RAIRO. RECHERCHE OPÉRATIONNELLE

SHEY-HUEI SHEU
CHUNG-MING KUO
TOSHIO NAKAGAWA

Extended optimal age replacement policy with minimal repair

RAIRO. Recherche opérationnelle, tome 27, nº 3 (1993), p. 337-351

<http://www.numdam.org/item?id=RO_1993__27_3_337_0>

Recherche opérationnelle/Operations Research
(vol. 27, n° 3, 1993, p. 337 à 351)

EXTENDED OPTIMAL AGE REPLACEMENT POLICY WITH MINIMAL REPAIR (*)

by Shey-Huei SHEU (1), Chung-Ming KUO (1) and Toshio NAKAGAWA (2)

Communicated by Shunsi OSAKI

Abstract. – *A generalization of the age replacement policy is proposed and analysed. Under such a policy, if an operating system fails at age $y \leq t$, it is either replaced by a new system (type II failure) with probability $p(y)$, or it undergoes minimal repair (type I failure) with probability $q(y) = 1 - p(y)$. Otherwise, a system is replaced when the first failure after t occurs or the total operating time reaches age T $(0 \leq t \leq T)$, whichever occurs first. The cost of the i-th minimal repair of a system at age y depends on the random part $C(y)$ and the deterministic part $c_i(y)$. The aim of the paper is to find the optimal (t^*, T^*) which minimizes the long-run expected cost per unit time of the policy. Various special cases are included and a numerical example is finally given.*

Keywords: Maintenance, Reliability, Repair, Replacement policy.

Résumé. – *Nous proposons et analysons une généralisation de la politique de renouvellement pour vieillissement; lorsque le système est en panne à l'instant $y \leq t$, il est, ou bien remplacé par un système neuf (panne du type II) avec une probabilité $p(y)$, ou bien soumis à une réparation minimale (panne du type I), avec une probabilité $q(y) = 1 - p(y)$. Sinon, le système est remplacé lorsque la première panne après l'instant t survient ou lorsque le temps total de bon fonctionnement atteint T $(0 \leq t \leq T)$. Le coût de la i-ième réparation minimale à l'instant y comprend une partie aléatoire $C(y)$ et une partie déterministe $c_i(y)$. L'objet de cet article est de trouver la paire optimale (t^*, T^*) qui minimise à la longue l'espérance du coût, par unité de temps, de la politique. Nous ajoutons quelques cas spéciaux et donnons finalement un exemple numérique.*

Mots clés : Maintenance, fiabilité, politique de renouvellement.

1. INTRODUCTION

A maintenance policy which includes replacements and minimal repairs has been first considered by Barlow and Hunter [2]. In the past three decades many modifications and generalizations of this policy have been proposed.

(*) Received February 1992.

(1) Department of Industrial Management, National Taiwan Institute of Technology, Taipei, Taiwan, Republic of China.

(2) Department of Industrial Engineering, Aichi Institute of Technology, Toyota 470-03, Japan.

Recherche opérationnelle/Operations Research, 0399-0559/93/03 /$ 4.00

For a complex system, it may be too expensive to replace or overhaul a system at any failure occasion. Naturally, we have to repair and use it again. When the system is repaired, it is restored to its functioning condition just prior to failure. This is called minimal repair. That is, if the original life distribution of the system when it was brand new was F, then the system upon repair may have survival function $\bar{F}_t$ where t is its age at failure and $\bar{F}_t(x)=\bar{F}(t+x)/\bar{F}(t)$.

In this paper a generalization of the age replacement policy is proposed which incorporates minimal repair and the replacement, and the cost of the i-th minimal repair at age y depends on the age-dependent random part $C(y)$ and the deterministic part $c_i(y)$, which depends on the age and the number of the minimal repair. The policy is described explicitly at the beginning of the next section. The long-run expected cost per unit time of the policy is derived and optimization results are obtained for the infinite-horizon case. As special cases, various results are obtained from Barlow and Hunter [2], Tahara and Nishida [15], Boland [7], Boland and Proschan [8], Cléroux, Dubuc, and Tilquin [9], Berg, Bienvenu and Cléroux [4], Block, Borges and Savits [6], Sheu [14], Muth [10], and Bai and Yun [1].

In the second section the policy is described, and then the long-run expected cost per unit time of the policy is found. A general optimization result for the infinite-horizon case is obtained. In the third section various special cases are discussed. In the last section a numerical example is given.

2. GENERAL MODEL

We consider a generalized age replacement model in which minimal repair or replacement takes place according to the following scheme.

