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RAIRO. Recherche opérationnelle, tome 31, nº 4 (1997), p. 343-362
<http://www.numdam.org/item?id=RO_1997__31_4_343_0>

Recherche opérationnelle/Operations Research
(vol. 31, n° 4, 1997, pp. 343 à 362)

THREE EASY SPECIAL CASES OF THE EUCLIDEAN TRAVELLING SALESMAN PROBLEM (*)

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Communicated by Philippe CHRÉTIENNE

Abstract. – *It is known that in case the distance matrix in the Travelling Salesman Problem (TSP) fulfills certain combinatorial conditions (e.g. the* Demidenko *conditions, the* Kalmanson *conditions or the* Supnick *conditions) then the TSP is solvable in polynomial time. This paper deals with the problem of recognizing Euclidean instances of the TSP for which there is a renumbering of the cities such that the corresponding renumbered distance matrix fulfills the Demidenko (Kalmanson, Supnick) conditions. We provide polynomial time recognition algorithms for all three cases.*

Keywords: Travelling salesman problem, Kalmanson condition, Demidenko condition, Supnick condition, Combinatorial optimization, Geometry, Polynomial algorithms.

Résumé. – *On sait que dans le cas où la matrice des distances du problème de voyageur de commerce (TSP) possède certaines propriétés combinatoires (par exemple les conditions de* Demidenko, *les conditions de* Kalmanson *ou les conditions de* Supnik*), alors le problème est polynomial. Cet article traite du problème de la reconnaissance d'instances euclidiennes de TSP pour lesquelles il existe un renumérotage des villes tel que la nouvelle matrice des distances satisfasse les conditions de Demidenko (Kalmanson, Supnick). Un algorithme polynomial est fourni pour chacun des trois cas.*

Mots clés : Problème du voyageur de commerce ; conditions de Kalmanson, Demidenko, Supnick ; optimisation combinatoire ; géométrie ; algorithmes polynomiaux.

1. INTRODUCTION

The *travelling salesman problem* (RSP) is defined as follows. Given an $n \times n$ distance matrix $C = (c_{ij})$ find a permutation $\pi \in S_n$ that minimizes

(*) Manuscript received March 1995.
This research has been supported by Spezialforschungsbereich F 003 "Optimierung und Kontrolle", Projektbereich Diskrete Optimierung.

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Recherche opérationnelle/Operations Research, 0399-0559/97/04/$ 7.00

the sum $\sum_{i=1}^{n-1} c_{\pi(i)\,\pi(i+1)} + c_{\pi(n)\,\pi(1)}$ (the salesman must visit the cities 1 to n in arbitrary order and wants to minimize the total travel length). This problem is one of the fundamental problems in combinatorial optimization and known to be NP hard. For more information, the reader is referred to the book by Lawler, Lenstra, Rinnooy Kan and Shmoys [8].

Several special cases of the TSP are solvable in polynomial time, due to special combinatorial structures in the distance matrix. A large class of such easy special cases is related to the concept of *pyramidal tours*, *i.e.* permutations $\pi \in S_n$ with $\pi = \langle 1, i_1, i_2, \ldots, i_r, n, j_1, \ldots, j_{n-r-2} \rangle$ where $i_1 < i_2 < \ldots < i_r$ and $j_1 > \ldots > j_{n-r-2}$ hold (for permutations, we use the notation $\pi = \langle x_1, x_2, \ldots, x_n \rangle$ for "$\pi(i) = x_i$ for $1 \leq i \leq n$"). Although the number of pyramidal tours on n cities is exponential in n, a minimum cost pyramidal tour can be determined in $0(n^2)$ time by a dynamic programming approach (*cf.* Gilmore, Lawler and Shmoys [5]). For several classes of specially structured matrices it is known that these matrices always possess an optimal TSP tour which is pyramidal. Among these classes are the class $\mathbb{D}$ of *Demidenko* matrices, the class $\mathbb{K}$ of *Kalmanson* matrices and the class $\$$ of *Supnick* matrices. A symmetric $n \times n$ matrix C is a *Demidenko* matrix ($C \in \mathbb{D}$) if

$$c_{ij} + c_{kl} \leq c_{ik} + c_{jl} \quad \text{for } 1 \leq i < j < k < l \leq n. \tag{1}$$

A symmetric matrix C is a *Kalmanson* matric ($C \in \mathbb{K}$), if it fulfills condition (1) and additionally

$$c_{il} + c_{jk} \leq c_{ik} + c_{jl} \quad \text{for } 1 \leq i < k < l \leq n. \tag{2}$$

A symmetric $n \times n$ matrix C is a *Supnik matrix* ($C \in \$$) if

$$c_{ir} + c_{js} \leq c_{is} + c_{jr} \quad \text{for } 1 \leq i < j \leq n, \quad 1 \leq r < s \leq n, \quad \{i, j\} \cap \{r, s\} = \emptyset. \tag{3}$$

In a famous paper in 1976, Demidenko [3] proved that for the TSP with Demindenko distance matrices there always exists an optimal tour that is pyramidal. Consequently, the TSP with Demidenko distance matrices is efficiently solvable. Since $\mathbb{K} \subseteq \mathbb{D}$, this result immediately carries over to Kalmanson matrices. However, here an even stronger statement holds: For symmetric Kalmanson distance matrices, the (pyramidal) identity permutation

$\langle 1, 2, 3, \ldots, n\rangle$ constitutes a shortest TSP tour (*cf.* Kalmanson [7]). Finally, for Supnick matrices the pyramidal permutation $\langle 1, 3, 5, 7, \ldots, 8, 6, 4, 2\rangle$, *i.e.* first the odd cities in increasing order and then the even cities in decreasing order, yields an optimal tour (*cf.* Supnick [12]).

Another important special case of the TSP is the *Euclidean* TSP: here the cities are points in the two-dimensional plane and their distances are measured according to the Euclidean metric. It is easy to see that in this case, the shortest TSP tour does not intersect itself (*cf.* Flood [4]) and hence, geometry makes the problem somewhat easier. Nevertheless, this special case is still NP-hard (*see* e.g. Papadimitriou [6) or chapter 3 in the TSP book [8]).

The subject of this paper is to identify easy instances of the Euclidean TSP based on the concept of Demidenko (Kalmanson, Supnick) matrices: trivially, the length of the optimum TSP tour does not depend on the original numbering of the cities. However for some of the numberings, the distance matrix may fulfill the Demidenko (Kalmanson, Supnick) conditions whereas for other numberings it does not. Hence, the problem arises of finding numberings of the cities such that the resulting matrix fulfills the Demidenko (Kalmanson, Supnick) conditions. The corresponding algorithmic problem is called "recognition of *permuted* Euclidean Demidenko (Kalmanson, Supnick) matrices". In this paper, we will derive the following results.

