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PROBABILISTIC BOUNDS ON ONE STEP OBJECTIVE/POTENTIAL FUNCTION IMPROVEMENT IN KARMARKAR'S ALGORITHM (*)

by M. MINOUX ([1])

Abstract. – *A detailed probabilistic analysis of the current step of Karmarkar's algorithm is presented. It does not rely on asymptotic probabilistic results and hence its validity is not restricted to "sufficiently large" values of n (the dimension of the space). The main results obtained are probabilistic bounds for both the decrease of the objective function value and the decrease of the potential function value at one single step of the algorithm.*

When compared with those classically derived from worst case analysis, these bounds show that much larger figures of the decrease are obtained with high probability; this may be viewed as a partial explanation of the very good practical behaviour of Karmarkar's algorithm. Finally it is shown that, contrasting with our analysis, results derived from asymptotic analysis only feature poor accuracy in the range of practical interest (n between 1000 and 10^7).

Keywords: Linear programming, interior point methods, probabilistic analysis of algorithms.

Résumé. – *Cet article présente une analyse probabiliste détaillée d'une itération courante de l'algorithme de Karmarkar. L'analyse ne repose pas sur des résultats probabilistes asymptotiques, sa validité n'est donc pas limitée à des valeurs « suffisamment grandes » de n (nombre de variables). Les résultats obtenus sont des bornes probabilistes sur la décroissance de la fonction objectif et la décroissance de la fonction potentiel au cours d'une itération de l'algorithme.*

Lorsqu'elles sont comparées aux résultats classiques de l'analyse du pire cas, ces bornes conduisent à des décroissances beaucoup plus rapides avec une probabilité élevée ; ceci peut expliquer en partie le très bon comportement observé en pratique de l'algorithme de Karmarkar. Finalement on montre que, contrairement aux résultats de notre analyse, des résultats fondés sur une analyse asymptotique sont peu précis pour les valeurs de n intéressantes en pratique (n compris entre 1 000 et 10^7).

Mots clés : Programmation linéaire, méthodes de points intérieurs, analyse probabiliste des algorithmes.

1. INTRODUCTION

The by now widely acknowledged practical efficiency of Karmarkar's algorithm [5] for solving large linear programming problems is primarily due

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to the fact that the number of necessary iterations (to reach a specified final accuracy) is most often very low and appears to increase only very slowly with the space dimension n (number of variables). However worst-case analysis of the algorithm, suggests that the number of necessary iterations may have to grow linearly with n.

In the present paper this important difference between worst-case and practical behaviour is (at least partially) explained through a detailed probabilistic analysis of one typical step of the algorithm.

Earlier investigations on the average behaviour of Karmarkar's algorithm can be found in [7] and [11]. Both authors make use of previously known asymptotic results in probability theory, therefore their conclusions are only valid for "large enough" n. However this kind of analysis does not provide any information about how large n should be in order to get sufficiently accurate estimates on the decrease of the objective function or of the potential function.

In the present paper, a much more detailed analysis is presented, which leads to probabilistic bounds on the decrease of both the objective function value (*see* section 3) and the potential function value (*see* section 4). When compared with those classically derived from worst-case analysis, these bounds show that much larger figures of the decrease are obtained with high probability (numerical results are presented for n ranging from 1000 to 10^7). This provides a partial explanation of the very good practical behaviour of Karmarkar's algorithm.

Finally, the behaviour of our bounds for n growing arbitrary large, is studied in section 5, and the asymptotic values are compared numerically with the exact values of the bounds obtained in sections 3 and 4. This comparison shows that asymptotic analysis only provides poor accuracy in the range of practical interest, namely for n between 1000 and 10^7.

2. KARMARKAR'S ALGORITHM IN BRIEF

Following Karmarkar (1984) the linear program to be solved is supposed to be available in the form:

$$\begin{cases} \text{Minimize } z = c^T \cdot x \\ \text{subject to :} \\ \qquad A \cdot x = 0 \\ \qquad x \in S = \{x / e^T \cdot x = 1,\ x \geq 0\} \end{cases} \qquad (LP)$$

where, if n denotes the number of variables:
$x \in \mathbb{R}^n$ is the n-vector whose coordinates are the variables of the problem,
$c \in \mathbb{R}^n$ is the cost vector,
A is a $m \times n$ real matrix of full rank,
$e = (1, 1, \ldots, 1)^T$ is the n-vector with all coordinates equal to 1,
$S = \{x/x \in \mathbb{R}^n,\ e^T x = 1,\ x \geq 0\}$ is the simplex of $\mathbb{R}^n$.

We also make the standard assumptions:

ASSUMPTION 1: *The optimum objective function value z^* in (LP) is supposed to be known and equal to $z^* = 0$.*

ASSUMPTION 2: *The point $x^0 = \frac{1}{n} e$ (the center of the simplex) is a feasible solution to (LP) or equivalently* $A \cdot e = 0$.

Karmarkar's algorithm generates a sequence of solutions $x^0, x^1, x^2, \ldots, x^k$ starting from an initial feasible solution $x^0 \left(= \frac{1}{n} e\right)$. The current iteration computes x^{k+1} from x^k as follows.

CURRENT ITERATION. Let x^k $(x^k > 0)$ be the current solution

$$D = \operatorname{diag}(x^k) = \begin{pmatrix} x_1^k & & & O \\ & x_2^k & & \\ & & \ddots & \\ O & & & x_n^k \end{pmatrix}.$$

Let T be the *projective transformation* which maps every $x \in S$ to $y \in S$ defined by: $y = \dfrac{D^{-1} x}{e^T \cdot D^{-1} x}$. Through this transformation, (LP) is transformed into the equivalent (*i.e.* having the same optimal solution set) linear program:

$$\begin{cases} \text{Minimize } \bar{c}^T \cdot y \\ \text{subject to :} \\ \qquad \bar{A} \cdot y = 0 \\ \qquad y \in S \end{cases} \qquad (\overline{LP})$$

where: $\bar{c} = D \cdot c$, $\bar{A} = A \cdot D$. Moreover the current solution x^k is mapped to $y^k = \dfrac{D^{-1} x^k}{e^T \cdot D^{-1} x^k} = \dfrac{1}{n} \cdot e$.

Compute g as the projection of $\bar{c}$ onto the subspace: $\{y/\bar{A} \cdot y = 0; e^T \cdot y = 0\}$. Take a step of length θ in the direction $-g$ in transformed space to find:

$$y^{k+1} = \frac{1}{n} e - \theta \frac{g}{\| g \|}.$$

Apply the inverse projective transformation to obtain:

$$x^{k+1} = \frac{D\, y^{k+1}}{e^T D\, y^{k+1}}.$$

(end of current iteration).

Up to now, convergence analysis of Karmarkar's algorithm has essentially been carried out in terms of *worst-case analysis*. Convergence proofs make use of the so-called *potential function*:

$$f(x) = n \log_e c^T \cdot x - \sum_{i=1}^{n} \log_e x_i.$$

[5] has shown that if $r = \dfrac{1}{\sqrt{n(n-1)}}$ (the radius of the largest sphere inscribed in the simplex S), then taking a *fixed step length* $\theta = \alpha \cdot r$ with $\alpha = \dfrac{1}{4}$ results, at each iteration, in a decrease of the potential function f by at least $\delta \geq \dfrac{1}{8} = 0.125$ (for sufficiently large n).

