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# SOLUTION BEHAVIOUR FOR PARAMETER-DEPENDENT QUASI-VARIATIONAL INEQUALITIES † (*)

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Abstract. – *In the paper the solution behaviour of a class of parameter-dependent quasi-variational inequalities is analysed. By using sensitivity and stability results for monotone variational inequalities and the Implicit Function Theorem of F. H. Clarke, we derive conditions under which the perturbed solution of a parametric quasi-variational inequality is locally unique, lipschitzian and directionally differentiable. These results are particularized in the case of parametric implicit complementary problems.*

Keywords: Quasi-variational inequalities, sensitivity and stability analysis.

Résumé. – *Dans cet article, nous analysons le comportement en fonction des paramètres, de la solution d'un problème d'inéquation quasi-variationnelle. En utilisant les résultats de stabilité et sensibilité sur les inéquations variationnelles monotones et le théorème des fonctions implicites de F. H. Clarke, nous obtenons les conditions sous lesquelles la solution de l'inéquation quasi-variationnelle paramétrée est localement unique, lipschitzienne et directionnellement différentiable. Nous spécialisons ces résultats au cas du problème de complémentarité implicite paramétré.*

Mots clés : Inéquations quasi-variationnelles, analyse de sensibilité et de stabilité.

## INTRODUCTION

The quasi-variational inequalities (QVI's) or, in particular, the implicit complementarity problems (ICP's) represent a very useful framework for modelling of various complicated equilibria, encountered e.g. in mechanics [13, 2, 7] or in mathematical economics [5, 6]. At present there is a considerable literature dealing with this subject, preferably with the existence and uniqueness questions both in the finite- as well as in the infinite-dimensional setting (e.g. [13, 17, 3, 2]). A substantially less number of works

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is devoted to their numerical solution [18, 16] and, as to our knowledge, in only few papers the solution behaviour of a parameter-dependent ICP is analysed [12, 8]. However, such results are important not only from the point of view of stability analysis, but also with respect to optimization problems in which parameter-dependent ICP's arise as side constraints.

The aim of this paper is

(i) to derive conditions under which the map, assigning to the parameter the (set of) solutions of a QVI, is locally unique and lipschitzian in a neighbourhood of a fixed value of the parameter, and

(ii) to show that, under these conditions, the considered selection of the mentioned map is even directionally differentiable and to provide formulae for the evaluation of the directional derivative.

To this purpose we will utilize some strong results concerning monotone variational inequalities (VI's) [19, 20, 14] and some results from the nonsmooth analysis [4, 10]. The obtained results may be directly used in the *a posteriori* analysis of the solved QVI's. Further, they have important implications for the numeical solution of optimization problems with QVI constraints [8]. In the case of ICP's these results could be somewhat simplified.

Throughout the paper we do not pay any attention to the existence questions; it is assumed that the considered QVI's or ICP's possess solutions for the examined values of the parameter.

The reading requires a certain basic knowledge of the theory of variational inequalities and of the lipschitzian analysis. For the reader's convenience we state here at least the definitions of the generalized Jacobian and the contingent derivative.

DEFINITION 1 [4]: Let the operator $H\,[\mathbb{R}^n \to \mathbb{R}^m]$ be lipschitzian near $x_0 \in \mathbb{R}^n$ and let $\sigma_H$ denote the set of points at which $H$ fails to be differentiable. The *generalized Jacobian* of $H$ at $x_0$ is the set of $[m \times n]$ matrices, given by

$$\partial H(x_0) = \operatorname{conv}\{\lim_{i\to\infty} \nabla H(x_i) | x_i \to x_0,\, x_i \notin \sigma_H\}.$$

For $m = 1$, $\partial H(x_0)$ is termed the *generalized gradient* of $H$ at $x_0$.

DEFINITION 2 [1]: Let $H$ be an operator mapping $\mathbb{R}^n$ into $\mathbb{R}^m$, $x_0 \in \mathbb{R}^n$, and $h \in \mathbb{R}^n$ be a given direction. The *contingent derivative* of $H$ at $x_0$ in the direction $h$, denoted $DH(x_0;\, h)$, is the set of all limits of the difference quotient $\dfrac{H(x_0 + \lambda_i h_i) - H(x_0)}{\lambda_i}$, where $\{h_i\}$ is a sequence of directions tending to $h$ and $\{\lambda_i\}$ is a sequence of positive numbers tending to 0.

It is clear that for $H$ being lipschitzian near $x_0$, all vectors from the contingent derivative of $H$ at $x_0$ in the direction $h$ can be obtained as the limits of $\dfrac{H(x_0 + \lambda_i h) - H(x_0)}{\lambda_i}$ for all sequences $\lambda_i \downarrow 0$. If $DH(x_0; h)$ reduces to a singleton for each direction $h$, we say that $H$ is *directionally differentiable* at $x_0$. Instead of $DH(x_0; h)$ we write then $H'(x_0; h)$.

Throughout the paper $\mathbb{R}^n_+$ is the nonnegative orthant of $\mathbb{R}^n$, $A^i$ is the $i$-th row of a matrix $A$ and $x^i$ is the $i$-th component of a vector $x \in \mathbb{R}^n$. $E$ is the unit matrix and $e_i$ is the $i$-th vector of the canonical basis in $\mathbb{R}^n$.