(a) Permuted $n \times n$ Euclidean Demidenko matrices can be recognized in $O(n^4)$ time.

(b) Permuted $n \times n$ Euclidean Kalmanson matrices can be recognized in $O(n^2)$ time.

(c) Permuted $n \times n$ Euclidean Supnick matrices are trivial to recognize: with a small number of exceptions only point sets in one-dimensional subspaces have Supnick distance matrices.

Our methods strongly exploit geometric structures in the problems like convex subsets and orderings along convex hulls, points lying on the branch of certain hyperbolas, intersection points of certain related hyperbolas and so on.

Organization of the paper. Sections 2 and 3 summarize elementary results and definitions for Kalmanson and Deminko matrices: Section 2 deals with combinatorial preliminaries, Section 3 with geometric preliminaries. The recognition problem of permuted Euclidean Kalmanson matrices is treated in Section 4 and permuted Euclidean Demidenko matrices are treated in Section 5. Section 6 gives a full characterization of Euclidean Supnick matrices. Finally, Section 7 closes with the discussion.

2. COMBINATORIAL PRELIMINARIES AND DEFINITIONS

In this section, several basic definitions for permutations and matrices are given and elementary properties of Demidenko, Kalmanson and Supnick matrices are summarized.

For an $n \times n$ matric C, denote by $I = \{1, \ldots, n\}$ the set of rows (columns). A row i *precedes* a row j in C ($i \prec j$ for short), if row i occurs before row j in C. For two sets K_1 and K_2 of rows, we write $K_1 \prec K_2$ iff $k_1 \prec k_2$ for all $k_1 \in K_1$ and $k_2 \in K_2$.

For $V = \{v_1, v_2, \ldots, v_r\}$ a subset of I, we denote by $C[V]$ the $r \times r$ submatrix of C which is obtained by deleting all rows and columns not contained in V.

The identity permutation is denoted by ε, *i.e.* $\varepsilon(i) = i$ for all $i \in I$. For a permutation ϕ, the permutation ϕ^- defined by $\phi^-(i) = \phi(n - i + 1)$ is called the *reverse permutation* of ϕ. Permutation ϕ is called a *cyclic shift* or *a rotation* if there exists a $k \in I$ such that $\phi = \langle k, k+1, \ldots, n, 1, \ldots, k-1\rangle$.

By C_ϕ we denote the matrix which is obtained from matrix C by permuting its rows and columns according to ϕ, *i.e.* $C_\phi(c_{\phi(i), \phi(j)})$. A permutation ϕ is called a *Demidenko* (*Kalmanson, Supnick*) *permutation for some matrix* C iff C_ϕ is a Demidenko (Kalmanson, Supnick) matrix.

For a partition $X = \langle X_1, \ldots, X_x\rangle$ of I into x subsets, the set $\mathrm{S_{TR}}(X_1, \ldots, X_x)$ contains all permutations ϕ that fulfill $\phi(x_1) \prec \phi(x_j)$ for all $x_i \in X_i$ and $x_j \in X_j$ with $1 \le i < j \le x$. $\mathrm{S_{TR}}(X_1, \ldots, X_x)$ is called the set of permutations induced by the sequence of *stripes* $X_1, \ldots, X_x$. Readers that are familiar with the concept of PQ-trees (Booth and Lueker [1]) may observe that the set $\mathrm{S_{TR}}(X_1, \ldots, X_x)$ can be represented by a PQ-tree of height two: the root is a Q-node with x sons. All sons of the root are P-nodes, where the i-th son has the elements in X_i as children.

PROPOSITION 2.1: (*Booth and Lueker* [1]) *For two partitions* $\langle X_1, \ldots, X_x\rangle$ *and* $\langle Y_1, \ldots, Y_y\rangle$ *of* I, *the set* $\mathrm{S_{TR}}(X_1, \ldots, X_x) \cap \mathrm{S_{TR}}(Y_1, \ldots, Y_y)$ *either equals* $\mathrm{S_{TR}}(Z_1, \ldots, Z_Z)$ *for an appropriate partition* $Z = \langle Z_1, \ldots, Z_z\rangle$ *of* I *or it is empty. The partition* Z *can be computed in* $O(|I|)$ *time.* ■

OBSERVATION 2.2: *Let* $D \in \mathbb{D}$, $K \in \mathbb{K}$ *and* $S \in \$$. *Then* $D_{\varepsilon^-} \in \mathbb{D}$, $K_{\varepsilon^-} \in \mathbb{K}$ *and* $S_{\varepsilon^-} \in \$$ *holds, and for any set* $J \subseteq I$, $D[J] \in \mathbb{D}$, $K[J] \in \mathbb{K}$, *and* $S[J] \in \$$. *Moreover, for any cyclic shift* σ, $K_\sigma \in \mathbb{K}$. ■

In other words, reversing a matrix does not destroy the combinatorial structures we are interested in and cyclically renumbering the rows and columns transforms Kalmanson matrices into Kalmanson matrices. For two rows (columns) i and j of C, define

$$\mathcal{M}(i, j) = \{k \in I \backslash \{i, j\} | c_{ik} - c_{jk} = \min_{l \neq i, j} \{c_{il} - c_{jl}\}\}.$$

LEMMA 2.3: *Let C be a symmetric $n \times n$ Kalmanson matrix with $n \geq 4$. Let i and j be two rows of C with $i \prec j$, $K = \mathcal{M}(i, j) \cup \{i\}$ and $K' = I \backslash K$. Then there exists a cyclic shift ϕ such that $C_\phi \in \mathbb{K}$ and $K \prec K'$ in C_ϕ.*

Proof: By definition $i \in K$ and $j \in K'$. Consider some $k \in \mathcal{M}(i, j)$. Then $c_{ik} - c_{jk} = c_{il} - c_{jl}$ for all $l \in K \backslash \{i\}$ and $c_{ik} - c_{jk} < c_{il} - c_{jl}$ for all $l \in K' \backslash \{j\}$. Let $I' = I \backslash \{i, j, k\}$. Distinguish the following three cases on the relative position of i, j and k: (i) $k \prec i \prec j$. The condition (2) implies $p \in K$ for all $p \in I'$ with $k \prec p \prec i$. (ii) $i \prec k \prec j$. By condition (1) $p \in K$ for all $p \in I'$ with $i \prec p \prec k$. (iii) $i \prec j \prec k$. Since $C \in \mathbb{K}$, $p \in K$ for all $p \in I'$ with $k \prec p$ or $p \prec i$.