Later on, [3] and, independently, [2], have proved that, at each iteration, there exists a step length value insuring a decrease in the potential function by at least 0.72148... As a consequence of the above results, the number of iterations necessary to obtain:

$$\frac{c^T x^k}{c^T x^0} \leq 2^{-q}$$

for given q (required number of accuracy digits) is $k = \mathcal{O}(qn)$. In other words, the number of iterations is, *in the worst-case*, proportional to the space dimension. However, in practice, Karmarkar's algorithm seems to behave much more efficiently. Table 1 shows some computational results reported in [10] on test problems from from NETLIB $\left(\text{a fixed step length } \theta = 0.99 \cdot r \text{ is used where } r = \dfrac{1}{\sqrt{n(n-1)}}\right)$. Table 2 shows similar results obtained by [1] on a larger sample of NETLIB test problems. In both cases, we display the number of iterations necessary to reach a final accuracy equal to $10^{-6} \times$ initial objective function value, starting from a feasible solution.

These results, as well as many others, support the statement that, in practice, the number of iterations should grow only very slowly with problem size (in terms of number of variables and/or constraints).

TABLE 1

Computational results from [10] on test problems from NETLIB. Results obtained with fixed step length $0.99 \times r$ *(r = radius of largest sphere inscribed in the simplex). Final accuracy* $10^{-6} \times$ *initial function value.*

	n	m	Number of iterations*
GNET 20	44	20	15
ADLITTLE	138	56	19
SHARE 2B	162	96	17
ISRAEL	316	174	21
BRANDY	303	220	21
BAND M	472	305	35

* (starting from a feasible solution)

TABLE 2

Computational results from [1] on test problems from NETLIB. Final accuracy $10^{-6} \times$ *initial objective function value.*

	n	m	Number of iterations*
ADLITTLE	138	56	23
SHARE 2B	162	96	24
SCAGR 25	671	471	25
SCS D1	760	77	17
BAND M	472	305	26
SCS D6	1 350	147	20
SHIP 041	2 166	402	26
SCTAP 2	2 500	1 090	26
SCTAP 3	3 340	1 480	29
SHIP 121	5 533	1 151	29

* (starting from a feasible solution)

In the following, we are going to show that these experimental results may be, at least partially, explained by studying the behaviour of Karmarkar's algorithm from a probabilistic point of view.

3. PROBABILISTIC ANALYSIS OF THE DECREASE IN THE OBJECTIVE FUNCTION VALUE ON ONE ITERATION

In the analysis to follow, we make the following basic probabilistic assumption (BPA).

ASSUMPTION (BPA): *The directions* $d = \frac{g}{\| g \|}$ *obtained by carrying out the first step of Karmarkar's algorithm on randomly chosen instances of problem (LP) are* random directions *in the subspace* $\{d / e^T \cdot d = 0\}$ *drawn from a* uniform probability distribution *(a probability distribution with hyperspherical symmetry around the origin in* $\mathbb{R}^{n-1}$*).*

We observe that (BPA) actually amounts to assuming the existence of an underlying probability distribution on the parameters A and c of (LP) inducing a uniform probability distribution on the directions d.

Let g be the projection of $\bar{c} = D \cdot c$ on the subspace $\{y / \bar{A} y = 0; e^T y = 0\}$. Define:

$$g_{\max} = \max_{j=1 \cdots n} \{g_j\}$$

$$g_{\min} = \min_{j=1 \cdots n} \{g_j\}$$

and, for $0 \leq \lambda \leq 1$ let $y(\lambda) \in S$ be defined by:

$$y(\lambda) = \frac{1}{n} e - \frac{\lambda}{n} \frac{g}{g_{\max}}.$$

The point $y(\lambda)$ is the one obtained after a move in the direction $-g$ in transformed space, according to the step size λ (note that, for $\lambda = 1$, $y(\lambda)$ hits the boundary of the simplex).

We will make use of the following lemma:

LEMMA 1 [2]: *For* $0 \leq \lambda \leq 1$*:*

$$\frac{\bar{c}^T y(\lambda)}{\frac{1}{n} \bar{c}^T \cdot e} \leq 1 - \frac{\lambda}{n} \frac{\| g \|^2}{| g_{\min} | \times | g_{\max} |}.$$

[2] shows that *in the worst case*, the term $\frac{\| g \|^2}{| g_{\min} | \times | g_{\max} |}$ may be as small as 2, therefore:

$$\frac{\bar{c}^T y(\lambda)}{\frac{1}{n} \bar{c}^T \cdot e} \leq 1 - \frac{2\lambda}{n}$$

moreover this bound is *tight.*

As a matter-of-fact, we are going to prove that, under assumption (BPA) the ratio $\frac{\bar{c}^T y(\lambda)}{\frac{1}{n} \bar{c}^T \cdot e}$ may be bounded, in a high proportion of cases (*i.e.* with probability $\geq 1 - 2\eta - 4\varepsilon$ for small prescribed η and ε) by:

$$1 - \lambda \times \varphi(\eta, \varepsilon, n)$$

where for fixed η and ε, $\varphi(\eta, \varepsilon, n)$ decreases to 0 much slower than $2/n$.

A natural way of drawing at random uniformly distributed directions d in the subspace: $\{d / e^T d = 0\}$ is as follows. Consider $v_1, v_2, \ldots, v_n$ to be n realizations of n independent normal variables $V_1, V_2, \ldots, V_n$, each drawn from a reduced normal distribution $\mathcal{N}(0, 1)$. If we denote v the n-vector with coordinates v_i $(i = 1, \ldots, n)$ it is well-known that the directions $\frac{v}{\| v \|}$ are uniformly distributed random directions in $\mathbb{R}^n$.

Let $\mu = \frac{1}{n} \sum_{i=1}^{n} v_i$, then $\tilde{v} = v - \mu e$ is the projection of v onto the subspace $\left\{ v / \sum_{i=1}^{n} v_i = 0 \right\}$, and the directions defined by the vectors $\tilde{v}$ are uniformly distributed random directions in this subspace. In view of the above, we want to analyze the probabilistic behaviour of the term:

$$\rho = \frac{\| \tilde{v} \|^2}{| \tilde{v}_{\min} | \times | \tilde{v}_{\max} |}$$

with $\tilde{v}_{\min} = v_{\min} - \mu$; $\tilde{v}_{\max} = v_{\max} - \mu$ and $v_{\min} = \min_{j=1 \cdots n} \{v_j\}$; $v_{\max} = \max_{j=1 \cdots n} \{v_j\}$.

Let $\varepsilon > 0$ be a (small) specified probability and let $\alpha > 0$ be defined through the equation:

$$\varepsilon = p(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\infty} e^{-t^2/2} \, dt$$

(the "tail" of the reduced normal distribution). Table A.1 in Appendix A shows some values of α corresponding to various (small) values of ε.