## 1. PROBLEM FORMULATION AND PRELIMINARIES

Let $\mathcal{A}$ be an open set in $\mathbb{R}^n$, $F[\mathcal{A} \times \mathbb{R}^m \to \mathbb{R}^m]$ be a continuously differentiable operator and $g^i[\mathcal{A} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}]$, $i = 1, 2, \ldots, s$, be twice continuously differentiable functions, convex in the third variable. Functions $g^i$ define the closed- and convex-valued multifunction

$$\Gamma(x, y) = \{z \in \mathbb{R}^m | g^i(x, y, z) \leq 0, \quad i = 1, 2, \ldots, s\}. \tag{1.1}$$

Consider now the parameter-dependent QVI:

For a given $x \in \mathcal{A}$, find a vector $y \in \Gamma(x, y)$ such that

$$\langle F(x, y), y' - y \rangle \geq 0 \quad \text{for all } y' \in \Gamma(x, y). \tag{1.2}$$

It is well-known [2] that (1.2) may be equivalently written as the nonsmooth equation

$$y = \mathrm{Proj}_{\Gamma(x, y)}(y - F(x, y)). \tag{1.3}$$

In what follows, we denote $\mathrm{Proj}_{\Gamma(x, y)}(y - F(x, y)) = Z(x, y)$ and $y - Z(x, y) = H(x, y)$ so that (1.3) can be written as $H(x, y) = 0$. To be able to analyse the properties of the implicit map, defined by this equation, we use the directional differentiability of $Z$ [11], an upper estimate of the generalized Jacobian of $H$ as provided by [14] and the Implicit Function Theorem of Clarke [4].

Let $x_0 \in \mathcal{A}$ and $y_0$ be a solution of (1.2) for $x = x_0$. We denote by

$$I(x_0, y_0) = \{i \in \{1, 2, \ldots, s\} | g^i(x_0, y_0, y_0) = 0\} \tag{1.4}$$

the set of "active" indices, and impose the standard "linear independence constraint qualification", known from mathematical programming:

(LI) The partial gradients $\nabla_z g^i(x_0, y_0, y_0)$, $i \in I(x_0, y_0)$, are linearly independent.

If (LI) holds, then the Karush-Kuhn-Tucker theory implies the existence of a unique multiplier vector $\lambda_0 \in \mathbb{R}^s_+$ such that

$$0 \in \begin{bmatrix} F(x_0, y_0) + (\nabla_z G(x_0, y_0, y_0))^T \lambda_0 \\ -G(x_0, y_0, y_0) \end{bmatrix} + \begin{bmatrix} 0 \\ N_{\mathbb{R}^s_+}(\lambda_0) \end{bmatrix}, \tag{1.5}$$

where

$$G(x, y, z) = \begin{bmatrix} g^1(x, y, z) \\ g^2(x, y, z) \\ \vdots \\ g^s(x, y, z) \end{bmatrix}$$

and

$$N_{\mathbb{R}^s_+}(\lambda) = \begin{cases} \text{normal cone to } \mathbb{R}^s_+ \text{ at } \lambda, \text{ provided } \lambda \in \mathbb{R}^s_+ \\ \emptyset \text{ otherwise.} \end{cases}$$

$\lambda_0$ is the Karush-Kuhn-Tucker (K.K.T.) vector associated to the constraints $g^i(x_0, y_0, y_0) \leq 0$, $i = 1, 2, \ldots, s$. The uniqueness of $\lambda_0$ enables us to define another index set

$$J(x_0, y_0) = \{i \in I(x_0, y_0) | \lambda_0^i > 0\} \tag{1.6}$$

which plays an important role in further consideration. Indices from (1.6) specify so-called *strongly active* inequality constraints, whereas the inequalities for $i \in I(x_0, y_0) \backslash J(x_0, y_0)$ are sometimes termed *semiactive*. We denote the single subsets of $I(x_0, y_0) \backslash J(x_0, y_0)$ by $T_i(x_0, y_0)$, where $i$ runs through a suitably chosen index set $\mathbb{K}(x_0, y_0)$. It will also be convenient to work with the Lagrangian

$$L(x, y, z, \lambda) = z - y + F(x, y) + (\nabla_z G(x, y, z))^T \lambda$$

corresponding to the projection operator $Z$.

For an index set $K \subset \{1, 2, \ldots, s\}$ and a vector $d \in \mathbb{R}^s$, we denote by $d_K$ the subvector composed from the components $d^i$, $i \in K$. Analogously, for a matrix $D$ with $s$ rows, $D_K$ denotes the submatrix composed from the rows $D^i$, $i \in K$. To shorten the notation, we will also sometimes drop the arguments at $I$, $J$ and $T_i$.

The following assertion relies on results of Robinson [19, 20].

