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ON THE NUMERICAL RESOLUTION OF ISAAC'S INEQUALITIES (*)

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Abstract. – *This paper deals with the numerical solution of the bilateral Isaacs' inequality associated to stopping time games. We present two algorithms which converge in a finite numbers of steps. The numerical computational task is considerably reduced by starting at special initial conditions.*

Keywords: Isaacs' inequality, bilateral variational inequalities, accelerated algorithms.

Résumé. – *On étudie la résolution numérique de l'inégalité bilatérale de Isaacs associée aux jeux différentiels avec temps d'arrêt. On présente deux algorithmes qui convergent en un nombre fini d'itérations. Le calcul numérique est réduit grâce aux choix de points de départ convenables.*

Mots clés : Inégalité de Isaacs, inéquation variationnelle bilatérale, algorithmes améliorés.

1. INTRODUCTION

Differential games with stopping times are a well known problem in the field of optimal dynamic decisions (*see* [1, 5, 11]). They originate bilateral variational inequalities (BVI) (*see* [6, 7, 9]) when analyzed with the dynamic programming methodology. Such BVI are solved numerically using fixed point algorithms. The convergence of those algorithms depends on the actualization rate and, in some cases the procedure may converge very slowly. Several algorithms have been devised to accelerate such convergence. For example, Tidball-González in [12], have obtained an algorithm which converges in a finite number of steps, (3^N iterations in the worst case, where N is the cardinal of the discretized space). In this work – continuing the developments presented in [2] – we devise algorithms that finish in a finite

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and small number of steps (number smaller than N in one case and smaller than $2N$ in the other one).

We consider the solution of the system (1)

$$M(w) = P_{[\phi,\,\psi]}(Bw + f), \tag{1}$$

where $P_{[\phi,\,\psi]}(x)$ denotes the Euclidean projection of $x \in \mathbb{R}^N$ on the box

$$[\phi_1,\, \psi_1] \times [\phi_2,\, \psi_2] \times \ldots \times [\phi_N,\, \psi_N].$$

The expression (1) is a bilateral inequality that results from discretizing the Isaacs' inequality associated to a zero sum differential game problem with stopping times (*see* [4, 5, 6, 9, 12]), when the methodology of discretization described in [8] is applied. The vector w represents an approximation to the *value* of the game. Besides, the system (1) can be considered as the dynamic programming equation for the value of a stopping time game defined on a finite Markov chain.

The problem consists in finding the unique fixed point of the operator M, which is a contraction, (*see* [12]); *i.e.*, finding $\overline{w}$ such that

$$\overline{w} = M\,\overline{w}. \tag{2}$$

We analyze the methodology obtained by starting at some suitable initial conditions and we obtain some properties that enables us to modify the usual fixed point algorithm for obtaining convergence in a small number of steps.

2. ELEMENTS OF THE PROBLEM

In what follows B is a $N \times N$ matrix, ϕ, ψ and $f \in \mathbb{R}^N$, and we assume there are $\alpha > 0$ and $\gamma < 1$ such that

$$\left|\begin{array}{ll} b_{ij} \geq 0 & \forall i,\, j = 1,\, \ldots,\, N, \\ Be \leq \gamma\, e, & \\ \psi - \phi \geq \alpha\, e, & \end{array}\right. \tag{3}$$

where e denotes the vector of $\mathbb{R}^N$ such that $e_j = 1$ for all j. Then, from the assumptions on B, it follows that M is a contraction and therefore there exists a unique $\overline{w}$ such that $M\overline{w} = \overline{w}$.

From (1), we deduce that the components $\overline{w}_i$ take only one of the following values

$$\overline{w}_i = \begin{cases} \phi_i & \forall i\, :\, (B\overline{w} + f)_i \leq \phi_i, \\ \psi_i & \forall i\, :\, (B\overline{w} + f)_i \geq \psi_i, \\ (B\overline{w} + f)_i & \forall i\, :\, \phi_i < (B\overline{w} + f)_i < \psi_i. \end{cases} \tag{4}$$

According to these three possible values, we define three index sets

$$S_1 = \{i : \overline{w}_i = \phi_i\},$$
$$S_2 = \{i : \overline{w}_i = \psi_i\},$$
$$C = \{i : \phi_i < \overline{w}_i < \psi_i\}.$$

3. ALGORITHMS WITH SPECIAL INITIAL CONDITIONS

3.1. Preliminary definitions

We now introduce some definitions to describe more easily the properties used in the algorithms presented in this section. Let $\mathbb{N}$ denote the set of non negative integer numbers.