Choosing ε in the range [0.1, 5 10^{-6}] results in α values in the range [1.28, 4.75].

LEMMA 2: *Assume that* $n \geq 1000$ *and let* α *be chosen such that* $p(\alpha) = \varepsilon$ *and* $1.28 \leq \alpha \leq 4.75$. *Then, with probability* $\geq 1 - 4\varepsilon$ *we have:*

$$n - \alpha\sqrt{2n} - \alpha^2 \leq \|\tilde{v}\|^2 \leq n + \alpha\sqrt{2n} + \frac{2}{3}\alpha^2.$$

Proof:

$$\|\tilde{v}\|^2 = \sum_i (v_i - \mu)^2 = \|v\|^2 - n\mu^2.$$

μ is a centered normal variable with variance $1/n$, therefore:

$$\text{Prob}\{n\mu^2 > \alpha^2\} \leq \frac{2}{\sqrt{2\pi}} \int_\alpha^\infty e^{-t^2/2}\, dt = 2\varepsilon$$

$\|v\|^2 = \sum_i v_i^2$ is a χ^2 distribution with n degrees of freedom. Using the formula due to Hilferty and Wilson (1931) (*see* [4]) which very accurately represents such a distribution as a cubic function of a reduced normal distribution, it is possible to prove that with probability $\geq 1 - \varepsilon$:

$$\|v\|^2 \leq n + \alpha\sqrt{2n} + \frac{2}{3}\alpha^2$$

and with probability $\geq 1 - \varepsilon$:

$$\|v\|^2 \leq n - \alpha\sqrt{2n}.$$

From this, the Lemma easily follows. □

We now study the probabilistic behaviour of $v_{\max}$, which is a random variable defined as the maximum of n independent realizations of the reduced normal distribution $\mathcal{N}(0, 1)$ (extreme order statistics). The distribution function of $v_{\max}$ is therefore:

$$F_n(u) = \text{Prob}\{v_{\max} \leq u\} = [1 - p(u)]^n \tag{1}$$

where,

$$p(u) = \frac{1}{\sqrt{2\pi}} \int_u^\infty e^{-t^2/2}\, dt$$

is the "tail" of the normal distribution $\mathcal{N}(0, 1)$.

In a similar way, the distribution function of $|v_{\min}|$ is also $F_n(u)$ given by (1) (however observe that the random variables $v_{\min}$ and $v_{\max}$ are *not independent*, and this is taken into account in our analysis).

$\eta > 0$ being a (small) prescribed probability, we denote $u_n^+(\eta)$ the value $\bar{u}$ of u such that $F_n(\bar{u}) = 1 - \frac{\eta}{2}$ and $u_n^-(\eta)$ the value $\bar{u}'$ of u such that $F_n(\bar{u}') = \frac{\eta}{2}$.

The following lemma provides accurate lower and upper bounds on $u_n^+(\eta)$ and $u_n^-(\eta)$ as functions of n and η.

LEMMA 3: *Let $\eta > 0$ ($\eta \ll 1$) be given and define:*

$$r^+ = -2\log_e\left(\sqrt{2\pi}\left[1-\left(1-\frac{\eta}{2}\right)^{\frac{1}{n}}\right]\right)$$
$$r^- = -2\log_e\left(\frac{\sqrt{2\pi}}{0.89}\left[1-\left(1-\frac{\eta}{2}\right)^{\frac{1}{n}}\right]\right).$$

Then, assuming $\sqrt{r^- - \log_e(r^+)} \geq 3$, we have:

$$\sqrt{r^- - \log_e(r^+)} \leq u_n^+(\eta) \leq \sqrt{r^+ - \log_e(r^- - \log_e(r^+))}. \qquad (2)$$

Similarly, taking:

$$s^+ = -2\log_e\left(\sqrt{2\pi}\left[1-\left(\frac{\eta}{2}\right)^{\frac{1}{n}}\right]\right)$$
$$s^- = -2\log_e\left(\frac{\sqrt{2\pi}}{0.89}\left[1-\left(\frac{\eta}{2}\right)^{\frac{1}{n}}\right]\right).$$

and assuming $\sqrt{s^- - \log_e(s^+)} \geq 3$, the following holds:

$$\sqrt{s^- - \log_e(s^+)} \leq u_n^-(\eta) \leq \sqrt{s^+ - \log_e(s^- - \log_e(s^+))}. \qquad (3)$$

Proof: Assuming that $u \geq 3$ (the conditions stated in Lemma 3 ensure that this assumption is valid), we use the approximation for $p(u)$ obtained in Appendix A:

$$p(u) = \frac{\omega}{\sqrt{2\pi}}\,\frac{e^{-u^2/2}}{u}$$

with $0.89 \leq \omega \leq 1$.

$u_n^+(\eta)$ is the solution in u of the equation:

$$\left[1 - \frac{\omega}{\sqrt{2\pi}} \frac{e^{-u^2/2}}{u}\right]^n = 1 - \frac{\eta}{2} \tag{4}$$

which may be rewritten as:

$$\frac{e^{-u^2/2}}{u} = a \tag{5}$$

with $a = \dfrac{\sqrt{2\pi}}{\omega}\left[1 - \left(1 - \dfrac{\eta}{2}\right)^{\frac{1}{n}}\right]$.

Analogously, $u_n^-(\eta)$ is the solution of a similar equation where the value of a is changed to

$$a' = \frac{\sqrt{2\pi}}{\omega}\left[1 - \left(\frac{\eta}{2}\right)^{\frac{1}{n}}\right].$$

Equation (5) is equivalent to

$$-\frac{u^2}{2} - \log_e u = \log_e(a)$$

or:

$$u^2 + \log_e(u^2) = 2\log_e\left(\frac{1}{a}\right). \tag{6}$$

Letting $x = u^2$ and $r = 2\log_e\left(\dfrac{1}{a}\right)$ we find an instance of the equation $x + \log_e(x) = r$, studied in Appendix B. In view of Proposition B.1 of Appendix B we get the following bounds for the solution to (5):

$$\sqrt{r - \log_e(r)} \le u \le \sqrt{r - \log_e(r - \log_e(r))}$$

with $r = 2\log_e\left(\dfrac{1}{a}\right)$. Since $0.89 \le \omega \le 1$, let:

$$a^+ = \frac{\sqrt{2\pi}}{0.89}\left[1 - \left(1 - \frac{\eta}{2}\right)^{\frac{1}{n}}\right]$$
$$a^- = \sqrt{2\pi}\left[1 - \left(1 - \frac{\eta}{2}\right)^{\frac{1}{n}}\right]$$

and

$$r^+ = 2\log_e\left(\frac{1}{a^-}\right) \qquad r^- = 2\log_e\left(\frac{1}{a^+}\right)$$

then:

$$\sqrt{r^- - \log_e(r^+)} \leq u_n^+(\eta) \leq \sqrt{r^+ - \log_e(r^- - \log_e(r^+))}.$$

Bounds for $u_n^-(\eta)$ are obtained in exactly the same way, just replacing r^+ by

$$s^+ = -2 \log_e \left(\sqrt{2\pi}\left[1 - \left(\frac{\eta}{2}\right)^{\frac{1}{n}}\right]\right)$$

and r^- by

$$s^- = -2 \log_e \left(\frac{\sqrt{2\pi}}{0.89}\left[1 - \left(\frac{\eta}{2}\right)^{\frac{1}{n}}\right]\right) \quad \square$$

Table C.3 in Appendix C shows, for a wide range of values for u and for the value $\eta = 10^{-4}$, that the inequalities (2) and (3) provide accurate confidence intervals for $u^+(\eta)$ and $u^-(\eta)$. From these results, confidence intervals for u subject to:

$$\frac{\eta}{2} \leq F_n(u) \leq 1 - \frac{\eta}{2}$$

can be obtained. These are shown in Table 3.