Summarizing, there exist two elements r and s such that either $K = \{r, \ldots, i, \ldots, s\}$ or $K' = \{s + 1, \ldots, j, \ldots, r - 1\}$. By Observation 2.2 every cyclic shift of C yields again a Kalmanson matrix. Choosing $\phi = \langle r, \ldots, s, \ldots, n, 1, \ldots, r-1 \rangle$ or $\phi = \langle r, \ldots, n, 1, \ldots, s, s+1, \ldots, r-1 \rangle$ completes the argument. ■

Sometimes it is useful to use other, equivalent characterizations of the specially structured matrices. One such characterization of $\mathbb{D}$ was given in [5]:

OBSERVATION 2.4: ([5]) *A symmetric $n \times n$ matrix C is a Demidenko matrix iff*

$$c_{ij} + c_{j+1, l} \leq c_{i, j+1} + c_{j, l} \quad \text{for all } 1 \leq i < j < j + 1 < l \leq n. \tag{4}$$

Below, we use another characterization of $\mathbb{D}$ and $\mathbb{K}$ which is formulated in the following proposition.

PROPOSITION 2.5: *A symmetric $n \times n$ matrix C is a Demidenko matrix iff*

$$\max_{1 \leq i \leq j-1} \{c_{ij} - c_{i, j+1}\} \leq \min_{j+2 \leq l \leq n} \{c_{j, l} - c_{j+1, l}\} \quad \text{for all } 2 \leq j \leq n - 2. \tag{5}$$

A symmetric $n \times n$ matrix C is a Kalmanson matrix iff

$$c_{i,\,j+1}+c_{i+1,\,j}\leq c_{ij}+c_{i+1,\,j+1} \quad for\ all\ 1\leq i\leq n-2,\ i+2\leq j\leq n-1 \tag{6}$$

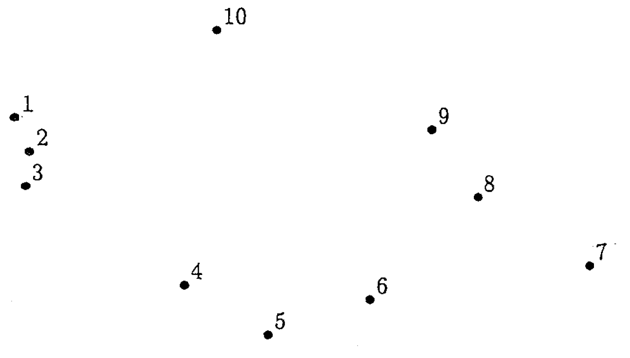
$$c_{i,\,j}+c_{i+1,\,n}\leq c_{n}+c_{i+1,\,1} \quad for\ all\ 2\leq i\leq n-2. \tag{7}$$

OBSERVATION 2.6: *For a symmetric $n \times n$ matrix C, it can be decided in $O(n^2)$ time whether C is a Kalmanson matrix (Demidenko matrix, respectively).*

Proof: Characterization (5) for Demidenko matrices and characterization (6) and (7) for Kalmanson matrices both can be verified in $O(n^2)$ time. ■

3. GEOMETRIC PRELIMINARIES AND DEFINITIONS

This section deals with planar Euclidean point sets whose distance matrices are permuted Demidenko, Kalmanson or Supnick matrices. Let $P = \langle v_1, v_2, \ldots, v_n \subseteq \mathbb{R}^2$ be a sequence of points in the Euclidean plane and let C denote its distance matrix defined by $c_{ij} = d(v_i, v_j)$ where $d(x, y)$ denotes the Euclidean distance between points x and y. If the distance matrix C fulfills the Demidenko (Kalmanson, Supnick) conditions, it is called a *Euclidean Demidenko (Kalmanson, Supnick) matrix* the sequence P is called a *Demidenko (Kalmanson, Supnick) point sequence*, and the points in P are said to form a *Demidenko (Kalmanson, Supnick) point set.* A permutation of P that transforms the distance matrix into a Demidenko (Kalmanson, Supnick) matrix is called a *Demidenko (Kalmanson, Supnick) permutation* for P. For any rearranged subsequence P' of the points in P, we denote by $\Diamond P'$ the sequence of indices in P'.



1	(13,184)
2	(24,157)
3	(21,129)
4	(143, 48)
5	(209, 8)
6	(290, 36)
7	(467, 63)
8	(377,119)
9	(340,173)
10	(169,254)

Figure 1. – A Kalmanson point set.

For $x, y \in \mathbb{R}^2$ and $\Delta \in \mathbb{R}$, denote by $h(x, y, \Delta) = \{p \in \mathbb{R}^2 | d(x, p) - d(y, p) = \Delta\}$ the set of points $p \in \mathbb{R}$ which lie on one (uniquely determined) branch of the hyperbola with focal points at x and y and by $H(x, y, \Delta) = \{p \in \mathbb{R}^2 | d(x, p) - d(y, p) \geq \Delta\}$ the set of points $p \in \mathbb{R}^2$ in the infinite region bounded by $h(x, y, \Delta)$ that does not contain the focal point x. Finally, define $\Delta_i^k = d(v_{k-1}, v_i) - d(v_k, v_i)$ for $2 \leq k \leq n$.

THEOREM 3.1: *A point sequence $P = \langle v_1, \ldots, v_n \rangle$ is a Demidenko point sequence if and only if for each, $4 \leq p \leq n$, the point v_p lies within the region*

$$H_p = \bigcap_{k=3}^{p-1} H(v_{k-1}, v_k, \Delta^k),$$

where $\Delta^k = \max\{\Delta_i^k | i = 1, \ldots, k-2\}$.

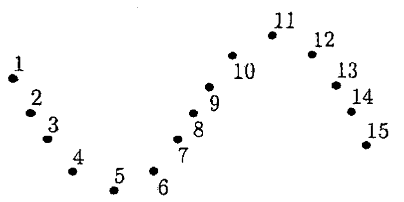
Proof: The proof is done by induction on $p \geq 4$. For $p = 4$, condition (1) must be satisfied, *i.e.* v_4 must be located such that the relation $d(v_2, v_1) + d(v_3, v_4) \leq d(v_2, v_4) + d(v_1, v_3)$ holds. This inequality is equivalent to $v_4 \in H(v_2, v_3, \Delta^3)$ with $\Delta^3 = d(v_2, v_1) - d(v_2, v_1)$.

Next, assume that the statement is true up to $p - 1$ and that the point sequence $\langle v_1, \ldots, v_{p-1} \rangle$ is a Demidenko point sequence. Then we only have to deal with those inequalities where point v_p is involved. By Observation 2.4, it is sufficient to show that condition (4) is fulfilled, *i.e.* that $d(v_j, v_i) - d(v_{j+1}, v_i) \leq d(v_j, v_p) - d(v_{j+1}, v_p)$ for all i and j with $1 \leq i < j + 1 \leq p - 1$ is equivalent to $v_p \in H_p$. Let $k = j + 1$. Then $d(v_{k-1}, v_i) - d(v_k, v_i) \leq d(v_{k-1}, v_p) - d(v_k, v_p)$ is equivalent to v_p in $H(v_{k-1}, v_k, \Delta_i^k)$. Since H_p is the intersection of all $H(v_{k-1}, v_k, \Delta_i^k)$ for $k = 3, \ldots, p - 1$ and $i = 1, \ldots, k - 2$, the theorem follows. ■

In the geometric interpretation, conditions (1) and (2) both correspond to hyperbolas. Taking into account the characterization of $\mathbb{K}$ in Proposition 2.5, Kalmanson point sequences may be characterized in analogy to the above theorem.