PROPOSITION 1.1: *Under (LI) the operator Z is lipschitzian near* $(x_0, y_0)$ *and directionally differentiable at* $(x_0, y_0)$. *For a pair of directions* $(h, k) \in \mathbb{R}^n \times \mathbb{R}^m$, *the directional derivative* $Z'(x_0, y_0; h, k) = v$, *where* $(v, u)$ *is the unique pair satisfying the system of equalities and inequalities*

$$
\begin{aligned}
&\nabla_z L(x_0, y_0, y_0, \lambda_0)\, v + (\nabla_z G_I(x_0, y_0, y_0))^T u_I \\
&\quad = -\nabla_x L(x_0, y_0, y_0, \lambda_0)\, h - \nabla_y L(x_0, y_0, y_0, \lambda_0)\, k \\
&\nabla_z G_J(x_0, y_0, y_0)\, v \\
&\quad = -\nabla_x G_J(x_0, y_0, y_0)\, h - \nabla_y G_j(x_0, y_0, y_0)\, k \\
&\nabla_z G_{I \setminus J}(x_0, y_0, y_0)\, v \\
&\quad \leq -\nabla_x G_{I \setminus J}(x_0, y_0, y_0)\, h - \nabla_y G_{I \setminus J}(x_0, y_0, y_0)\, k \\
&u^i = 0 \qquad \text{for} \quad i \notin I(x_0, y_0) \\
&u_{I \setminus J} \geq 0 \\
&(\langle \nabla_x g^i(x_0, y_0, y_0), h \rangle + \langle \nabla_y g^i(x_0, y_0, y_0), k \rangle \\
&\qquad\qquad + \langle \nabla_z g^i(x_0, y_0, y_0), v \rangle)\, u^i = 0 \\
&\quad \mathit{for} \quad i \in I(x_0, y_0) \setminus J(x_0, y_0).
\end{aligned} \tag{1.7}
$$

*Proof*: The operator $z \mapsto z - y + F(x, y)$ is strongly monotone in $z$ uniformly with respect to $x$ and $y$ [5]. This, together with the convexity of functions $g^i(x, y, \cdot)$ and the (LI) condition, imply that the generalized equation, corresponding to operator $Z$, satisfies the Strong Regularity Condition (SRC) [9, 11] at $(x_0, y_0, y_0, \lambda_0)$. Thus $Z$ is lipschitzian near $(x_0, y_0)$. The directional differentiability of $Z$ and the system (1.7) for the computation of the directional derivative follow either from [11, Theorem 2.1] (also based on the results of Robinson) or directly from [20].

For $i \in \mathbb{K}(x_0, y_0)$ we introduce now the matrices

$$
\begin{aligned}
&D_{J \cup T_i}(x_0, y_0, y_0, \lambda_0) \\
&\quad = \begin{bmatrix} \nabla_z L(x_0, y_0, y_0, \lambda_0) & (\nabla_z G_{J \cup T_i}(x_0, y_0, y_0))^T \\ -\nabla_z G_{J \cup T_i}(x_0, y_0, y_0) & 0 \end{bmatrix}, \\
&B^1_{J \cup T_i}(x_0, y_0, y_0, \lambda_0) = \begin{bmatrix} -\nabla_x L(x_0, y_0, y_0, \lambda_0) \\ \nabla_x G_{J \cup T_i}(x_0, y_0, y_0) \end{bmatrix}, \\
&B^2_{J \cup T_i}(x_0, y_0, y_0, \lambda_0) = \begin{bmatrix} -\nabla_y L(x_0, y_0, y_0, \lambda_0) \\ \nabla_y G_{J \cup T_i}(x_0, y_0, y_0) \end{bmatrix}.
\end{aligned}
$$

It can easily be shown [15] that under the imposed assumptions all matrices $D_{J \cup T_i}(x_0, y_0, y_0, \lambda_0)$, $i \in \mathbb{K}(x_0, y_0)$, are nonsingular.

PROPOSITION 1.2: *Let the condition (LI) be fulfilled and assume that for each $i \in \mathbb{K}(x_0, y_0)$, an $[m \times n]$ matrix $P_i^1$ together with a matrix $Q_i^1$ (uniquely) solve the matrix linear equation*

$$D_{J \cup T_i}(x_0, y_0, y_0, \lambda_0)\begin{bmatrix} -P_i^1 \\ Q_i^1 \end{bmatrix} = B^1_{J \cup T_i}(x_0, y_0, y_0, \lambda_0)$$

*and the $[m \times m]$ matrix $P_i^2$ together with a matrix $Q_i^2$ (uniquely) solve the matrix linear equation*

$$D_{J \cup T_i}(x_0, y_0, y_0, \lambda_0)\begin{bmatrix} E - P_i^2 \\ Q_i^2 \end{bmatrix} = B^2_{J \cup T_i}(x_0, y_0, y_0, \lambda_0). \tag{1.8}$$

*Then one has*

$$\partial H(x_0, y_0) \subset \operatorname{conv}\{[P_i^1, P_i^2] \mid i \in \mathbb{K}(x_0, y_0)\}. \tag{1.9}$$

*Proof*: On the basis of the main result from [14] one easily deduces that

$$\partial Z(x_0, y_0) \subset \operatorname{conv}\{[-P_i^1, E - P_i^2] \mid i \in \mathbb{K}(x_0, y_0)\}.$$

As $H(x, y) = y - Z(x, y)$, the assertion follows. $\square$

*Remark*: In the case of strict complementarity $(I(x_0, y_0) = J(x_0, y_0))$ one easily concludes that $H$ is (strictly) differentiable at $(x_0, y_0)$ and

$$\nabla H(x_0, y_0) = [P^1, P^2],$$

where $P^1$, $P^2$ are the unique matrices solving (together with some matrices $Q^1$, $Q^2$) the equations

$$D_I(x_0, y_0, y_0, \lambda_0)\begin{bmatrix} -P^1 \\ Q^1 \end{bmatrix} = B_I^1(x_0, y_0, y_0, \lambda_0),$$

$$D_I(x_0, y_0, y_0, \lambda_0)\begin{bmatrix} E - P^2 \\ Q^2 \end{bmatrix} = B_I^2(x_0, y_0, y_0, \lambda_0),$$

respectively.