TABLE 3
Confidence intervals $[u_n^-(\eta), u_n^+(\eta)]$ for u subject to $\eta/2 \leq F_n(u) \leq 1 - \eta/2$ for various values of n.

n	$\eta = 10^{-4}$ range for u subject to $\eta/2 \leq F_n(u) \leq 1 - \eta/2$	
	$u_n^-(\eta)$	$u_n^+(\eta)$
1 000	2.27	5.33
5 000	2.83	5.61
10 000	3.04	5.73
50 000	3.50	6.00
100 000	3.68	6.11
500 000	4.07	6.36
10^6	4.23	6.47
10^7	4.72	6.80

From the various preceding lemmas, we can deduce

PROPOSITION 1: *Let* $\varepsilon > 0$ *and* $\eta > 0$ *be small chosen tolerances,* $\varepsilon \in [0.1, 5 \cdot 10^{-6}]$, *and let* α *be such that* $p(\alpha) = \varepsilon$, *(hence* $1.28 \leq \alpha \leq 4.75$*). Then, for* $n \geq 1\,000$, *the double sided inequality:*

$$\rho_{\min} \leq \frac{\| \tilde{v} \|^2}{| \tilde{v}_{\min} | \times | \tilde{v}_{\max} |} \leq \rho_{\max}$$

holds with probability $\geq 1 - 2\eta - 4\varepsilon$ *where:*

$$\rho_{\min} = \frac{n - \alpha\sqrt{2n} - \alpha^2}{\left[u_n^+(\eta)\right]^2 + \frac{\alpha}{\sqrt{n}}\left[u_n^+(\eta) - u_n^-(\eta)\right]}$$

$$\rho_{\max} = \frac{n + \alpha\sqrt{2\pi} + (2/3)\,\alpha^2}{\left[u_n^-(\eta)\right]^2 - \frac{\alpha}{\sqrt{n}}\left[u_n^+(\eta) - u_n^-(\eta)\right] - \frac{\alpha^2}{n}}.$$

Proof: From Lemma 2, all the following inequalities

$$n - \alpha\sqrt{2n} - \alpha^2 \leq \| \tilde{v} \|^2 \leq n + \alpha\sqrt{2n} + (2/3)\,\alpha^2$$

$$| \mu | \leq \frac{\alpha}{\sqrt{n}}$$

simultaneously hold with probability $\geq 1 - 4\varepsilon$. On the other hand, each of the inequalities:

$$u_n^-(\eta) \leq v_{\max} \leq u_n^+(\eta) \tag{7}$$

$$u_n^-(\eta) \leq | v_{\min} | \leq u_n^+(\eta) \tag{8}$$

holds with probability $\geq 1 - \eta$. Now

$$\begin{aligned} \tilde{v}_{\min} \times \tilde{v}_{\max} &= (v_{\min} - \mu)(v_{\max} - \mu) \\ &= v_{\min} \times v_{\max} - \mu\,(v_{\min} + v_{\max}) + \mu^2. \end{aligned}$$

Since $\sum_{i=1}^{n} \tilde{v}_i = 0$, we have that $\tilde{v}_{\max} \geq 0$ and $\tilde{v}_{\min} \leq 0$, $| \tilde{v}_{\min} | = -\tilde{v}_{\min}$.
Therefore

$$| \tilde{v}_{\min} | \times | \tilde{v}_{\max} | = -v_{\min} \times v_{\max} + \mu\,(v_{\min} + v_{\max}) - \mu^2.$$

Assuming both (7) and (8) hold (which occurs with probability $\geq 1 - 2\eta$) we have:

$$u_n^-(\eta) - u_n^+(\eta) \leq v_{\max} + v_{\min} \leq u_n^+(\eta) - u_n^-(\eta).$$

From all the above inequalities, we conclude that, with probability $\geq 1 - 2\eta - 4\varepsilon$, we have:

$$|\tilde{v}_{\min}| \times |\tilde{v}_{\max}| \leq [u_n^+(\eta)]^2 + \frac{\alpha}{\sqrt{n}}[u_n^+(\eta) - u_n^-(\eta)]$$

$$|\tilde{v}_{\min}| \times |\tilde{v}_{\max}| \geq [u_n^-(\eta)]^2 - \frac{\alpha}{\sqrt{n}}[u_n^+(\eta) - u_n^-(\eta)] - \frac{\alpha^2}{n}$$

and

$$\rho_{\min} \leq \frac{\|\tilde{v}\|^2}{|\tilde{v}_{\min}| \times |\tilde{v}_{\max}|} \leq \rho_{\max}.$$

Q.E.D. □

We are now ready to state the main result of this section:

THEOREM 1: *Let $\varepsilon > 0$ and $\eta > 0$ be small chosen probabilities, $\varepsilon \in [0.1,\ 5 \cdot 10^{-6}]$, and let α be such that*

$$p(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\infty} e^{-t^2/2}\, dt = \varepsilon.$$

Then, for $n \geq 1\,000$ and for all $\lambda \in [0, 1]$ the inequality

$$\frac{\bar{c}^T \cdot y(\lambda)}{\bar{c}^T \cdot e/n} \leq 1 - \frac{\lambda}{n} \frac{n - \alpha\sqrt{2n} - \alpha^2}{[u_n^+(\eta)]^2 + \frac{\alpha}{\sqrt{n}}[u_n^+(\eta) - u_n^-(\eta)]}$$

holds with probability $\geq 1 - 2\eta - 4\varepsilon$.

Proof: using Lemma 1 with $g = \tilde{v}$

$$\frac{\bar{c}^T \cdot y(\lambda)}{\bar{c}^T \cdot e/n} \leq 1 - \frac{\lambda}{n} \frac{\|\tilde{v}\|^2}{|\tilde{v}_{\min}| \times |\tilde{v}_{\max}|}.$$

From Proposition 1, $\dfrac{\|\tilde{v}\|^2}{|\tilde{v}_{\min}| \times |\tilde{v}_{\max}|} \geq \rho_{\min}$ with probability $\geq 1 - 2\eta - 4\varepsilon$. Therefore:

$$\frac{\bar{c}^T \cdot y(\lambda)}{\bar{c}^T \cdot e/n} \leq 1 - \frac{\lambda}{n}\rho_{\min}$$

with probability $\geq 1 - 2\eta - 4\varepsilon$.