THEOREM 3.2: *A point sequence $P = \langle v_1, \ldots, v_n \rangle$ is a Kalmanson point sequence if and only if it is a Demidenko point sequence and if each point $v_p \in P$, $p \geq 4$, belonfs to the region*

$$H_p' = \bigcap_{k=3}^{p-1} \bigcap_{i=1}^{k-2} H(v_{i+1}, v_i, -\Delta_k^{i+1}). \quad ■$$



1	(-22.0, 4.4)	9	(2.0, 3.4)
2	(-20.0, 0.0)	10	(5.0, 7.5)
3	(-18.0, -3.4)	11	(10.0, 10.0)
4	(-15.0, -7.5)	12	(15.0, 7.5)
5	(-10.0,-10.0)	13	(18.0, 3.4)
6	(-5.0, -7.5)	14	(20.0, 0.0)
7	(-2.0, -3.4)	15	(22.0, -4.4)
8	(0.0, 0.0)		

Figure 2. – A Demindenko point set.

Figure 1 gives an illustration for Kalmanson point sequences. Note the depicted point set is "almost convex" and that the numbering gollows the "almost convex hull". All Kalmanson point sets that we constructed in our computation experiments had a similar shape. Figure 2 depicts a Demidenko point sequence. The optimum TSP tour for this point set is $\langle 1, 2, 3, 4, 5, 6, 7, 8, 15, 14, 13, 12, 11, 10, 9\rangle$.

A point set P is called *degenerate* if all points in P lie on a common line and *non-degenerate* otherwise. A point set P is called *convex* if each of its points lies on the boundary of the convex hull. A sequence of points is called *cyclically ordered*, if its points form a convex set and if the numbering corresponds to the clockwise or counterclockwise order along the convex hull. In the case of a degenerated set, a cyclic ordering is one of the two orderings along the line.

Observation 3.3: *Assume that the points v_1, v_2, v_3 and v_4 (in this order) form a non-degenerate convex quadrangle. Then*

(i) $d(v_1, v_3) + d(v_2, v_4) \geq d(v_1, v_2) + d(v_3, v_4)$ *and* $d(v_1, v_3) + d(v_2, v_4) \geq d(v_2, v_3) + d(v_1, v_4)$ (i.e. *the total length of the diagonals is greater than the total length of two opposite sides).*

(ii) *Up to cyclic shifts,* $\langle v_1, v_2, v_3, v_4\rangle$ *and* $\langle v_4, v_3, v_2, v_1\rangle$ *are the only permutations that yield Kalmanson sequences.* ■

The following proposition is an easy consequence of Observation 3.3(i) above.

Proposition 3.4: *(Kalmanson* [7], *folklore). If* $P = \langle v_1, \ldots, v_n\rangle \subseteq \mathbb{R}^2$ *is a non-degenerate, convex, cyclically ordered sequence of points, then its distance matrix is a Kalmanson matrix. Moreover, up to cyclic shifts, this* *and its reverse permutation are the only orderings for the points in P that yield Kalmanson sequences.* ■

In case the Euclidean coordinates of all n points of a convex set are explicitly given, a cyclic ordering (and thus a numbering that makes the point set a Kalmanson sequence) can be found in $O(n \log n)$ time by applying a standard convex hull algorithm (*see* e.g. Preparata and Shamos [10]). In case the coordinates of the points are not given explicitly, but only implicitly via the distance matrix, numerical and computational difficulties arise: In order to compute the exact coordinates from the distances, computations with irrational numbers are to be performed. This will lead to rounding errors and to numerical instabilities. Moreover, the computational standard models (Turing machine, random access machine) cannot cope with irrational numbers. For these reasons, all algorithms in this paper will be designed in such a way that they work directly with the distance matrix and without intermediate computation of Euclidean coordinates.

LEMMA 3.5: *For the Euclidean distance matrix of a convex point set P, the index sequence of a cyclic ordering of the points in P can be computed in $O(n \log n)$ time without intermediate computation of Euclidean coordinates.*

Proof: The cyclic ordering is easy to find if one has two adjacent points x and y on the convex hull. One can check that $d(x, v) - d(y, v)$ must not decrease as we visit the points v by walking on the hull from x to y (the difference may remain constant for some time, for points in P on the line through x and y, but else it increases). Therefore, the correct ordering can be found by sorting. In order to find x and y, we start with two arbitrary points x and z and select $y \in P\backslash\{x\}$ so that $d(x, y) - d(z, y)$ becomes minimum. ■

LEMMA 3.6: *If all points of a Euclidean point set P lie on a common line, then the distance matrix of P is a permuted Demidenko, Kalmanson and Supnick matrix.*

Proof: Verify that if the points are sorted along the line, then the resulting distance matrix fulfills all conditions (1), (2), and (3). ■

4. PERMUTED EUCLIDEAN KALMANSON MATRICES

This section deals with the problem of recognizing permuted Euclidean Kalmanson matrices. For our purposes, the most important case of Kalmanson point sequences consists of two points v_1 and v_n and $n - 2$ points lying on some hyperbola branch $h(v_1, v_n, \Delta)$. The following two lemmas deal with this case.

LEMMA 4.1: *Let $P = \langle v_1, v_2, \ldots, v_n \rangle$ be a Kalmanson sequence for which all points, v_i, $2 \leq i \leq n-1$, lie on $h(v_1, v_n, \Delta)$. If $\Delta \geq 0$, then the points in $P \backslash \{v_1\}$ form a convex set and if $\Delta \leq 0$, the points in $P \backslash \{v_n\}$ form a convex set.*

Proof: We only deal with $\Delta \geq 0$, since the other case is symmetric. Hence, let $\Delta \geq 0$ and suppose that $P \backslash \{v_1\}$ is not a convex set. Let V_1 contain those points of P which lie above or on the line L through v_1 and v_n, and let V_2 contain those points below L. Since $P \backslash \{v_1\}$ is not convex, $V_1 \neq \emptyset$. Let $v_a \in V_1$ and $v_b \in V_2$ be the points at minimum distance to L. The line through v_a and v_b crosses the line segment connecting v_1 to v_n (otherwise $P \backslash \{v_1\}$ would be convex). This yields that v_1, v_a, v_n and v_b (in this order) form a convex quadrangle and contradict Observation 3.3 (ii). ■