*Remark*: Equation (1.8) can equivalently be written as

$$D_{J \cup T_i}(x_0, y_0, y_0, \lambda_0)\begin{bmatrix} P_i^2 \\ -Q_i^2 \end{bmatrix} = \begin{bmatrix} \nabla_z L(x_0, y_0, y_0, \lambda_0) + \nabla_y L(x_0, y_0, y_0, \lambda_0) \\ -\nabla_z G_{J \cup T_i}(x_0, y_0, y_0) - \nabla_y G_{J \cup T_i}(x_0, y_0, y_0) \end{bmatrix}. \tag{1.10}$$

## 2. THE DIRECTIONAL DIFFERENTIABILITY

Throughout this section it is assumed that $x_0 \in \mathcal{A}$, $y_0$ solves the nonsmooth equation $H(x_0, y) = 0$ and $\lambda_0$ is the corresponding K.K.T. vector in the sense of (1.5). Further, we suppose that the assumption (LI) holds.

THEOREM 2.1: *Suppose that all matrices from*

$$\operatorname{conv}\{P_i^2 \mid i \in \mathbb{K}(x_0, y_0)\}$$

*are nonsingular. Then there exists a neighbourhood $\mathcal{O}$ of $x_0$ and a unique lipschitzian operator $\mathcal{S}[\mathcal{O} \to \mathbb{R}^m]$ such that $y_0 = S(x_0)$ and, for every $x \in \mathcal{O}$,*

$$H(x, S(x)) = 0.$$

The above assertion is a direct consequence of the Implicit Function Theorem of Clarke [4] and Prop. 1.2. In the following simple example the appropriate assumptions may be easily verified.

EXAMPLE 2.1: Consider the QVI (1.2) with $m = 2$, $n = 1$,

$$F(x, y) = \begin{bmatrix} -\dfrac{100}{3}\,x + 2\,y^1 + \frac{8}{3}\,y^2 \\ -22.5\,x + 2\,y^2 + \dfrac{5}{4}\,y^1 \end{bmatrix} \tag{2.1}$$

and

$$\Gamma(x, y) = \{z \in \mathbb{R}^2 \mid z^1 \leq 15 - y^2,\ z^2 \leq 20 - y^1\} \tag{2.2}$$

at $x_0 = 1$ and at its solution $y_0 = (10, 5)$. Evidently, the (LI) condition is satisfied, $I(x_0, y_0) = \{1\}$ and $J(x_0, y_0) = \emptyset$. After the appropriate computations one gets

$$\operatorname{conv}\{P_i^2 \mid i \in \mathbb{K}(x_0, y_0)\} = \left\{ \begin{bmatrix} 1 + \alpha & 1 + \dfrac{5}{3}\,\alpha \\ \dfrac{5}{4} & 2 \end{bmatrix} \middle|\ \alpha \in [0, 1] \right\}.$$

Therefore, the nonsingularity assumption of Theorem 2.1 holds and so the QVI, given by (2.1) (2.2), defines on a neighbourhood of $x_0 = 1$ a unique implicit operator $S$ for which $S(x_0) = (10, 5)$.

We turn now our attention to the directional differentiability of $S$. Thereby we employ another Lagrangian

$$\mathcal{L}(x, y, \lambda) = F(x, y) + (\nabla_z G(x, y, y))^T \lambda$$

which may be related directly to the QVI (1.2). Evidently

$$\mathcal{L}(x, y, \lambda) = L(x, y, y, \lambda)$$

and for the derivatives at $(x_0, y_0, \lambda_0)$ one has

$$\begin{aligned} \nabla_x L(x_0, y_0, y_0, \lambda_0) &= \nabla_x \mathcal{L}(x_0, y_0, \lambda_0), \\ \nabla_z L(x_0, y_0, y_0, \lambda_0) + \nabla_y L(x_0, y_0, y_0, \lambda_0) &= \nabla_y \mathcal{L}(x_0, y_0, y_0). \end{aligned} \tag{2.3}$$

