}}\}$ | Var $\{\tilde{Z}\}$ |
|---|---|---|---|---|---|
| 20 | 0.985506 | 0.985520 | $2.180 \times 10^{-7}$ | $9.495 \times 10^{-8}$ | $5.069 \times 10^{-10}$ |
| 36 | 0.909608 | 0.909620 | $1.255 \times 10^{-6}$ | $6.692 \times 10^{-7}$ | $2.616 \times 10^{-9}$ |
| 50 | 0.667960 | 0.667936 | $3.384 \times 10^{-6}$ | $2.452 \times 10^{-7}$ | $1.064 \times 10^{-9}$ |
| 60 | 0.456285 | 0.456283 | $3.785 \times 10^{-6}$ | $3.130 \times 10^{-8}$ | $3.210 \times 10^{-11}$ |
| 71 | 0.372807 | 0.372810 | $3.568 \times 10^{-6}$ | $2.015 \times 10^{-8}$ | $5.113 \times 10^{-12}$ |

Table 2 compares the performance of $\hat{\Phi}(\vec{C})$, $\tilde{\mathcal{F}}$ and $\tilde{Z}$ defined by (5), (7) and (13), respectively. In Column 2, we give the variance-reduction ratio achieved by the estimator $\tilde{\mathcal{F}}$ when compared to the standard sampling $\hat{\Phi}(\vec{C})$. In Column 3, we present the variance-reduction ratio achieved by the estimator $\tilde{Z}$ relatively to $\hat{\Phi}(\vec{C})$. This ratio is always greater than the variance-reduction ratio achieved by the estimator $\tilde{\mathcal{F}}$. It results that $\tilde{Z}$ is more accurate than $\tilde{\mathcal{F}}$. In Column 4, the quantity $T(\hat{\Phi}(\vec{C}))/T(\tilde{Z})$ gives the ratio of the time required to collect $2^{16}$ trials for $\hat{\Phi}(\vec{C})$ and the time required to collect $2^{16}$ trials for $\tilde{Z}$. Because this ratio is always smaller than 1, the reduced variance has been achieved at a higher cost per trial for $\tilde{Z}$ than for the standard estimator. A concise performance parameter that takes into account both variance-reduction ratio and time ratio is the product

$$\frac{W(\hat{\Phi}(\vec{C}))}{W(\tilde{Z})} = \frac{\text{Var}\{\hat{\Phi}(\vec{C})\}}{\text{Var}\{\tilde{Z}\}} \times \frac{T(\hat{\Phi}(\vec{C}))}{T(\tilde{Z})}.$$

This parameter is called the *speedup* of the estimator $\tilde{Z}$ with respect to $\hat{\Phi}(\vec{C})$ and gives the time that the estimator $\hat{\Phi}(\vec{C})$ requires to obtain the same variance obtained by the estimator $\tilde{Z}$ in one unit of time. The speedup

TABLE 2

*Evolution of the speedup of the estimator $\tilde{Z}$ with respect to $\hat{\Phi}(\vec{C})$ and $\tilde{\mathcal{F}}$ [13], as a function of the demand d on the network in Figure 1, with $p = 0.9$, $M = 1$ and $K = 2^{16}$.*

| d | $\dfrac{\text{Var}\{\hat{\Phi}(\vec{C})\}}{\text{Var}\{\tilde{\mathcal{F}}\}}$ | $\dfrac{\text{Var}\{\hat{\Phi}(\vec{C})\}}{\text{Var}\{\tilde{Z}\}}$ | $\dfrac{T(\hat{\Phi}(\vec{C}))}{T(\tilde{Z})}$ | $\dfrac{W(\hat{\Phi}(\vec{C}))}{W(\tilde{Z})}$ | $\dfrac{W(\tilde{\mathcal{F}})}{W(\tilde{Z})}$ |
|---|---|---|---|---|---|
| 20 | 2.30 | 430 | 0.156 | 67 | 29 |
| 36 | 1.87 | 479 | 0.215 | 103 | 55 |
| 50 | 13.80 | 3180 | 0.238 | 758 | 55 |
| 60 | 120.94 | 117928 | 0.327 | 38610 | 319 |
| 71 | 180.11 | 697790 | 0.461 | 321988 | 1787 |

values, presented in Column 5, are always greater than 1, showing a better performance of the estimator $\tilde{\mathcal{Z}}$.

As the cost of the estimating by $\tilde{\mathcal{F}}$ is about the same as the cost of estimating by $\hat{\Phi}(\vec{\mathcal{C}})$ [13] with the same sample size, the speedup of $\tilde{\mathcal{F}}$ with respect to the standard estimator is close to the variance-reduction ratio, and the speedup of $\tilde{\mathcal{Z}}$ with respect to the stratified sampling estimator can be estimated by the ratio of the values in Columns 5 and 2, for each case. These speedups are tabulated in Column 6. They show that our methodology is more attractive than the stratified sampling in [13].

The speedup of $\tilde{\mathcal{Z}}$ with respect to $\tilde{\mathcal{F}}$ is improved in the case of highly reliable components. For instance, when $p$ is set to 0.99, we obtain the speedups from 18159 to 2048026.