We consider sets S_1^n, S_2^n which are approximations of S_1, S_2. They are suitably defined for each algorithm in order to get a sequence $\{w^n\}$ converging to $\overline{w}$ in a fast way. Starting with B^0, f^0 and w^0 defined in each particular case, we generate sequences $\{B^n\}$, $\{f^n\}$ and $\{w^n\}$ as follows.

$$(b^{n+1})_{ij} = \begin{cases} 0 & \forall\, i \in S_1^n \cup S_2^n \quad \forall\, j \in \{1, \dots, N\}, \\ b_j & \text{otherwise}; \end{cases} \tag{5}$$

$$(f^{n+1})_i = \begin{cases} \phi_i & \forall\, i \in S_1^n, \\ \psi_i & \forall\, i \in S_2^n, \\ f_i & \text{otherwise}, \end{cases} \tag{6}$$

$$w^{n+1} = (I - B^{n+1})^{-1} f^{n+1}; \tag{7}$$

Remark 3.1: It is clear that the sequence $\{w^n\}$ is well defined since, by construction, $(I - B^n)$ is nonsingular. Furthermore w^n is the unique fixed point of the transformation $\xi \mapsto B^n \xi + f^n$.

Remark 3.2: For each $n \in \mathbb{N}\backslash\{0\}$, the vector w^n verifies

$$(w^n)_i = \begin{cases} \phi_i & \forall\, i \in S_1^{n-1}, \\ \psi_i & \forall\, i \in S_2^{n-1}. \end{cases} \tag{8}$$

3.2. Algorithm A1

Let $B^{\circ} = B$, $f^{\circ} = f$ and $w^{\circ} = (I - B)^{-1} f$ be the initial conditions to generate the sequences defined in (5), (6) and (7). Moreover, for each $n \in \mathbb{N}$ we define

$$\hat{S}_1^n = \{j : (w^n)_j < \phi_j, (w^n - \phi)_j = \min_i (w^n - \phi)_i\}; \tag{9}$$

$$\hat{S}_2^n = \{j : (w^n)_j > \psi_j, (w^n - \psi)_j = \max_i (w^n - \psi)_i\}; \tag{10}$$

$$S_1^n = \bigcup_{m=0}^{n} \hat{S}_1^m; \tag{11}$$

$$S_2^n = \bigcup_{m=0}^{n} \hat{S}_2^m; \tag{12}$$

LEMMA 3.1: *For each* $n \in \mathbb{N}$, $S_2^n \subseteq S_2$ *and* $S_1^n \subseteq S_1$.

Proof: We only prove $S_2^\circ \subseteq S_2$. The remaining inclusions can be proved in a similar way.

We denote $\delta = \max_i (w^\circ - \psi)_i^+$, and with the same symbol the vector δe.

It is easy to see that

$$(w^\circ - \delta) \leq \psi \tag{13}$$

$$(w^\circ - \delta)_i = \psi_i, \quad \forall\, i \in S_2^\circ. \tag{14}$$

From (3) and (7), we have $B\delta \leq \delta$ and

$$B(w^\circ - \delta) + f = w^\circ - B\delta \geq w^\circ - \delta. \tag{15}$$

The operator M is monotone, *i.e.* $v_1 \leq v_2 \Rightarrow Mv_1 \leq Mv_2$. From this property, (13) and (15), it follows that

$$M(w^\circ - \delta) = P_{[\phi,\psi]}(B(w^\circ - \delta) + f) \geq P_{[\phi,\psi]}(w^\circ - \delta) \geq w^\circ - \delta. \tag{16}$$

Therefore, for each $k \in \mathbb{N}\backslash\{0\}$

$$M^{(k)}(w^\circ - \delta) \geq M(w^\circ - \delta) \geq w^\circ - \delta, \tag{17}$$

where $M^{(k)}$ denote the composition of the operator M, k times.

As M is a contraction, we have

$$\lim_{k \to \infty} M^{(k)}(w^\circ - \delta) = \overline{w}. \tag{18}$$

Then, from (4), (14), (17) and (18),

$$\psi_i \geq \overline{w}_i \geq (w^\circ - \delta)_i = \psi_i. \tag{19}$$

Then, $\overline{w}_i = \psi_i$, *i.e.* $i \in S_2$. □

Let $M^n w = P_{[\phi,\psi]}(B^n w + f^n)$. Then, M^n is a contraction which has the same fixed point than M.