THEOREM 2.2: *Under the assumption of Theorem 2.1 the map $S$ is directionally differentiable at $x_0$. For $h \in \mathbb{R}^n$ the directional derivative $v = S'(x_0; h)$ satisfies with a vector $u_I$ the system of equations and inequalities*

$$\begin{aligned} &\nabla_y \mathcal{L}(x_0, y_0, \lambda_0)\, v + (\nabla_z G_I(x_0, y_0, y_0))^T u_I \\ &\quad = -\nabla_x \mathcal{L}(x_0, y_0, \lambda_0)\, h \\ &[\nabla_z G_J(x_0, y_0, y_0) + \nabla_y G_J(x_0, y_0, y_0)]\, v \\ &\quad = -\nabla_x G_J(x_0, y_0, y_0)\, h \\ &[\nabla_z G_{I\setminus J}(x_0, y_0, y_0) + \nabla_y G_{I\setminus J}(x_0, y_0, y_0)]\, v \\ &\quad \leq -\nabla_x G_{I\setminus J}(x_0, y_0, y_0)\, h \\ &u^i = 0 \quad \text{for } i \notin I(x_0, y_0) \\ &u_{I\setminus J} \geq 0 \\ &(\langle \nabla_x g^i(x_0, y_0, y_0), h\rangle + \langle \nabla_y g^i(x_0, y_0, y_0) \\ &\quad + \nabla_z g^i(x_0, y_0, y_0), v\rangle)\, u^i = 0 \\ &for\ \ i \in I(x_0, y_0)\setminus J(x_0, y_0). \end{aligned} \tag{2.4}$$

*Proof*: In the proof we essentially follow the ideas used in the proof of Lemma 1 in [10].

Consider a sequence of positive numbers $t_j \downarrow 0$ which generates a vector from $DS(x_0; h)$, *i.e.*

$$\lim_{j\to\infty} \frac{S(x_0 + t_j h) - S(x_0)}{t_j} = v \in \mathbb{R}^m.$$

(As $S$ is lipschitzian near $x_0$ due to Theorem 2.1, such a sequence exists.) Evidently, by (1.3) for $j$ sufficiently large

$$S(x_0 + t_j h) = Z(x_0 + t_j h, S(x_0 + t_j h)).$$

The operator $Z$ is directionally differentiable due to Proposition 1.1 and thus

$$\begin{aligned} S(x_0 + t_j h) =& Z(x_0, S(x_0)) \\ &+ t_j Z'\left(x_0, S(x_0); h, \frac{S(x_0 + t_j h) - S(x_0)}{t_j}\right) + o(t_j), \end{aligned}$$

again due to the lipschitzian nature of $S$. However $Z(x_0, S(x_0)) = S(x_0)$ and so one gets

$$\begin{aligned} &\frac{S(x_0 + t_j h) - S(x_0)}{t_j} \\ &\quad = Z'\left(x_0, S(x_0); h, \frac{S(x_0 + t_j h) - S(x_0)}{t_j}\right) + o(t_j). \end{aligned}$$

By letting $j \to \infty$ and using the fact that the directional derivative of a lipschitzian map is continuous in the direction variable, we obtain that

$$v = Z'(x_0, y_0; h, v). \tag{2.5}$$

Thus, we have just to modify the system (1.7) accordingly. The introduction of the Lagrangian $\mathcal{L}$ enables to simplify the first equation due to relations (2.3).

To prove the directional differentiability of $S$, it remains to show that (2.4) admits a unique solution $v$ for each direction $h$. Assume by contradiction that (2.5) possesses for a given $h$ two different solutions $v_1$, $v_2$. Evidently, (2.5) may be rewritten to the form

$$\begin{bmatrix} v - Z'(x_0, y_0; u, v) \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ h \end{bmatrix} \tag{2.6}$$

and thus we assume that Equation (2.6) possesses two different pairs of solutions $(u_1, v_1)$ and $(u_2, v_2)$. As $Z$ is directionally differentiable, it implies that

$$\left\| \begin{matrix} y_0 + tv_1 - Z(x_0 + tu_1, y_0 + tv_1) - (y_0 + tv_2) + Z(x_0 + tu_2, y_0 + tv_2) \\ x_0 + tu_1 - (x_0 + tu_2) \end{matrix} \right\| \le o(t). \tag{2.7}$$

Equation (1.3) may also be written in the "inverse function" form

$$\begin{bmatrix} y - Z(w, y) \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ x \end{bmatrix}.$$

The Clarke Inverse Function Theorem implies the existence of a lipschitzian inverse function $G$ defined on a neighbourhood of $(0, x_0)$ such that for $i = 1, 2$

$$G\begin{pmatrix} y_0 + tv_i - Z(x_0 + tu_i, y_0 + tv_i) \\ x_0 + tu_i \end{pmatrix} = \begin{pmatrix} y_0 + tv_i \\ x_0 + tu_i \end{pmatrix}$$

for $t$ sufficiently small. Therefore

$$\begin{aligned} &\left\| \begin{pmatrix} y_0 + tv_1 \\ x_0 + tu_1 \end{pmatrix} - \begin{pmatrix} y_0 + tv_2 \\ x_0 + tu_2 \end{pmatrix} \right\| \\ &\leq L \left\| \begin{matrix} y_0 + tv_1 - Z(x_0 + tu_1, y_0 + tv_1) - (y_0 + tv_2) + Z(x_0 + tu_2, y_0 + t v_2) \\ x_0 + tu_1 - (x_0 + tu_2) \end{matrix} \right\|, \end{aligned}$$

where $L$ is the Lipschitz modulus of $G$. By combining the last inequality with (2.7) we get

$$t \left\| \begin{pmatrix} v_1 - v_2 \\ u_1 - u_2 \end{pmatrix} \right\| \leq o(t),$$

which is the needed contradiction. Thus $DS(x_0; h)$ shrinks to a singleton for each direction $h$ and we are done. ■

As $Z$ is in fact Bouligand-differentiable ($B$-differentiable) (*cf.* [20]), we could also apply Theorem 3.2.3 from [21]. According to this theorem, roughly speaking, the local inverse to a local Lipschitz $B$-differentiable homeomorphism $f$ is $B$-differentiable and its $B$-derivative is the inverse of the $B$-derivative of $f$. In our case it would imply the our implicit map $S$ is even $B$-differentiable.