LEMMA 4.2: *Let $P = \{v_1, v_2, \ldots, v_n\}$ be a point set for which all points v_i, $2 \leq i \leq n-1$, lie on $h(v_1, v_n, \Delta)$. Then there exist at most two Kalmanson permutations for P that have v_1 as first point and v_n as last point. These two permutations can be computed in $O(n \log n)$ time.*

Proof: Lemma 4.1 yields that in case a Kalmanson permutation with the stated properties exists, then $\{v_2, \ldots, v_{n-1}\}$ forms a convex set together with, say, point v_1. By Proposition 3.4, the only orderings that turn a convex set into a Kalmanson sequence, are the clockwise and counterclockwise orderings along the convex hull and cyclic shifts of these permutations. Since v_1 is the first point in the sequence, the cyclical ordering is anchored at v_1 and thus fixed up to orientation. Lemma 3.5 yields the time bound. ■

Next, a polynomial time recognition algorithm for permuted Euclidean Kalmanson matrices will be designed in two phases. In the first phase, we investigate the special case where the index p of the first point and the index q of the last point in the Kalmanson permutation are *a priori* known. The second phase treats the general problem without any restrictions.

LEMMA 4.3: *Let C be the Euclidean distance matrix of some planar point set $P = \{v_1, \ldots, v_n\}$ and let v_p and v_q be two points in P. Then it can be decided in $O(n^2)$ time whether there is a Kalmanson permutation that has v_p as first point and v_q as last point.*

Proof: The algorithm is mainly based on the above Lemmata 4.1 and 4.2, and it uses the fact that every cyclic shift of a Kalmanson sequence again is a Kalmanson sequence (*cf.* Observation 2.2). The algorithm consists of the following five Steps (A1)-(A5).

(A1) Compute $\Delta(v) = d(v_p, v) - d(v_q, v)$ for all points v in $P \backslash \{v_p, v_q\}$, and sort them by increasing $\Delta(v)$ values. By grouping points with identical Δ-values together, a partition of the set into $m \leq n - 2$ subsets P_i obtained, $1 \leq i \leq m$, such that all points in P_i have the same $\Delta(v)$ value Δ_i and $\Delta_i < \Delta_{i+1}$ for $1 \leq i \leq m - 1$.

Since v_p and v_q are the first and the last point in the Kalmanson point sequence, $d(v_p, v) - d(v_q, v) \leq d(v_p, w) - d(v_q, w)$ must hold for all points v preceding point w just to fulfill condition (1). Hence, each set P_i must precede set P_{i+1} in a Kalmanson sequence, and the set of potentially feasible permutations is described by $\mathrm{STR}(\{p\}, \diamond P_1, \ldots, \diamond P_m, \{q\})$

(A2) For every set P_i with $s = |P_i| > 1$, $1 \leq i \leq m$ do: if $\Delta_i \leq 0$, construct a cyclic ordering σ_i' of the points $P_i \cup \{v_p\}$, otherwise construct a cyclic ordering σ_i' of the points $P_i \cup \{v_q\}$. This yields a permutation $\sigma_i' = \langle p, x_1, \ldots, x_s \rangle$ or $\sigma_i' = \langle x_1, \ldots, x_s, q \rangle$ of the indices of the points in P_i. Set $\sigma_i'' = \langle x_1, \ldots, x_s \rangle$.

If $m = 1$, compute two permutations according to Lemma 4.2. Check whether one of them indeed yields a Kalmanson sequence. Stop.

Note that every set P_i is located on the branch of a hyperbola. Lemma 4.1 yields that for every i, $P_i \cup \{v_p\}$ or $P_i \cup \{v_q\}$ is a convex set (depending on the sign of Δ_i. Similarly as in Lemma 4.2 this implies that for every such convex set the only orderings that turn the set into a Kalmanson sequence, are the clockwise and counterclockwise orderings along the convex hull. These orderings are computed (up to orientation) in Step (A.2), and it remains to determine the right orientation for every ordering.

(A3) For every permutation $\sigma_i'' = \langle x_1, \ldots, x_s \rangle$ of a set P_i with $|P_i| = s > 1$ do: compute the value $\Psi_i = d(v_p, v_{x_1}) - d(v_p, v_{x_s})$.

If $d(v, v_{x_1}) - d(v, v_{x_s}) = \Psi_i$ for all $v \in P \backslash P_i$, then find two permutations for $P \backslash P_i \cup \{v_{x_1}, v_{x_s}\}$ as in Lemma 4.2. In both permutations, replace the sequence x_1, x_s by σ_i'' (respectively, x_s, x_1 by $(\sigma_i'')^-$). Check whether one of them indeed yields a Kalmanson sequence and whether it (or one of its cyclic shifts) has v_p and v_q as first and last point. Stop.

Consider the branch $h(v_{x_1}, v_{x_s}, \Psi_i)$. It contains v_p by definition and it is not hard to see that it also contains v_q. In case this branch also covers all other points in $P \backslash P_i$. Lemma 4.1 applies to the set $P \backslash P_i \cup \{v_{x_1}, v_{x_s}\}$. We know that in any feasible Kalmanson sequence, P_i is a contiguous subsequence and hence we may replace the two indices x_1 and x_s by an appropriate cyclic ordering of P_i.

(A4) Otherwise, there exists some point v with $d(v, v_{x_1}) - d(v, v_{x_s}) \neq \Psi_i$. If $v \in P_1 \cup \ldots \cup P_{i-1}$ and $d(v, v_{x_1}) - d(v, v_{x_s}) < \Psi_i$ or if $v \in P_{i+1} \cup \ldots \cup P_m$ and $d(v, v_{x_1}) - d(v, v_{x_s}) > \Psi_i$ then $x_1 \prec x_s$ in σ_i and otherwise $x_s \prec x_1$ in σ_i. Set $\sigma_i = \sigma_i''$ or $\sigma_i = (\sigma_i'')^-$, depending on the relative placement of x_1 and x_s.

(A5) For every P_i of cardinality one, $P_i = \{v_z\}$, define $\sigma_i = (z)$. Compute σ by glueing together $p, \sigma_1, \ldots, \sigma_m, q$. Test if $C_\sigma \in \mathbb{K}$. Stop.

If the algorithm branches into (A4), then there exists some points $v \notin h(v_{x_1}, v_{x_s}, \Psi_i)$. Assume without loss of generality that $v \in P_1 \cup \ldots \cup P_{i-1}$ and that $d(v, v_{x_1}) - d(v, v_{x_s}) < \Psi_i$ (all other cases are symmetric). The problem boils down to deciding whether the ordering $\langle v_p, v, v_{x_1}, v_{x_s}, v_q \rangle$ or whether $\langle v_p, v, v_{x_s}, v_{x_1}, v_q \rangle$ is the correct ordering. Since $d(v, v_{x_1}) - d(v, v_{x_s}) < \Psi_i = d(v_p, v_{x_1}) - d(v_p, v_{x_s})$, the second ordering contradicts condition (2). Thus, it is infeasible and v_{x_1} must precede v_{x_s}. Exactly this check is performed in Step (A4).