Q.E.D. □

TABLE 4

Comparison of the probabilistic and worst-case upper bounds for $\frac{\bar{c}^T\, y(1)}{\bar{c}^T\, e/n}$. *Here we have taken* $\varepsilon = 5 \cdot 10^{-5}$ *and* $\eta = 10^{-4}$, *hence the probabilistic bounds hold with probability* $\geq 1 - 4 \cdot 10^{-4}$ (= 0.9996).

n	Upper bound on $\frac{\bar{c}^T\, y(1)}{\bar{c}^T\, e/n}$ with probability $\geq 1 - 2\eta - 4\epsilon$ (Theorem 1)	Anstreicher's worst-case upper bound on $\frac{\bar{c}^T\, y(1)}{\bar{c}^T\, e/n}$
1 000	0.971	0.998
5 000	0.970	0.9996
10 000	0.971	0.9998
50 000	0.972	0.99996
100 000	0.973	0.99998
500 000	0.975	0.999996
10^6	0.976	0.999998
10^7	0.978	0.9999998

The numerical results displayed on Table 4 show for $\lambda = 1$ how the probabilistic upper bound on $\frac{\bar{c}^T \cdot y(\lambda)}{\bar{c}^T \cdot e/n}$ given by **Theorem 1** compares with the worst case (tight) upper bound $1 - \frac{2}{n}$ from [2]. It is observed that, with very high probability (≥ 0.9996), the relative decrease on the objective function value is better than 2.2% for n ranging from 1 000 to 10^7 variables; moreover the bound on the improvement ratio deteriorates only extremely slowly for increasing n. By contrast, the worst-case bound constantly increases and comes very close to 1 for large n.

4. PROBABILISTIC ANALYSIS OF THE DECREASE IN THE POTENTIAL FUNCTION VALUE ON ONE ITERATION

We now extend our analysis to the so-called *potential function* defined as:

$$f(x) = n \log_e c^T \cdot x - \sum_{i=1}^{n} \log_e (x_i).$$

The potential function value at the initial point $\frac{1}{n} \cdot e$ is:

$$f\left(\frac{1}{n} e\right) = n \log_e \left(\frac{c^T e}{n}\right) - n \log_e \left(\frac{1}{n}\right).$$

Let $d = \frac{g}{\| g \|}$ be the direction taken at the first step of Karmarkar's algorithm. For a step size $\lambda \in [0, 1]$ the new point is $y(\lambda) = \frac{1}{n} e - \frac{\lambda}{n} \frac{g}{g_{\max}}$ and the potential function value at this point is:

$$f(y(\lambda)) = n \log_e (c^T \cdot y(\lambda)) - \sum_{i=1}^{n} \log_e \left(1 - \frac{\lambda g_i}{g_{\max}}\right) - n \log_e \left(\frac{1}{n}\right).$$

Therefore, denoting $\Delta f = f(y(\lambda)) - f\left(\frac{1}{n} e\right)$ we have:

$$\Delta f = n \log_e \left(\frac{c^T y(\lambda)}{c^T \cdot e/n}\right) - \sum_{i=1}^{n} \log_e \left(1 - \frac{\lambda g_i}{g_{\max}}\right).$$

A probabilistic bound for $\frac{c^T y(\lambda)}{c^T e/n}$ has been obtained in the previous section, so let us concentrate on the second term in the above expression.

Under the probabilistic assumption (BPA) (section 3) we are therefore interested in the probabilistic behaviour of

$$\sum_{i=1}^{n} \log_e \left(1 - \lambda \frac{\tilde{v}_i}{\tilde{v}_{\max}}\right) \tag{9}$$

where $\tilde{v}_i = v_i - \mu$ $(i = 1 \cdots n)$, v_i being n independent realizations of the reduced normal distribution $\mathcal{N}(0, 1)$, and $\mu = \frac{1}{n} \sum_{i=1}^{n} v_i$.

We make use of the following lemma.

LEMMA 4: *Let* $0 < \lambda < 1$. *Then for all* $\theta \geq -\lambda$

$$\log_e (1 + \theta) \geq \theta - \frac{\theta^2}{2} \times \frac{1}{1 - \lambda}.$$

Proof: Let $\varphi(\theta) = \log_e (1+\theta) - \theta + \frac{\theta^2}{2} \times \frac{1}{1-\lambda}$

$$\frac{d\varphi}{d\theta} = \frac{1}{1+\theta} - 1 + \frac{\theta}{1-\lambda} = -\frac{\theta}{1+\theta} + \frac{\theta}{1-\lambda} = \frac{\theta[\theta+\lambda]}{(1+\theta)(1-\lambda)}.$$

For $-\lambda \le \theta \le 0$, $\frac{d\varphi}{d\theta} \le 0$ and $\varphi(0) = 0$, hence $\varphi(\theta) \ge 0$ for all $\theta \in [-\lambda, 0]$. For $\theta \ge 0$, $\frac{d\varphi}{d\theta} \ge 0$ and $\varphi(0) = 0$, hence $\varphi(\theta) \ge 0$ for all $\theta \ge 0$. □

Now, defining: $\theta_i = -\lambda \frac{\tilde{v}_i}{\tilde{v}_{\max}}$, expression (9) may be rewritten as

$$\sum_{i=1}^{n} \log_e (1+\theta_i) \tag{10}$$

and we note that, $\forall i : \theta_i \ge -\lambda$. Therefore, applying Lemma 4 to each term of (10) we get the inequality:

$$\sum_{i=1}^{n} \log_e \left(1 - \lambda \frac{\tilde{v}_i}{\tilde{v}_{\max}}\right) \ge -\frac{\lambda}{\tilde{v}_{\max}} \sum_{i=1}^{n} \tilde{v}_i - \frac{\lambda^2}{2(1-\lambda)(\tilde{v}_{\max})^2} \sum_{i=1}^{n} (\tilde{v}_i)^2.$$

The first term on the right handside is zero, therefore:

$$\sum_{i=1}^{n} \log_e \left(1 - \lambda \frac{\tilde{v}_i}{\tilde{v}_{\max}}\right) \ge -\frac{\lambda^2}{2(1-\lambda)} \times \frac{\|\tilde{v}\|^2}{(\tilde{v}_{\max})^2}.$$

From this we deduce the following bound for Δf:

$$\Delta f \le n \log_e \left(\frac{c^T y(\lambda)}{c^T e/n}\right) + \frac{\lambda^2}{2(1-\lambda)} \frac{\|\tilde{v}\|^2}{(\tilde{v}_{\max})^2} \tag{11}$$

and we can state:

THEOREM 2: *Under the same assumptions as for Theorem 1 the inequality:*

$$\Delta f \le -\lambda \rho_{\min} + \frac{\lambda^2}{2(1-\lambda)} \rho_{\max} \tag{12}$$

holds with probability $\geq 1 - 2\eta - 4\varepsilon$ *where:*

$$\rho_{\min} = \frac{n - \alpha\sqrt{2n} - \alpha^2}{[u_n^+(\eta)]^2 + \frac{\alpha}{\sqrt{n}}[u_n^+(\eta) - u_n^-(\eta)]}$$

and

$$\rho_{\max} = \frac{n + \alpha\sqrt{2n} + (2/3)\alpha^2}{[u_n^-(\eta)]^2 - \frac{\alpha}{\sqrt{n}}[u_n^+(\eta) - u_n^-(\eta)] - \frac{\alpha^2}{n}}.$$