The second topology in Figure 2 is a more interesting example. Its evaluation by the exact algorithm of Doulliez and Jamoulle takes more than week (we have aborted the execution after 8 days). For each arc $j$ of this network, the random discrete capacity $C_j$ takes 0 with probability $q$ and the value on the arc $j$, or 1 if there is no value on the arc, with probability $p = 1 - q$. When the capacity of every arc is fixed at its upper value, the
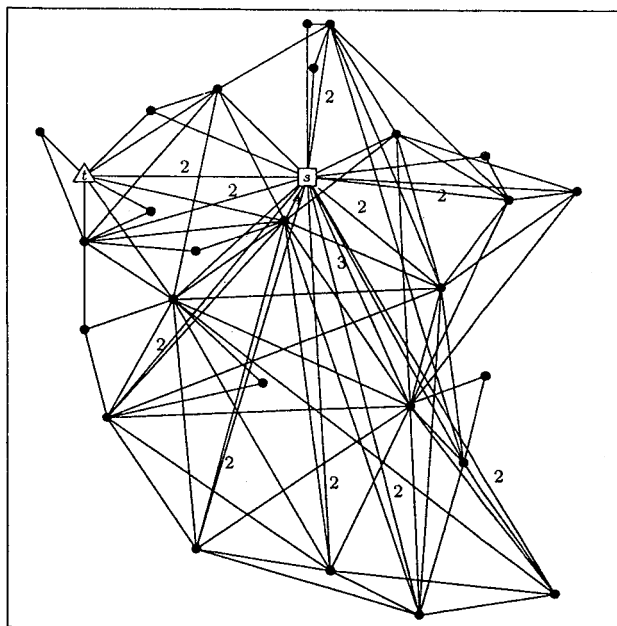


Figure 2. – A version of the French packet switching network TRANSPAC.

TABLE 3
*Evolution of the speedup of the estimator $\tilde{Z}$ with respect to $\tilde{\mathcal{F}}$ [13] and $\tilde{Z}$ [4] on the French packet switching network TRANSPAC, with $p = 0.99$, $M = 1$ and $K = 10^{16}$.*

| $d$ | $\tilde{Z}$ | $\dfrac{T(\tilde{\mathcal{F}})}{T(\tilde{Z})}$ | $\dfrac{\mathrm{Var}(\tilde{\mathcal{F}})}{\mathrm{Var}(\tilde{Z})}$ | $\dfrac{W(\tilde{\mathcal{F}})}{W(\tilde{Z})}$ | $\dfrac{W(\tilde{\mathcal{F}})}{W(\tilde{Z})}$ |
|---|---|---|---|---|---|
| 4 | 0.99999999 | 0.0243 | 2816612.22 | 68498.15 | 308.82 |
| 7 | 0.99906758 | 0.0398 | 2972.05 | 118.38 | 205.98 |
| 9 | 0.90429149 | 0.0273 | 129711.04 | 3544.65 | 996.24 |

maximum *st*-flow is equal to 9. Then a demand greater than 9 at the node $t$ can not be satisfied. For this example, we consider three demands $d = 4, 7, 9$ and $p$ equal to 0.99. For each case, we tabulate in Column 2 the estimate of $g(\Omega)$ by the proposed estimator $\tilde{Z}$ with $K = 10^6$ and $M = 1$. The CPU execution time consummed by each estimate is about 15 hours. In Column 3, we give the ratio of the time required to collect $10^6$ trials for $\tilde{\mathcal{F}}$ and the time required to collect $10^6$ trials for $\tilde{Z}$. In Columns 5 and 6, we tabulate the speedups of the estimator $\tilde{Z}$ with respect to the estimator $\tilde{\mathcal{F}}$ of Fishman and Shaw [13] and the estimator $\tilde{Z}$ [4] respectively. Each speedup value gives the time needed by the corresponding estimator to obtain the same variance obtained by the estimator $\tilde{Z}$ in one unit of time.

## 7. CONCLUSIONS

The problem of the exact evaluation of the probability that the maximum *st*-flow exceeds a fixed value $d$ in a stochastic flow network is an NP-hard problem. Consequently, algorithms to resolve it exactly have a high computational cost. When exact algorithms fail or when their computational time is prohibitive, Monte Carlo methods can supply an estimate in a reasonable time. In this paper, we have proposed to use the decomposition procedure of Doulliez and Jamoulle in order to construct a new stratified sampling estimator. We have shown that it belongs to the variance-reduction family and that it is more accurate than the previous stratified sampling estimator based on the same decomposition [13]. We deduce from the results of several tests that our methodology offers substantial gains with respect to previous simulation methods. Further research can be performed in two directions. On one hand, one can attempt to accomplish the $M$ decompositions such that the bounds are close, at each call of the procedure *Bounds* ( ). This will lead to more accurate estimates. On the other hand, one can quantify the effects of varying the number of decompositions $M$ in order

to establish some rules to suitably fix this parameter. An other interesting point is to study the incorporation of recursive stratification within the undetermined sets.

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