Finally, in Step (A5) the orderings for the sets P_i are composed to a potential solution permutation σ. Since σ was computed just by investigating *necessary* conditions, we must verify in the end whether it indeed yields a Kalmanson sequence.

The correctness of the algorithm is clear by the above arguments, and it remains to prove the claimed time complexity. The sorting and grouping in Step (A1) is done in $O(n \log n)$ time. Computing the orderings along the convex hulls of all m sets P_i in Step (A2) is done in overall time $O(n \log n)$ by applying the algorithm described in Lemma 3.5. The case $m = 1$ is handled according to Lemma 4.2 in $O(n \log n)$ time. Steps (A3) and (A4) together cos at most $O(n)$ time per set P_i and thus are performed in $O(n^2)$ time. By Observation 2.6, testing permutation σ in Step (A5) takes $O(n^2)$ time. Summarizing, this yields an overall time complexity of $O(n^2)$ and the proof of Lemma 4.3 is complete. ■

THEOREM 4.4: *For the $n \times n$ distance matrix C of a Euclidean point set P, it can be decided in $O(n^2)$ time whether C is a permuted Kalmanson matrix.*

Proof: Applying the algorithm in Lemma 4.3 to each of the $O(n^2)$ pairs of indices (p, q) yields a naive $O(n^4)$ time algorithm for recognizing permuted Euclidean Kalmanson matrices. To improve on this, we generate a small (constant size) set $\mathcal{S}$ of candidate pairs with the following property: in case C is a permuted Euclidean Kalmanson matrix, then there exists a Kalmanson sequence of P in which at least one of the pairs in $\mathcal{S}$ is adjacent. Then we call the algorithm designed in Lemma 4.3 for every index pair $(p, q) \in \mathcal{S}$. By the definition of $\mathcal{S}$, the procedure will succeed for at least one pair and yield a permutation that transforms C into a Kalmanson matrix.

Hence, it remains to explain how to generate the constant size set $\mathcal{S}$ in at most $O(n^2)$ time: choose two arbitrary indices i and j and compute the set $\mathcal{M}(i, j)$. If $|\mathcal{M}(i, j)| \leq 2$, then let $\mathcal{S}$ contain all pairs over $\mathcal{M}(i, j) \cup \{i\}$. If $|\mathcal{M}(i, j)| \geq 3$, observe that all points with index in $\mathcal{M}(i, j)$ lie on $h(v_i, v_j, \Delta)$ for some appropriate Δ and thus form a convex set. Compute the indices k, l and m of three consecutive points on the hull and let $\mathcal{S}$ contain all pairs over $\{k, l, m\}$.

By Lemma 2.3, there exists a cyclic shift that makes the points corresponding to $\mathcal{M}(i, j) \cup \{i\}$ a prefix of some Kalmanson sequence. This justifies the definition of $\mathcal{S}$ in case $|\mathcal{M}(i, j)| \leq 2$ holds. If $|\mathcal{M}(i, j)| \geq 3$ holds, then the ordering of this convex set within a Kalmanson sequence must follow the convex hull (*cf.* Observation 3.4) and thus is fixed up to orientation and up to cyclic shifts. By Lemma 2.3, there exists a Kalmanson permutation that has a prefix this convex set with the point v_i somewhere inbetween. Out of three consecutive points on the hull, at most one pair can be separated by the point v_i in the Kalmanson sequence. ∎

5. PERMUTED EUCLIDEAN DEMIDENKO MATRICES

This section deals with the recognition of permuted Euclidean Demidenko matrices. Our approach is conceptually similar to the approach for Euclidean Kalmanson matrices described in the preceding section. The main difference (and main difficulty) arises from the fact that condition (2) need not be fulfilled by Demidenko matrices. Hence, less combinatorial structure is imposed. e.g. Lemmata 4.1 and 4.2 are not necessarily true for Demidenko point sequences and (worst of all!) cyclic shifts of Demidenko permutations do not necessarily yield Demidenko permutations.

For an Euclidean point set P two points $f_1, f_2 \in P$ are called *a pair of focal points* for P, if there exists a real Δ such that all other points in $P \backslash \{f_1, f_2\}$ lie on $h(f_1, f_2, \Delta)$. We will make use of the following two observations (where the first observation is elementary and the second observation is an easy consequence of the first one).

OBSERVATION 5.1: *Two branches of two not-identical hyperbolas intersect in at most four points.* ■

OBSERVATION 5.2: *A set P of $n \geq 9$ points in the Euclidean plane possesses at most one pair $\{f_1, f_2\}$ of focal points for P.* ■

The polynomial time recognition algorithm is designed in three phases. In the first phase, we deal with the special case where (i) the index p of the first point and the index q of the last point in the Demidenko permutation are *a priori* known and where (ii) v_p and v_q are not a pair of focal points for the underlying point set P. This forms the main part of this section. The second phase treats the complementary case where v_p and v_q are a pair of focal points for P. Finally, in the third phase the general problem without any restrictions is solved.

LEMMA 5.3: *Let C be the Euclidean distance matrix of some planar point set $P = \{v_1, \ldots, v_n\}$, let v_p and v_q be two points in P that are not a pair of focal points for P. Then it can be decided in $O(n^2)$ time whether there is a Demidenko permutation that has v_p as first point and v_q as last point.*

Proof: We will call a Demidenko permutation that has v_p as first point and v_q as last point an *appropriate* Demidenko permutation. The algorithm consists of three STEPS (B1), (B2), and (B3). Recall that Step (A1) in the preceding section only exploited the Demidenko condition (1). Hence, we may start the same way and have (B1) identical to (A1).

(B1) Compute $\Delta(v) = d(v_p, v) - d(v_q, v)$ for all points v in $P \backslash \{v_p, v_q\}$, and sort them by increasing $\Delta(v)$ values. By grouping points with identical Δ-values together, a partition of the set into $m \leq n - 2$ subsets P_i is obtained, $1 \leq i \leq m$, such that all points in P_i have the same $\Delta(v)$ value Δ_i and $\Delta_i < \Delta_{i+1}$ for $1 \leq i \leq m - 1$.

Again in any appropriate Demidenko permutation, set P_i must precede set P_{i+1}. Hence all appropriate Demidenko permutations are contained in $\mathrm{STR}_1 = \mathrm{STR}(\{p\}, \Diamond P_1, \ldots, \Diamond P_m, \{q\})$. Moreover, subset P_i is situated on the hyperbola branch $h(v_p, v_q, \Delta_i)$ with $\Delta_i = \Delta(v)$ for $v \in P_i$. Note that $m \geq 2$, since v_p and v_q are not focal points for P. Consequently, $P_1 \neq P_m$.