Proof: Using (11) together with Proposition 1 and Theorem 1, we have with probability $\geq 1 - 2\eta - 4\varepsilon$

$$\Delta f \leq n \log_e\left(1 - \frac{\lambda}{n}\rho_{\min}\right) + \frac{\lambda^2}{2(1-\lambda)}\rho_{\max}.$$

Observing that $\log_e\left(1 - \frac{\lambda}{n}\rho_{\min}\right) \leq -\frac{\lambda}{n}\rho_{\min}$ the result follows. □

Since, for very large n, $\rho_{\min}$ and $\rho_{\max}$ are almost equal, a good choice for λ is obtained when

$$\varphi(\lambda) = -\lambda + \frac{\lambda^2}{2(1-\lambda)}$$

is minimized. The exact minimum value is -0.2679 and is obtained for $\lambda = 0.42 \left(= 1 - \frac{\sqrt{3}}{3}\right)$. However, as shown on Table 5 for $50\,000 \leq n \leq 10^7$

$$2 \leq \frac{\rho_{\max}}{\rho_{\min}} \leq 3$$

and therefore, in the range $[50\,000,\ 10^7]$, λ should be chosen between 0.29 and 0.22.

Table 5 shows numerical values of the bound (12) on the decrease Δf of the potential function. It is seen that, $|\Delta f|$ *increases almost linearly with* n; this is to be contrasted with worst-case analysis (*see* [5]) where the bound on Δf is a *constant* (*i.e.* does not depend on n).

TABLE 5

Bound on the improvement Δf in the potential function with probability $\geq 1 - 4 \cdot 10^{-4}$, (here, again $\varepsilon = 5 \cdot 10^{-5}$ and $\eta = 10^{-4}$).

n	$\frac{\rho_{\min}}{n}$	$\frac{\rho_{\max}}{n}$	probabilistic bound on $\frac{\Delta f}{n}$	probabilistic bound on Δf
1 000	0.028	0.246	$-1.5\ 10^{-3}$ (for $\lambda = 0.1$)	-1.5
5 000	0.029	0.137	$-2.4\ 10^{-3}$ (for $\lambda = 0.2$)	-12
10 000	0.028	0.115	$-2.3\ 10^{-3}$ (for $\lambda = 0.25$)	-23
50 000	0.027	0.083	$-3.3\ 10^{-3}$ (for $\lambda = 0.25$)	-165
100 000	0.026	0.075	$-3.5\ 10^{-3}$ (for $\lambda = 0.25$)	-350
500 000	0.024	0.060	$-3.6\ 10^{-3}$ (for $\lambda = 0.25$)	$-1\ 800$
10^6	0.023	0.056	$-3.5\ 10^{-3}$ (for $\lambda = 0.25$)	$-3\ 500$
10^7	0.021	0.044	$-3.4\ 10^{-3}$ (for $\lambda = 0.25$)	$-34\ 000$

5. ASYMPTOTIC ANALYSIS

For fixed ε and η, it is easily seen that for n growing arbitrarily large,

$$\frac{\rho_{\min}}{n} [u_n^+ (\eta)]^2 \to 1$$

and

$$\frac{\rho_{\max}}{n} [u_n^- (\eta)]^2 \to 1.$$

Therefore the asymptotic analysis of the bounds given by Theorem 1 and Theorem 2 will follow from the asymptotic behaviour of $u_n^+ (\eta)$ and $u_n^- (\eta)$.

THEOREM 3:

(i) $$\lim_{n \to \infty} \frac{u_n^+ (\eta)}{\sqrt{2 \log_e \left(\frac{n}{\sqrt{\frac{\pi}{2}}\, \eta} \right)}} = 1$$

(ii) $$\lim_{n \to \infty} \frac{u_n^- (\eta)}{\sqrt{2 \log_e \left(\frac{n}{\sqrt{2\pi} \log_e (2/\eta)} \right)}} = 1$$

(iii) $$\lim_{n \to \infty} \frac{u_n^+ (\eta) - u_n^- (\eta)}{\sqrt{2 \log_e (n)}} = 0.$$

Proof: From Lemma 3, a lower bound for $u_n^+(\eta)$ is

$$\sqrt{r^- - \log_e(r^+)} = \sqrt{r^-} \times \sqrt{1 - \frac{\log_e(r^+)}{r^-}}$$

where

$$r^+ = -2 \log_e \left(\sqrt{2\pi}\left[1 - \left(1 - \frac{\eta}{2}\right)^{1/n}\right]\right)$$

and

$$r^- = r^+ + 2 \log_e(0.89).$$

When $n \to +\infty$, both r^+ and r^- tend to $+\infty$ and $\dfrac{\log_e(r^+)}{r^-} \to 0$. From this it is seen that $u_n^+(\eta) \to +\infty$. Similarly, it may be proved that $u_n^-(\eta) \to +\infty$.

For any fixed value of η, $u_n^+(\eta)$ is the root of the equation

$$\left[1 - \frac{\omega(u)}{\sqrt{2\pi}} \frac{e^{-u^2/2}}{u}\right]^n = 1 - \frac{\eta}{2}$$

with $\omega(u) \to 1$ when $u \to +\infty$. Since $u_n^+(\eta) \to +\infty$ $(n \to +\infty)$, the asymptotic value of $u_n^+(\eta)$ is the root of the equation:

$$\left[1 - \frac{1}{\sqrt{2\pi}} \frac{e^{-u^2/2}}{u}\right]^n = 1 - \frac{\eta}{2}$$

which may be rewritten

$$u^2 + \log_e(u^2) = r$$

with $r = -2 \log_e \left(\sqrt{2\pi}\left[1 - \left(1 - \frac{\eta}{2}\right)^{1/n}\right]\right)$.

When $n \to +\infty$

$$\log_e\left(1 - \left(1 - \frac{\eta}{2}\right)^{1/n}\right) \to \log_e\left(\frac{\eta}{2n}\right)$$

therefore:

$$r \to r_\infty(n) = -2 \log_e\left(\sqrt{2\pi}\right) - 2 \log_e\left(\frac{\eta}{2n}\right) = 2 \log_e\left(\frac{n}{\sqrt{\frac{\pi}{2}}\,\eta}\right).$$

Using Proposition B.1 of Appendix B, we deduce for the asymptotic value of $u_n^+(\eta)$ the double sided inequality:

$$\sqrt{r_\infty(n) - \log_e(r_\infty(n))} \leq u_n^+(\eta) \leq \sqrt{r_\infty(n) - \log_e(r_\infty(n) - \log_e(r_\infty(n)))}$$

and, since $r_\infty(n) \to +\infty$, $(n \to +\infty)$ we deduce that:

$$\lim_{n \to +\infty} \frac{u_n^+(\eta)}{\sqrt{r_\infty(n)}} = 1$$

which proves (i).