It can easily be shown that the system (2.3) is equivalent to the linear QVI:

Find $v \in \Omega(v)$ such that

$$\begin{gathered} \langle \nabla_y \mathcal{L}(x_0, y_0, \lambda_0) v + \nabla_x \mathcal{L}(x_0, y_0, \lambda_0) h, v' - v \rangle \geq 0 \\ \text{for all } v' \in \Omega(v), \end{gathered} \tag{2.8}$$

where

$$\begin{aligned} \Omega(v) = \{w \in \mathbb{R}^m | &\langle \nabla_y g^i(x_0, y_0, y_0), h \rangle \\ &+ \langle \nabla_y g^i(x_0, y_0, y_0), v \rangle + \langle \nabla_z g^i(x_0, y_0, y_0), w \rangle = 0 \\ &\text{for } i \in J(x_0, y_0), \\ &\langle \nabla_x g^i(x_0, y_0, y_0), h \rangle + \langle \nabla_y g^i(x_0, y_0, y_0), y \rangle \\ &+ \langle \nabla_z g^i(x_0, y_0, y_0), w \rangle \leq 0 \\ &\text{for } i \in I(x_0, y_0) \backslash J(x_0, y_0)\}. \end{aligned}$$

Indeed, by writing the QVI (2.8) in the form (1.5), we get exactly the system (2.4).

We illustrate now the application of Theorem 2.2 by a simple QVI of Ex. 2.1.

EXAMPLE 2.2: Consider the QVI of Example 2.1 at $x_0 = 1$ and at its solution $y_0 = (10, 5)$. The system (2.4) attains the form

$$\begin{aligned} 2\,v^1 + \frac{8}{3}\,v^2 + u^1 &= \frac{100}{3}\,h \\ \frac{5}{4}\,v^1 + 2\,v^2 &= 22.5\,h \\ v^1 + v^2 &\le 0 \\ u^2 &= 0 \\ u^1 &\ge 0 \\ (v^1 + v^2)\,u^1 &= 0. \end{aligned} \tag{2.9}$$

One easily computes that for $h = 1$ and $h = -1$ system (2.9) possesses the (unique) solutions $(v^1,\, v^2) = (-30,\, 30)$ and $(v^1,\, v^2) = (-10,\, -5)$, respectively. Thus, as $S'(x_0;\, 1) \neq S'(x_0;\, -1)$, the map $S$ is nonsmooth at $x_0$.

In general, solving (2.4) amounts (similarly as in the case of the system (1.7)) to finding an index set $T_i(x_0,\, y_0)$, $i \in \mathbb{K}(x_0,\, y_0)$, such that

$$\begin{gathered} \begin{bmatrix} \nabla_y \mathcal{L}(x_0,\, y_0,\, \lambda_0) \\ \nabla_z G_{J \cup T_i}(x_0,\, y_0,\, y_0) + \nabla_y G_{J \cup T_i}(x_0,\, y_0,\, y_0) \end{bmatrix} v \\ + \begin{bmatrix} \nabla_x \mathcal{L}(x_0,\, y_0,\, \lambda_0) \\ \nabla_x G_{J \cup T_i}(x_0,\, y_0,\, y_0) \end{bmatrix} h \\ + \begin{bmatrix} (\nabla_z G_{J \cup T_i}(x_0,\, y_0,\, y_0))^T \\ 0 \end{bmatrix} u_{J \cup T_i} = 0 \\ [\nabla_z G_{I \setminus (J \cup T_i)}(x_0,\, y_0,\, y_0) + \nabla_y G_{I \setminus (J \cup T_i)}(x_0,\, y_0,\, y_0)]\, v \\ + \nabla_x G_{I \setminus (J \cup T_i)}(x_0,\, y_0,\, y_0)\, h < 0 \\ u_{J \cup T_i} \ge 0. \end{gathered} \tag{2.10}$$

If the cardinality of $\mathbb{K}(x_0,\, y_0)$ is not too large, one has just to solve a few linear systems and check the remaining strict inequalities.

The result of Theorem 2.2 plays an important role in optimization problems with QVI constraints because it essentially enables to apply some effective nondifferentiable optimization methods to composite functions $\Theta(x) := f(x, S(x))$, where $f[\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}]$ is a given (smooth) objective. This approach is particularly advisable, if for all admissible values of $x$ the corresponding QVI's possess unique solutions, *cf.* [8].

Alternatively to the approach presented in this section one could apply the stability theory of Robinson [19, 20] directly to the generalized equation (1.5) which is under a suitable constraint qualification equivalent to the QVI (1.2). In this way the assumption of Theorem 2.1 would be replaced by a different one, but we would get exactly the same formulae for the directional derivative $S'(x_0; h)$.