LEMMA 3.2: *For each $n \in \mathbb{N}$, $\overline{w}$ is the fixed point of operator M^n*, i.e.

$$M^n \overline{w} = \overline{w}. \tag{20}$$

Proof: Let $i \in S_2^{n-1}$. So, $(b^n)_{ij} = 0 \ \forall j$ and $(f^n)_i = \psi_i$. Hence, $M^n(\overline{w})_i = \psi_i$. From the previous lemma, we have $\overline{w}_i = \psi_i$. Then, $M^n(\overline{w})_i = \overline{w}_i$. Similarly, we can obtain that $i \in S_1^{n-1}$ implies $M^n(\overline{w})_i = \overline{w}_i$.

Let $i \notin S_1^{n-1} \cup S_2^{n-1}$. Then, for each $j \in \{1, \ldots, N\}$, $(b^n)_{ij} = b_{ij}$ and $(f^n)_i = f_i$. Hence, $M^n(\overline{w})_i = M(\overline{w})_i$. Since, $\overline{w}$ is the fixed point of the operator M, condition (20) follows. □

Algorithm A1

Step 0: $n = 0$;

Step 1: Compute w^n as in (7).
Construct $\hat{S}_1^n$ and $\hat{S}_2^n$ as in (9) and (10).

Step 2: If $\hat{S}_1^n \cup \hat{S}_2^n = \emptyset$, stop.
Else,
Construct $(b^{n+1})_{ij}$ and $(f^{n+1})_i \ \forall i, j$ as in (5) and (6).
Set $n = n + 1$ and go to Step 1.

Convergence of A1

LEMMA 3.3: *For each $n \in \mathbb{N}$ the following alternative holds*

$$\hat{S}_1^n \cup \hat{S}_2^n \neq \emptyset \quad or \quad w^n = \overline{w}. \tag{21}$$

Proof: $w^n = \overline{w}$ implies $\phi \leq w^n \leq \psi$ and therefore, $\hat{S}_1^n \cup \hat{S}_2^n = \emptyset$. On the other hand, $\hat{S}_1^n \cup \hat{S}_2^n = \emptyset$ implies $\phi \leq w^n \leq \psi$. Then,

$$M^n(w^n) = P_{[\phi,\psi]}(B^n w^n + f^n) = P_{[\phi,\psi]}(w^n) = w^n.$$

From Lemma 3.2, it follows that $w^n = \overline{w}$. □

From Lemma 3.3 the algorithm finds the fixed point of operator M. Let us prove that this occurs in a finite number of iterations.

THEOREM 3.1: *The algorithm converges in at most N steps.*

Proof: Let $\mathrm{card}(X)$ denote the cardinal of the set X. While $\hat{S}_1^n \cup \hat{S}_2^n \neq \emptyset$, we have $\mathrm{card}(C^n) < \mathrm{card}(C^{n-1})$. As there are at most N indices, there exists an index $\overline{n} \leq N$ such that

$$\mathrm{card}(C^{\overline{n}}) = \mathrm{card}(C^{\overline{n}-1}), \tag{22}$$

$$\hat{S}_1^{\overline{n}} \cup \hat{S}_2^{\overline{n}} \neq \emptyset. \tag{23}$$

From (23) and Lemma 3.3,

$$w^{\overline{n}} = \overline{w}.$$

□

3.3. Algorithm A2

Starting with $B^\circ = 0$, $f^\circ = w^\circ = \psi$, we generate sequences $\{B^n\}$, $\{f^n\}$ and $\{w^n\}$ as in (5), (6) and (7). The approximated sets are given by $S_2^\circ = \{1, \ldots, N\}$, $S_1^\circ = \emptyset$ and, for each $n \in \mathbb{N}$,

$$\hat{S}_1^n = \{j : (w^n)_j < \phi_j, (w^n - \phi)_j = \min_i (w^n - \phi)_i\}; \tag{24}$$

$$S_1^{n+1} = S_1^n \cup \hat{S}_1^n; \tag{25}$$

$$S_2^{n+1} = \{i \in S_2^n : (Bw^n + f)_i \geq \psi_i\}. \tag{26}$$

$$\hat{C}^{n+1} = \{i \in S_2^n : (Bw^n + f)_i < \psi_i\}. \tag{27}$$

Clearly,

- from construction we have that $S_2^{n+1} \subseteq S_2^n$;
- it can be proved as in Lemma 3.1 that $S_1^n \subseteq S_1$

LEMMA 3.4: *For each $n \in \mathbb{N}$, $\hat{C}^n \subseteq S_1 \cup C$.*

Proof: From construction, it follows that for each $i \in \hat{C}^n$, $(w^n)_i = \psi_i$. In consequence, $(Bw^n + f)_i < (w^n)_i$. Since M is a monotone operator, we have

$$M^{(k)}(w^n)_i \leq M^{(k-1)}(w^n)_i < (w^n)_i = \psi_i. \tag{28}$$

As M is a contraction, we deduce that $\lim_{k \to \infty} M^{(k)}(\psi) = \overline{w}$.