In a similar way, the asymptotic value of $u_n^-(\eta)$ is the root of the equation

$$u^2 + \log_e(u^2) = s$$

with

$$s = -2\log_e\left(\sqrt{2\pi}\left[1 - \left(\frac{\eta}{2}\right)^{1/n}\right]\right).$$

When $n \to \infty$:

$$1 - \left(\frac{\eta}{2}\right)^{1/n} \to -\frac{\log_e\left(\frac{\eta}{2}\right)}{n} = \frac{\log_e\left(\frac{2}{\eta}\right)}{n}$$

therefore

$$s \to s_\infty(n) = -2\log_e\left(\sqrt{2\pi}\right) - 2\log_e\left[\frac{\log_e\left(\frac{2}{\eta}\right)}{n}\right] = 2\log_e\left(\frac{n}{\sqrt{2\pi}\log_e\left(\frac{2}{\eta}\right)}\right).$$

Again, using Proposition B.1, we deduce

$$\lim_{n \to +\infty} \frac{u_n^-(\eta)}{\sqrt{s_\infty(n)}} = 1$$

which proves (ii).

Finally we have:

$$\frac{u_n^+(\eta)}{\sqrt{2\log_e(n)}} = \frac{u_n^+(\eta)}{\sqrt{2\log_e\left(\frac{n}{\sqrt{\frac{\pi}{2}}\eta}\right)}} \times \sqrt{1 - \frac{\log_e\left(\sqrt{\frac{\pi}{2}}\eta\right)}{\log_e(n)}}$$

and

$$\frac{u_n^-(\eta)}{\sqrt{2\log_e(n)}} = \frac{u_n^-(\eta)}{\sqrt{2\log_e\left(\frac{n}{\sqrt{2\pi}\log_e(2/\eta)}\right)}} \times \sqrt{1 - \frac{\log_e\left(\sqrt{2\pi}\log_e\left(\frac{2}{\eta}\right)\right)}{\log_e(n)}}$$

which show that, for fixed η, both $\dfrac{u_n^+(\eta)}{\sqrt{2\log_e(n)}}$ and $\dfrac{u_n^-(\eta)}{\sqrt{2\log_e(n)}}$ tend to 1, and this proves (iii). □

From Theorem 3 we easily deduce the asymptotic behaviour of the bounds given by Theorem 1 and Theorem 2. Thus, for n sufficiently large, and with probability $\geq 1 - 2\eta$

$$\frac{\bar{c}^T y(\lambda)}{\bar{c}^T e/n} \leq 1 - \frac{\lambda}{2\log_e\left(\frac{n}{\sqrt{\frac{\pi}{2}}\eta}\right)} \tag{13}$$

and

$$\frac{\Delta f}{n} \leq \frac{-\lambda}{2\log_e\left(\frac{n}{\sqrt{\frac{\pi}{2}}\eta}\right)} + \frac{\lambda^2}{4(1-\lambda)\log_e\left(\frac{n}{\sqrt{2\pi}\log_e(2/\eta)}\right)}. \tag{14}$$

The asymptotic probabilistic result used by [7] and [11] states that if $\xi^{(n)} \in \mathbb{R}^n$ is a random unit vector uniformly distributed in $\mathbb{R}^n$ then, when $n \to \infty$ the random variable $Z = \dfrac{\|\xi^{(n)}\|_{L2}^2}{\|\xi^{(n)}\|_{L\infty}^2}$ converges in probability to $\dfrac{n}{2\log_e(n)}$.

Since for n sufficiently large,

$$\frac{n}{[u_n^+(\eta)]^2} \leq \frac{\|\tilde{v}\|^2}{|\tilde{v}_{\min}| \times |\tilde{v}_{\max}|} \leq \frac{n}{[u_n^-(\eta)]^2}$$

(with probability $\geq 1 - \eta$) it may be seen that Theorem 3 above proves a similar asymptotic result for the random variable

$$Z' = \frac{\|\xi^{(n)}\|_{L2}^2}{|\xi_{\min}^{(n)}| \times |\xi_{\max}^{(n)}|}$$

where $\xi_{\min}^{(n)}$ and $\xi_{\max}^{(n)}$ respectively denote the largest and the smallest component of $\xi^{(n)}$.

(Note that though $|\xi_{\min}^{(n)}|$ and $|\xi_{\max}^{(n)}|$ have the same probability distribution, they are not independent, thus Z and Z' are distinct random variables).

Table 6 shows a numerical comparison of the asymptotic values for $u_n^+(\eta)$ and $u_n^-(\eta)$ as given by Theorem 3 and the exact values derived from Lemma 3 (the value of η is 10^{-4}). It is seen that for $u_n^+(\eta)$ the error ranges from 5.6% (for $n = 1\,000$) to 4% (for $n = 10^7$); and for $u_n^-(\eta)$ the error ranges from 87% (for $n = 1\,000$) to 23% (for $n = 10^7$).

These results show that, at least in the range $n = 10^3$ to $n = 10^7$, asymptotic analysis only provides poor accuracy for the quantities $u_n^+(\eta)$ and $u_{n\,(\eta)}^-$ involved in the probabilistic bounds of Theorem 1 and Theorem 2.

TABLE 6

Comparison of the asymptotic values for $u_n^+(\eta)$ and $u_n^-(\eta)$ as given by Theorem 3 with the exact values derived from Lemma 4 ($\eta = 10^{-4}$).

n	Exact values Confidence interval $u_n^+(\eta)$	Exact values Confidence interval $u_n^-(\eta)$	Asymptotic values $u_n^+(\eta)$	Asymptotic values $u_n^-(\eta)$
1 000	5.30 - 5.33	2.27 - 2.39	5.63	4.49
5 000	5.58 - 5.61	2.83 - 2.92	5.91	4.84
10 000	5.70 - 5.73	3.04 - 3.12	6.03	4.98
50 000	5.97 - 6.00	3.50 - 3.56	6.29	5.29
100 000	6.08 - 6.11	3.68 - 3.74	6.40	5.42
500 000	6.33 - 6.36	4.07 - 4.12	6.64	5.71
10^6	6.44 - 6.47	4.23 - 4.28	6.75	5.83
10^7	6.78 - 6.80	4.72 - 4.76	7.08	6.21

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APPENDIX

A. AN APPROXIMATION FORMULA FOR THE TAIL OF THE REDUCED NORMAL DISTRIBUTION

Let X be a centered reduced and normally distributed random variable. In this Appendix, we derive a good approximation to:

$$p(u) = \frac{1}{\sqrt{2\pi}} \int_u^{+\infty} e^{-t^2/2}\, dt$$

for sufficiently large u.

We first give a general upper bound, valid for all $u > 0$.

PROPOSITION A.1: *For all $u > 0$:*

$$p(u) \leq \frac{1}{\sqrt{2\pi}} \times \frac{e^{-u^2/2}}{u}.$$

Proof: For all $t \geq u$, $\dfrac{t}{u} \geq 1$, hence:

$$\int_u^{\infty} e^{-t^2/2}\, dt < \frac{1}{u} \int_u^{\infty} e^{-t^2/2}\, t\, dt = \frac{e^{-u^2/2}}{u}.$$

Q.E.D. □

Now, we derive lower and upper bounds, valid for all $u > 0$ and providing better approximations (note that the upper bound only improves over the one in Proposition A.1 for sufficiently large u, $u \geq 2,5$ say).