## 3. IMPLICIT COMPLEMENTARITY PROBLEMS

Consider the QVI given by (1.1), (1.2), where in (1.1) one has $s = m$ and

$$g^i(x, y, z) = \varphi^i(x, y) - z^i, \qquad i = 1, 2, \ldots, m.$$

We denote

$$D(x, y) = \begin{bmatrix} \varphi^1(x, y) \\ \varphi^2(x, y) \\ \vdots \\ \varphi^m(x, y) \end{bmatrix}$$

and observe that the equivalent nonsmooth Equation (1.3) attains now the form

$$H(x, y) = \min\{F(x, y), y - D(x, y)\} = 0, \tag{3.1}$$

where the minimum is taken componentwise.

*Remark*: In (3.1) one easily recognizes the standard form of a parameter-dependent ICP:

For a given $x \in \mathcal{A}$, find a vector $y \in \mathbb{R}^m$ such that

$$F(x, y) \geq 0, \qquad y \geq D(x, y), \qquad \langle F(x, y), y - D(x, y)\rangle = 0.$$

Let $x_0 \in \mathcal{A}$ be fixed and assume that $y_0$ solves the equation $H(x_0, y) = 0$. We introduce the index sets

$$\begin{aligned} M(x_0, y_0) &:= \{i \in \{1, 2, \ldots, m\} \mid F^i(x_0, y_0) < y^i - \varphi^i(x_0, y_0)\} \\ N(x_0, y_0) &:= \{i \in \{1, 2, \ldots, m\} \mid F^i(x_0, y_0) > y^i - \varphi^i(x_0, y_0)\} \\ L(x_0, y_0) &:= \{i \in \{1, 2, \ldots, m\} \mid F^i(x_0, y_0) = y^i - \varphi^i(x_0, y_0)\}, \end{aligned}$$

and similarly as in Section 1 we denote by $T_i(x_0, y_0)$ the single subsets of $L(x_0, y_0)$, where $i$ runs through a suitably chosen index set $\mathbb{K}(x_0, y_0)$. Let $P_i^1, P_i^2$ be the matrices, defined for $i \in \mathbb{K}(x_0)$ by

$$\left.\begin{aligned} (P_i^1)^j &= \left\langle \begin{array}{l} \nabla_x F^j(x_0, y_0) \\ \quad \text{if } j \in M(x_0, y_0) \cup T_i(x_0, y_0) \\ -\nabla_x \varphi^j(x_0, y_0) \\ \quad \text{if } j \in N(x_0, y_0) \cup (L(x_0, y_0) \backslash T_i(x_0, y_0)) \end{array} \right. \\ (P_i^2)^j &= \left\langle \begin{array}{l} \nabla_y F^j(x_0, y_0) \\ \quad \text{if } j \in M(x_0, y_0) \cup T_i(x_0, y_0) \\ e_j - \nabla_y \varphi^j(x_0, y_0) \\ \quad \text{if } j \in N(x_0, y_0) \cup (L(x_0, y_0) \backslash T_i(x_0, y_0)), \end{array} \right. \end{aligned}\right\} \tag{3.2}$$

$j = 1, 2, \ldots, m$. From the definition of the generalized Jacobian it is clear that

$$\partial H(x_0, y_0) \subset \operatorname{conv}\{[P_i^1, P_i^2] \mid i \in \mathbb{K}(x_0, y_0)\}.$$

Hence, due to the Implicit Function Theorem of Clarke, Theorem 1.2 with matrices $P_i^2$ given by (3.2) holds true and ensures thus the existence of the implicit operator $S$ possessing the mentioned properties.

*Remark*: Observe that in this case the (LI) condition automatically holds.

In the sequel we will assume that all matrices from $\operatorname{conv}\{P_i^2 \mid i \in \mathbb{K}(x_0, y_0)\}$ are nonsingular and turn our attention to the directional derivatives of the implicit map $S$ at $x_0$.

PROPOSITION 3.1: *The map $S$ is directionally differentiable at $x_0$. For $h \in \mathbb{R}^n$ the directional derivative $v = S'(x_0; h)$ satisfies the system of equations*

$$\left.\begin{aligned} \nabla_x F_M(x_0, y_0) h + \nabla_y F_M(x_0, y_0) v = 0, \\ -\nabla_x D_N(x_0, y_0) h + v_N - \nabla_y D_N(x_0, y_0) v = 0 \\ \min\{\nabla_x F_L(x_0, y_0) h + \nabla_y F_L(x_0, y_0) v, \\ -\nabla_x D_L(x_0, y_0) h + v_L - \nabla_y D_L(x_0, y_0) v\} = 0. \end{aligned}\right\} \tag{3.3}$$

*Proof*: Due to Theorem 2.2 we need just to show that the directional derivative is given by (3.3). To be able to proceed in the same way as in the proof of Theorem 2.2, we rewrite the Equation (3.1) in the form

$$H(x, y) = y - Z(x, y) = y - \max\{y - F(x, y), D(x, y)\} = 0.$$

It is well-known that

$$(Z^i)'(x_0, y_0; h, v) = \begin{cases} -\nabla_x F^i(x_0, y_0)\, h + v^i - \nabla_y F^i(x_0, y_0)\, v \\ \quad \text{if} \quad i \in M(x_0, y_0) \\ \nabla_x D^i(x_0, y_0)\, h + \nabla_y D^i(x_0, y_0)\, v \\ \quad \text{if} \quad i \in N(x_0, y_0) \\ \max\{-\nabla_x F^i(x_0, y_0)\, h + v^i - \nabla_y F^i(x_0, y_0)\, v, \\ \quad \nabla_x D^i(x_0, y_0)\, h + \nabla_y D^i(x_0, y_0)\, v\} \\ \quad \text{if} \quad i \in L(x_0, y_0). \end{cases}$$