Therefore, $\overline{w}_i < \psi_i$. □

Algorithm A2

Step 0: $n = 0$, $C^\circ = S_1^\circ = \emptyset$, $S_2^\circ = \{1, \ldots, N\}$, $w^\circ = \psi$

Step 1: Construct $\hat{C}^{n+1}$ as in (27).

If $\hat{C}^{n+1} = \emptyset$, stop.

Else, set

$C^{n+1} = C^n \cup \hat{C}^{n+1}$,

$S_2^{n+1} = S_2^n \backslash \hat{C}^{n+1}$, $\quad S_1^{n+1} = S_1^n$,

$$(b^{n+1})_{ij} = \begin{cases} 0 & \forall\, i \in S_2^{n+1} \quad \forall\, j \in \{1, \ldots, N\}, \\ b_{ij} & \text{otherwise} \end{cases}$$

$$(f^{n+1})_i = \begin{cases} \psi_i & \forall\, i \in S_2^{n+1} \\ f_i & \text{otherwise} \end{cases}$$
$n = n + 1$ and go to Step 2.

Step 2: Compute $w^n = (I - B^n)^{-1} f^n$.
Construct $\hat{S}_1^n$ as in (24).
If $\hat{S}_1^n = \emptyset$, go to Step 1.
Else, set
$$C^{n+1} = C^n \backslash \hat{S}_1^n,$$
$$S_1^{n+1} = S_1^n \cup \hat{S}_1^n, \quad S_2^{n+1} = S_2^n,$$
$$(b^{n+1})_{ij} = \begin{cases} 0 & \forall\, i \in \hat{S}_1^n \quad \forall\, j \in \{1, \ldots, N\}, \\ (b^n)_{ij} & \text{otherwise} \end{cases}$$
$$(f^{n+1})_i = \begin{cases} \phi_i & \forall\, i \in \hat{S}_1^n \\ (f^n)_i & \text{otherwise} \end{cases}$$
$n = n + 1$ and restart Step 2.

Convergence of A2

Now, we prove that when the stopping rule is verified, the fixed point has been found.

LEMMA 3.5: *If $\hat{C}^k = \emptyset$ for some $k \in \mathbb{N}\backslash\{0\}$, then*
$$w^k = \overline{w}. \tag{29}$$

Proof: If $\hat{C}^k = \emptyset$ for some $k \in \mathbb{N}$, we have for each $i \in S_2^k$, $(w^k)_i = \psi_i$ and $(Bw^k + f)_i \geq \psi_i$, so
$$M(w^k)_i = (w^k)_i, \quad i \in S_2^k. \tag{30}$$
For each $i \in S_1^k$, we have $(w^k)_i = \phi_i$ and we know from definition of S_1^k that $(Bw^k + f)_i \leq \phi_i$, which implies
$$M(w^k)_i = (w^k)_i, \quad i \in S_1^k. \tag{31}$$
Finally, from construction of w^k, for each $i \notin S_1^k \cup S_2^k$ it follows that $\phi_i < (w^k)_i < \psi_i$, so
$$M(w^k)_i = (w^k)_i, \quad i \notin S_1^k \cup S_2^k. \tag{32}$$
From (30), (31) and (32), we have
$$M(w^k) = w^k, \tag{33}$$
which implies
$$w^k = \overline{w}. \tag{34}$$
□

We have just proved that the algorithm converges to the fixed point of operator M. Let us prove that this occurs in a finite number of iterations.

THEOREM 3.2: *The algorithm converges in at most* $2N$ *steps.*

Proof: At each time Step 1 is executed, at least one index leaves S_2^t and enters C^t. At each execution of Step 2 at least one index leaves C^t and enters S_1^t. Then, at most $2N$ iterations are needed to complete the computation. □

Remark 3.3: In a similar way and with the same rate of convergence as in the last algorithm, it is possible to construct another one starting at $w^\circ = \phi$.