Next, select two arbitrary points $v_r \in P_1$ and $v_s \in P_m$. In any appropriate Demidenko permutation, point v_r precedes all points in $P_2 \cup \ldots \cup P_M \cup \{v_q\}$.

Analogously to Step (B1), sort the differences $d(v, v_r) - d(v, v_q)$ for $v \in \bigcup_{i=2}^{m} P_i$ increasingly and obtain another partition $R_1, \ldots, R_\mu$ by grouping points with identical values together. Symmetrically, point v_s comes after all points in $\{v_p\} \cup P_1 \cup \ldots \cup P_{m-1}$. Sorting the differences $d(v, v_p) - d(v, v_s)$ increasingly for all $v \in \bigcup_{i=1}^{m-1} P_i$ results in a third partition $S_1, \ldots, S_\nu$ of $\bigcup_{i=1}^{m-1} P_i$. The way we derived the three partitions implies that any appropriate Demidenko is contained in

$$\begin{aligned} &\mathrm{S}_{\mathrm{TR}_1} \cap \mathrm{S}_{\mathrm{TR}}(\{p\}, \diamond P_1, \diamond R_1, \ldots, \diamond R_\mu, \{q\}) \\ &\quad \cap \mathrm{S}_{\mathrm{TR}}(\{p\}, \diamond S_1, \ldots, S_\nu, \diamond P_m, \{q\}). \end{aligned}$$

These explanations clarify and justify the next Step (B2).

(B2) Select $v_r \in P_1$ and $v_s \in P_m$. Compute $\Delta^1(v) = d(v, v_\tau) - d(v, v_q)$ for all points v in $\bigcup_{i=2}^{m} P_i$ and construct the partition $R_1, \ldots, R_\mu$ by sorting and grouping the points according to their Δ^1-values. Set $\mathrm{S}_{\mathrm{TR}_2} = \mathrm{S}_{\mathrm{TR}}(\{p\}, \diamond P_1, \diamond R_1, \ldots, \diamond R_\mu, \{q\})$. Compute $\Delta^1(v) = d(v, v_p) - d(v, v_s)$ for all points v in $\bigcup_{i=1}^{m-1} P_i$ and construct the partition $S_1, \ldots, S_\nu$ by sorting and grouping the points according to their Δ^2-values. Set $\mathrm{S}_{\mathrm{TR}_3} = \mathrm{S}_{\mathrm{TR}}(\{p\}, \diamond P_1, \diamond S_1, \ldots, \diamond S_\nu, \{q\})$.

Compute $\mathrm{S}_{\mathrm{TR}} = \mathrm{S}_{\mathrm{TR}_1} \cap \mathrm{S}_{\mathrm{TR}_2} \cap \mathrm{S}_{\mathrm{TR}_3}$. In case S_{TR} is empty, stop with the answer "NO APPROPRIATE PERMUTATION EXISTS". Otherwise, there is a partition $T_1, \ldots, T_\kappa$, of P with $T_1 = \{v_p\}$, $T_\kappa = \{v_q\}$, and $|T_i| \leq 4$ for all $2 \leq i \leq \kappa - 1$ such that $\mathrm{S}_{\mathrm{TR}} = \mathrm{S}_{\mathrm{TR}}(\diamond T_1, \ldots, \diamond T_\kappa)$.

Intersecting $\mathrm{S}_{\mathrm{TR}_1}$, $\mathrm{S}_{\mathrm{TR}_2}$ and $\mathrm{S}_{\mathrm{TR}_3}$ is done according to Proposition 2.1 (this proposition also guarantees the existence of stripes T_i). Since the points of every T_i are intersection points of at least two non-identical hyperbola branches (the hyperbolas habe distinct focal points), Observation 5.1 yields $|T_j| \leq 4$ for all $1 \leq j \leq \kappa$. Hence, all that remains to do is to determine the internal orderings in every set T_i. Recall that by condition (5) for two neighboring points p_j and p_{j+1} in a Demidenko sequence

$$\max_{1 \leq i \leq j-1} \{c_{ij} - c_{i,j+1}\} \leq \min_{j+2 \leq l \leq n} \{c_{j,l} - c_{j+1,l}\} \tag{8}$$

must holds. Conversely, if (8) for all neighboring points p_j and p_{j+1} with $2 \leq j \leq n-2$ than this ordering indeed is a Demidenko ordering.

Now consider some fixed permutation π of the elements of some set T_i with $|T_i| \geq 2$. We test for every pair of neighboring indices x and y in this permutation (where x comes before y) whether they fulfill the inequality corresponding to (8) as follows: let Q_1 contain all numbers in $T_1 \cup \ldots \cup T_{i-1}$

together with all numbers in T_i that precede x and y in π (if any exist). Let Q_2 contain all numbers in $T_{i+1} \cup \ldots \cup T_\kappa$ together with all numbers in T_i that succeed x and y in π. The necessary condition to hold is

$$\max_{k \in Q_1}\{c_{kx} - c_{ky}\} \leq \min_{l \in Q_2}\{c_{xl} - c_{yl}\}.$$

The permutation π is called a *nice permutation* for T_i if all of its pairs of neighboring indices pass this test. For sets T_i with $|T_i| = 1$, the unique possible (trivial) permutation is nice by definition.

A similar test is performed for every index x in T_{i-1} and every index y in T_i (with $Q_1 = (T_1 \cup \ldots \cup T_{i-1})\backslash\{x\})$ and $Q_2 = (T_1 \cup \ldots \cup T_\kappa)\backslash\{y\})$. In cases the indices x and y pass this test, they are called *nicely adjacent.*

(B3) Construct a directed auxiliary graph $G = (V, E)$: for any nice permutation for any set T_i with $|T_i| \geq 2$, there is a corresponding vertex in V. If π_1 is nice for T_{i-1}, π_2 is nice for T_i and if the last element in π_1 is nicely adjacent to the first element in π_2, then there is an edge in E going from the vertex corresponding to π_1 to the vertex corresponding to π_2.

Test whether in G there is a directed path going from the (unique) vertex corresponding to T_1 to the (unique)vertex corresponding to T_κ. C is a permuted Demidenko matrix if an only if such a path exists. In case the path exists, a solution permutation can be computed by concatenating all nice permutations along this path.

It is easy to see that the existence of appropriate Demidenko is equivalent to the existence of a connecting path in the auxiliary graph G: in G there are only edges going from permutations corresponding to T_{i-1} to permutations corresponding to T_i. Because of this leveled structure, any path connecting T_1 to T_κ in G must visit exactly one nice permutation for every T_i. Hence, it spans the whole set P. By the definition of "nice" and "nicely adjacent", every pair of adjacent indices along this path fulfills condition (8) and hence, by Proposition 2.5, the corresponding permutation is a Demidenko permutation. On the other hand any Demidenko permutation trivially gives rise to a path connecting T_1 to T_κ.