PROPOSITION A.2: *For all* $u > 0$:

$$p(u) = \frac{\omega(u)}{\sqrt{2\pi}} \times \frac{e^{-u^2/2}}{u}$$

where

$$1 - \frac{1}{u^2} < \omega(u) < 1 - \frac{1}{u^2} + \frac{3}{u^4}$$

(hence $\omega(u)$ *is almost constant and equal to 1 for large* u*).*

Proof: (i) The lower bound is obtained by observing that:

$$\frac{d}{dx}\left[-\left(\frac{1}{x} - \frac{1}{x^3}\right) e^{-x^2/2}\right] = \left(1 - \frac{1}{x^2}\right) e^{-x^2/2} - \left(-\frac{1}{x^2} + \frac{3}{x^4}\right) e^{-x^2/2}$$
$$= e^{-x^2/2} - \frac{3}{x^4}\, e^{-x^2/2}$$

from which we deduce that:

$$\left(\frac{1}{u} - \frac{1}{u^3}\right) e^{-u^2/2} = \int_u^{\infty} e^{-x^2/2}\, dx - 3 \int_u^{\infty} \frac{e^{-x^2/2}}{x^4}\, dx$$

and since the last summation is strictly positive:

$$\int_u^{\infty} e^{-x^2/2}\, dx > \left(1 - \frac{1}{u^2}\right) \frac{e^{-u^2/2}}{u}.$$

(ii) Now, the upper bound is obtained as follows. We have that:

$$\frac{d}{dx}\left[-\left(\frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5}\right) e^{-x^2/2}\right] = \left(1 - \frac{1}{x^2} + \frac{3}{x^4}\right) e^{-x^2/2}$$
$$- \left(-\frac{1}{x^2} + \frac{3}{x^4} - \frac{15}{x^6}\right) e^{-x^2/2}$$
$$= e^{-x^2/2} + \frac{15}{x^6}\, e^{-x^2/2}.$$

From this we deduce that:

$$\left(\frac{1}{u} - \frac{1}{u^3} + \frac{3}{u^5}\right) e^{-u^2/2} = \int_u^\infty e^{-x^2/2}\, dx + 15 \int_u^\infty \frac{e^{-x^2/2}}{x^6}\, dx.$$

Since the second term of the right handside is strictly positive we have

$$\int_u^\infty e^{-x^2/2}\, dx < \left(1 - \frac{1}{u^2} + \frac{3}{u^4}\right) \times \frac{e^{-u^2/2}}{u}.$$

Q.E.D. □

As a direct consequence of Proposition A.2 it is observed that for all $u > 3$, $\omega(u) \in [0.89;\ 1]$. Table A.1 below shows that, as soon as $u \geq 3$ the approximation is very good.

TABLE A.1

u	Actual value of $p(u)$ from tables	Lower bound $\frac{1}{\sqrt{2\pi}}\left(1 - \frac{1}{u^2}\right)\frac{e^{-u^2/2}}{u}$	Upper bound $\frac{1}{\sqrt{2\pi}}\left(1 - \frac{1}{u^2} + \frac{3}{u^4}\right)\frac{e^{-u^2/2}}{u}$
1.28	0.1	0.053	0.206
2.32	0.01	0.0094	0.0106
2.57	5.10^{-3}	$4.8.10^{-3}$	$5.22.10^{-3}$
3.29	5.10^{-4}	$4.89.10^{-4}$	5.0310^{-4}
3.89	5.10^{-5}	$4.94.10^{-5}$	$5.01.10^{-5}$
4.26	10^{-5}	1.10^{-5}	$1.01.10^{-5}$
4.75	5.10^{-6}	10^{-6}	$1.005.10^{-6}$

B. SOLUTION OF THE EQUATION $x + \log_e(x) = r$

PROPOSITION B.1: *For $r \geq 1$, the equation $x + \log_e(x) = r$ has a unique solution $\bar{x}(r)$ which satisfies for all $r \geq 1$:*

$$r - \log_e(r) \leq \bar{x}(r) \leq r - \log_e(r - \log_e(r)).$$

Proof: The function $x \to x + \log_e(x)$ is monotone increasing in x, therefore for all $r > 0$, the equation has a unique solution.

For $x_1 = r - \log_e(r)$ we have:

$$\begin{aligned} x_1 + \log_e(x_1) &= r - \log_e(r) + \log_e(r - \log_e(r)) \\ &= r + \log_e\left(1 - \frac{\log_e(r)}{r}\right). \end{aligned}$$

Since $r \geq 1$, $\log_e(r) \geq 0$ hence: $\log_e\left(1 - \frac{\log_e(r)}{r}\right) \leq 0$ and we have $x_1 + \log_e(x_1) \leqq r$ which proves that x_1 is a lower bound to $\bar{x}(r)$.

Now since $\bar{x}(r) = r - \log_e(\bar{x}(r))$ and x_1 is a lower bound to $\bar{x}(r)$ we have

$$\bar{x}(r) \leq r - \log_e(x_1) = r - \log_e(r - \log_e(r)).$$

Q.E.D. □

The following table shows the values of $x_1 = r - \log_e(r)$ and $x_2 = r - \log_e(x_1)$ together with the exact values $\bar{x}(r)$ for various values of r ranging from 9 to 100. The exact values $\bar{x}(r)$ have been computed by means of the iteration:

$$\begin{cases} x^{(0)} = r \\ x^{(k+1)} = r - \log_e(x^{(k)}) \end{cases}$$

which, for the values $r \geq 9$, converges very rapidly (no more than 5 iterations are needed for a relative accuracy 10^{-5}).

TABLE B.2

Solutions to $x + \log_e(x) = r$ for various values of r.

Values of r	$x_1 = r - \log_e(r)$	$x_2 = r - \log_e(x_1)$	$\bar{x}(r)$
9	6.802	7.082	7.0473
10	7.697	7.959	7.9293
12	9.515	9.747	9.7252
14	11.360	11.569	11.5530
16	13.227	13.417	13.4044
20	17.004	17.166	17.1575
25	21.781	21.918	21.9129
30	26.598	26.719	26.7147
50	46.087	46.169	46.1677
100	95.394	95.441	95.4414
1000	993.092	993.099	993.0991

TABLE C.3

Confidence intervals for $u_n^+(\eta)$ and $u_n^-(\eta)$ derived from Lemma 3 [inequalities (2) and (3)] for $\eta = 10^{-4}$.

n	$\leq u_n^+(\eta) \leq$		$\leq u_n^-(\eta) \leq$	
1 000	5.30	5.33	2.27	2.39
5 000	5.58	5.61	2.83	2.92
10 000	5.70	5.73	3.04	3.12
50 000	5.97	6.00	3.50	3.56
100 000	6.08	6.11	3.68	3.74
500 000	6.33	6.36	4.07	4.12
10^6	6.44	6.47	4.23	4.28
10^7	6.78	6.80	4.72	4.76