4. EXAMPLES

In tables I and II we show how S_1^n, S_2^n and C^n evolve when A1 and A2 are used. These examples correspond to $N = 15$ and data randomly generated.

To describe the evolution, we define for each $n \in \mathbb{N}$ the function $I^n : \{1, \ldots, N\} \mapsto \{-1, 0, 1\}$,

$$(I^n)_i = \begin{cases} -1 & \text{if} \quad i \in S_1^n \\ 0 & \text{if} \quad i \in C^n \\ 1 & \text{if} \quad i \in S_2^n. \end{cases}$$

TABLE I
Evolution of A1

Index	Iteration				
	I^0	I^1	I^2	I^3	I^4
1	0	0	0	0	0
2	0	0	0	0	0
3	0	1	1	1	1
4	0	0	0	1	1
5	0	0	1	1	1
6	0	0	0	0	0
7	0	0	0	0	0
8	0	0	0	0	0
9	0	−1	−1	−1	−1
10	0	0	0	0	0
11	0	0	0	0	0
12	0	0	0	0	0
13	0	0	−1	−1	−1
14	0	0	0	0	0
15	0	0	0	0	0

TABLE II
Evolution of A2

Index	Iteration						
	I^0	I^1	I^2	I^3	I^4	I^5	I^6
1	1	1	1	1	0	0	0
2	1	0	0	0	0	0	0
3	1	1	1	1	1	1	1
4	1	1	1	1	1	1	1
5	1	1	1	1	1	1	1
6	1	0	0	0	0	0	0
7	1	0	0	0	0	0	0
8	1	1	1	1	0	0	0
9	1	0	− 1	− 1	− 1	− 1	− 1
10	1	0	0	0	0	0	0
11	1	1	1	1	1	0	0
12	1	1	1	1	0	0	0
13	1	0	0	0	− 1	− 1	− 1
14	1	0	0	0	0	0	0
15	1	0	0	0	0	0	0

Tables III and IV are comparisons of iteration number and CPU times between A1 and A2 for problems specially devised to show that neither algorithm has the best performance in all cases.

In table V we give an example where the algorithm developed in [12] by Tidball-González (algorithm TG) converges more slowly than the algorithms A1 and A2.

TABLE III
Best performance for A1.

	Iterations	Time
A1	4	03,89 s
A2	31	14,55 s

TABLE IV
Best performance for A2.

	Iterations	Time
A1	16	14,39 s
A2	11	05,16 s

TABLE V
Comparison with TG.

	Time
A1	03,89 s
A2	14,55 s
TG	31,94 s

These examples were computed in a PC 486, 50 Mhz, using the programming system MAT-LAB. In the last three examples the data are $N = 33$,

$$\left| \begin{array}{ll} b_{i,\,i+1} = 0.945 & i = 1, \ldots, 32 \\ b_{33,1} = 0.945 & \\ b_{i,\,j} = 0 & \text{otherwise,} \end{array} \right.$$

$$f_i = \begin{cases} -1 & i = 1, \ldots, 32 \\ 999 & i = 33. \end{cases}$$

For the examples shown in table 3 and 5, $\phi_i = -9\,i$ and $\psi_i = \max\{1, -999 + 99\,i\}$. For the example corresponding to table 4, $\phi_i = 0$ and $\psi_i = 999\,i$.

5. CONCLUSIONS AND COMMENTS

The algorithms developed in this paper to solve the discrete stopping time game problem improve the results obtained in [12] for the same problem. In fact, A1 and A2 have polynomial complexity while TG may have exponential complexity. Besides, in Table 5 we have shown an example where A1 and A2 are faster than TG. Our algorithms start at special initial conditions which can be computed easily from the data of the problem.

Actually, the problem is a stopping time game on a Markov chain (*see* [10]), defined as follows

- The chain has N nodes or states "i".
- The transition probabilities p_{ij} are defined by $p_{ij} = b_{ij} / \sum_{k=1}^{N} b_{ik}$.

(We only consider the case $\sum_{k=1}^{N} b_{ik} > 0$ because the other one can be transformed in the previous case through an additional variable).

The results presented here can be further improved by using the Markov chain structure associated to the problem. We present in [3] an algorithm on a hierarchical decomposition of the Markov chain associated to the problem.

The obtained decomposition – comprising the recurrent and transient states of the chain – enables us to divide the linear system appearing in (1) into sub-systems. By using A1 or A2, we obtain the solution of the system (1) after solving a finite number of sub-systems associated to each communicated states class.

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