It remains to analyze the time complexity of the above algorithm. The sorting in Step (B1) costs $O(n \log n)$ time, the grouping operations are done in $O(n)$ time. Analogously, computing $\mathrm{S}_{\mathrm{TR}_2}$ and $\mathrm{S}_{\mathrm{TR}_3}$ in (B2) costs $O(n \log n)$ time. According to Proposition 2.1, intersecting the three sets of permutations is performed in linear time. The auxiliary graph in Step (B3) has at most 24κ vertices and it is easy to verify that the number of edges is also $O(\kappa)$. Testing whether a permutation is nice for some T_i and whether two indices are nicely adjacent amounts to computing the minimum and maximum of two sets with $O(n)$ elements according to (8). Hence, G can be constructed in $O(n\kappa)$ time. Testing for the existence of a connecting path is solved e.g. by Depth-First-Search in time linear in the number of edges and vertices in a graph. Hence, $O(n)$ time is sufficient for this. Since $\kappa \leq n$, the overall time complexity is $O(n^2)$. This completes the proof of Lemma 5.3. ■

LEMMA 5.4: *Let C be the Euclidean distance matrix of some planar point set $P = \{v_1, \ldots, v_n\}$, let v_p and v_q be a pair of focal points for P. Then it can be decided in $O(n^3)$ time whether there is a Demidenko permutation that has v_p as first point and v_q as last point.*

Proof: There are $n - 2$ candidates for the second point p_x in a Demidenko permutation. For a fixed candidate point p_x, all appropriate Demidenko permutations are in $\mathrm{S_{TR}}(\{p\}, \{x\}, I\backslash\{p, x, q\}, \{q\})$. Hence, we are in a situation analogous to that one after Step (B1) in the algorithm in the preceding Lemma 5.3 (*i.e.* $m \geq 2$ holds and $P_1 \neq P_m$). Performing Steps (B2) and (B3) in $O(n^2)$ time per candidate results in $O(n^3)$ overall time as claimed above. ■

THEOREM 5.5: *For the distance matrix C of some Euclidean point set, it can be decided in $O(n^4)$ time whether C is a permuted Demidenko matrix.*

Proof: For $n \leq 8$ check all possible permutations of C in constant time. For $n \geq 9$, test for every pair v_p, $v_q \in P$ whether there is an appropriate Demidenko permutation with v_p as first point and v_q as last point. By Lemmata 5.3 and 5.4, this takes $O(n^2)$ time for every non-focal pair of points and $O(n^3)$ for focal pairs of points. Since by Observation 5.2, there is at most one pair of focal points for P the claimed overall time complexity $O(n^4)$ follows. ■

6. PERMUTED EUCLIDEAN SUPNIK MATRICES

In this section, it will be shown that the combinatorial structure of Supnick point sets is rather primitive: in case a Supnick set contains $n \geq 9$ points, all these points must lie on a common straight line. Hence, Supnick point sets are trivial to recognize. This result was also mentioned without proof in a paper by Quintas and Supnick [11] in 1965. The proof combines the following two propositions.

PROPOSITION 6.1: *Any non-degenerate point set P in the Euclidean plane with $|P| \geq 9$ contains a non-degenerate subset P^* such that (i) $|P^*| = 5$ and (ii) P^* is a convex set.* ■

PROPOSITION 6.2: *Let C be a 5×5 Supnick matrix. Then the our $\langle 1, 3, 5, 4, 2 \rangle$ yields a shortest travelling salesman tour.* ■

A proof for Proposition 6.1 can be found in the book by Lovász [9] (solution to problem 15.31). Proposition 6.2(b) follows from Supnick's result [12] as described in the introduction section.

LEMMA 6.3: *Let P be a Supnick point set with $|P| \geq 9$. Then all points in P lie on a common line.*

Proof: Suppose the contrary and let $\langle v_1, \ldots, v_n \rangle$ be a numbering of P such that the corresponding distance matrix is a Supnick matrix. P fulfills the conditions of Proposition 6.1 and hence contains a convex non-degenerate subsets P^* on five points, without loss of generality $P^* = \langle v_1, v_2, v_3, v_4, v_5 \rangle$. By Proposition 6.2, the induced ordering $\langle v_1, v_3, v_5, v_4, v_2 \rangle$ of the points in P^* yields a shortest tour and obviouly, this tour must follow the convex hull. This in turn implies that the points v_5, v_4, v_2, v_1 (in this order) form a convex quadrangle and also fulfill the Supnick condition $d(v_1, v_4) + d(v_2, v_5) \leq d(v_1, v_5) + d(v_2, v_4)$. This is a contradiction to Observation 3.3(i). ■

THEOREM 6.4: *For the distance matrix C of some Euclidean point set, it can be decided in $O(n^2)$ time whether C is a permuted Supnick matrix.*

Proof: For $n \leq 8$, check all possible permutations whether they yield a Supnick matrix. For $n \geq 9$, check whether C is the distance matrix of a point set on a line and apply Lemma 3.6. ■

7. CONCLUSION AND OPEN PROBLEMS

In this paper we have shown how to recognize in polynomial time Euclidean point sets whose distance matrices fulfill the Demidenko, Kalmanson, or Supnick condition for an appropriate numbering of the points. The applied methods heavily relied on geometric features of the problems and strongly exploited geometric properties like convexity.

Several related questions remain open: for which other "nice" classes of matrices is it polynomial time decidable whether the distance matrix of some given Euclidean point set belongs to this class? One potential candidate for such a nice class are the *symmetric Van der Veen* matrices [13] defined by

$$c_{ij} + c_{kl} \leq c_{il} + c_{jk} \quad \text{for } 1 \leq i < j < k < l \leq n.$$

There is a geometric characterization of Euclidean Van der Veen point sequences via hyperbolas analogous to the characterization in Theorems 3.1 and 3.2 for Demidenko and Kalmanson point sequences. However, we did not succeed in finding a polynomial time recognition algorithm for Van der Veen point sets.

Another problem consists in deriving polynomial time algorithms for recognizing *arbitrary* permuted Demidenko, Kalmanson, and Supnick matrices (that do not necessarily result from Euclidean point sets). Without the geometric structures, such recognition problems clearly become much harder. A first step towards a solution was taken by Deĭneko, Rudolf and Woeginger [2] who showed how to recognize permuted $n \times n$ Supnick matrices in $O(n^2 \log n)$ time. Note that compared to the geometric case, this running time is a $O(n \log n)$ factor slower.

ACKNOWLEDGEMENT

We would like to thank Bettina Klinz for carefully reading the paper and for many helpful comments.

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