Thus, it remains to use the above expression in the equation $v = Z'(x_0, y_0; h, v)$ and we arrive immediately at system (3.3). □

*Remark*: In [8] a different way is used to ensure the single-valuedness, the lipschitzian behaviour and the directional differentiability of the implicit map, defined by Equation (3.1). It relies on a transformation of the ICP to a strongly monotone variational inequality and then it suffices to apply the results from [19, 11].

We again illustrate the application of the above statement by a simple example.

EXAMPLE 3.1 : Consider the Equation (3.1) with $m = 4$, $n = 1$,

$$F(x, y) = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} y + \begin{bmatrix} 1-x \\ 1-x \\ 1-x \\ 1-x \end{bmatrix} \tag{3.4}$$

and

$$\varphi^i(x, y) = -3.9 + x + 0.1\,(y^i)^2, \qquad i = 1, 2, \ldots, 4, \tag{3.5}$$

at $x_0 = 0$ and at its solution $y_0 = (-2, -3, -3, -2)$. One easily computes that $M(x_0, y_0) = \{1, 4\}$, $N(x_0, y_0) = \emptyset$ and $L(x_0, y_0) = \{2, 3\}$. Therefore, to verify the assumption of the Implicit Function Theorem of Clarke, we have to check the nonsingularity of the matrices

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}, \qquad \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 1-0.2\,y_0^2 & 0 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix},$$

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & 0 & 1-0.2\,y_0^3 & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix}, \qquad \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 1-0.2\,y_0^2 & 0 & 0 \\ 0 & 0 & 1-0.2\,y_0^3 & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

and their convex combinations. All these matrices are indeed nonsingular and so there exists an neighbourhood $\mathcal{O}$ of 0 (in $\mathbb{R}$) and a lipschitzian operator $S\,[\mathcal{O} \to \mathbb{R}^4]$ such that $y_0 = S\,(x_0)$ and for each $x \in \mathcal{O}$, $S\,(x)$ is a solution of the ICP given by (3.4), (3.5). System (3.3) attains the form

$$\begin{aligned} -h + 2\,v^1 - v^2 &= 0 \\ -h - v^3 + 2\,v^4 &= 0 \\ \min\left\{ \begin{bmatrix} -h - v^1 + 2\,v^2 - v^3 \\ -h - v^2 + 2\,v^3 - v^4 \end{bmatrix}, \begin{bmatrix} -h + (1 - 0.2\,y_0^2)\,v^2 \\ -h + (1 - 0.2\,y_0^3)\,v^3 \end{bmatrix} \right\} &= 0. \end{aligned}$$

For $h = \pm 1$ its solutions are $S'\,(x_0;\,1) = (2,\,3,\,3,\,2)$ and $S'\,(x_0;\,-1) = (-0{,}8125,\,-0{,}625,\,-0.625,\,-0.8125)$. Therefore, expectantly, $S$ is nonsmooth at $x_0$.

In [12] a different approach is applied to the study of the solution behaviour for parameter-dependent ICP's, based on the Implicit Function Theorem of Robinson [20]. To fulfil the appropriate requirements, however, one needs to assume that

$$\nabla_y F^i\,(x_0,\,y_0) = e_i - \nabla_y\,\varphi^i\,(x_0,\,y_0) \quad \text{for all } i \in L\,(x_0,\,y_0).$$

One immediately observes that this assumption simplifies substantially also the verification of the nonsingularity requirement in the Implicit Function Theorem of Clarke, because one has to examine only one matrix. This shows its considerable severity (in Example 3.1 for $i = 2$ one has $\nabla_y F^2\,(x_0,\,y_0) = (-1,\,2,\,-1,\,0)$ and $e_2 - \nabla_y\,\varphi^2\,(x_0,\,y_0) = (0,\,1.6,\,0,\,0)$). Our approach is applicable in more general situations, but the effort, connected with the analysis of the solution behaviour, may be rather considerable.

## CONCLUSION

The assumptions of Theorem 2.1 could be somewhat weakened by replacing the generalized Jacobian by the directional derivative of Kummer.

This derivative, for a function $f$ at $x$ in the direction $h$, is the set of all limits of the difference quotient $1/t_i\,[f(x_i + t_i h) - f(x_i)]$, where $x_i \to x$ and $t_i \downarrow 0$. In the appropriate inverse function Theorem [9], one has to require that this derivative does not contain the zero operator for any nonvanishing direction $h$. However, as the evaluation of the appropriate limits could be rather difficult in our case, we have preferred to retain the approximation by generalized Jacobians, developed in [14, 15].

From the viewpoint of both the *a posteriori* solution analysis as well as optimization with QVI constraints it would be desirable to obtain certain stability and sensitivity results also for the case, where $S$ does not admit locally unique solutions. For this generalization nonsmooth analysis offers a variety of effective tools and so we hope that these results could be obtained in a near future.

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