Model:

If an operating system fails at age $y \leq t$, it is either replaced by a new system (type II failure) with probability $p(y)$ at a cost c_u, or it undergoes minimal repair (type I failure) with probability $q(y)=1-p(y)$. Otherwise an operating system is replaced by a new system at a cost c_r when the first failure after t occurs, or preventively replaced by a new system at a cost c_p when the total operating time reaches age $T(0 \leq t \leq T)$, whichever occurs first. The cost of i-th minimal repair at age y is $g(C(y), c_i(y))$, where $C(y)$ is the age-dependent random part, $c_i(y)$ is the deterministic part which depends on the age and the number of the minimal repair, and g is a positive, non-decreasing and continuous function. Suppose that the random part $C(y)$ at age y has distribution $L_y(x)$, density function $l_y(x)$ and finite mean $E[C(y)]$. We assume all failures are instantly detected and repaired.

Let the lifetime X of a new system be a random variable with the distribution function $F(x)$, the survival function $\bar{F}(x)$, the density function $f(x)=F'(x)$, the failure rate $r(x)=f(x)/\bar{F}(X)$, and the hazard function $R(x) = \int_0^x r(y)\,dy$. It is well-known that $\bar{F}(x)=\exp\{-R(x)\}$. Note that throughout the paper for any function $H(x)$ the notation $\bar{H}(x)=1-H(x)$ is used. We assume that the failure rate $r(x)$ is differentiable and strictly increasing to infinity.

Let X_t denote the residual life of a system of which age is $t \geq 0$. X_0 means the life of a new system. Let $F_t(x)$ be the distribution function of X_t. If at time t a minimal repair is done, then

$$F_t(x) = \frac{F(t+x) - F(t)}{\bar{F}(t)}. \tag{1}$$

The corresponding survival function is

$$\bar{F}_t(x) = \frac{\bar{F}(t+x)}{\bar{F}(t)}.$$

Let Y_i^* denote the length of the i-th successive replacement cycle for $i=1, 2, 3, \ldots$ Let R_i^* denote the operational cost over the renewal interval Y_i^*. Thus $\{(Y_i^*, R_i^*)\}$ constitutes a renewal reward process. The pairs (Y_i^*, R_i^*), $i=1, 2, 3, \ldots$ are independent and identically distributed. If $D(t)$ denotes the expected cost of the operating system over the time interval $[0, t]$, then it is well-known that

$$\lim_{t \to \infty} \frac{D(t)}{t} = \frac{E[R_1^*]}{E[Y_1^*]}. \tag{2}$$

(*see*, e. g., Ross [11], p. 52). We shall denote the right-hand side of (2) by $B(t, T)$.

We now give a derivation of the expression for $E[R_1^*]$ and $E[Y_1^*]$. First, however, we must describe in more detail the failure process which governs the cost over the interval $[0, Y_1^*]$.

Consider a non-homogeneous Poisson process $\{N(t), t \geq 0\}$ with intensity $r(t)$ and successive arrival times $S_1, S_2, \ldots$ At time S_n we flip a coin. We designate the outcome by Z_n which takes the value one(head) with probability

$p(S_n)$ and the value zero(tail) with probability $q(S_n)$. Let $L(t) = \sum_{n=1}^{N(t)} Z_n$ and $M(t)=N(t)-L(t)$. Then it can be shown that the process $\{L(t), t \geq 0\}$ and $\{M(t), t \geq 0\}$ are independent non-homogeneous Poisson process with respective intensities $p(t)r(t)$ and $q(t)r(t)$. (*see*, e. g., Savits [12]). This is similar to the classical decomposition of a Poisson process for constant p. Let Y_1 denote the waiting time until the first type II failure for our model with $t=\infty$. Then $Y_1 = \inf\{t \geq 0; L(t)=1\}$. Note that Y_1 is independent of $\{M(t), t \geq 0\}$. Thus the survival distribution of Y_1 is given by

$$\bar{F}_p(y) = P(Y_1 > y) = P(L(y) = 0)$$
$$= \exp\left\{-\int_0^y p(x)\, r(x)\, dx\right\}. \quad (3)$$

We also require the following Lemma from Sheu [13].

Lemma 1: *Let* $\{M(t), t \geq 0\}$ *be a non-homogeneous Poisson process with intensity* $\lambda(t), t \geq 0$ and $\Lambda(t) = E[M(t)] = \int_0^t \lambda(u)\, du$. *Denote the successive arrival times by* S_1, S_2, *Assume that at time* S_i *a* cost *of* $g(C(S_i), c_i(S_i))$ *is incurred. Suppose that* $C(y)$ *at age* y *is random variables with finite mean* $E[C(y)]$ *and* g *is a positive, non-decreasing and continuous function. If* $A(t)$ *is the total coast incurred over* $[0, t]$, *then*

$$E[A(t)] = \int_0^t h(y)\, \lambda(y)\, dy, \quad (4)$$

where $h(y) = E_{M(y)}[E_{C(y)}[g(C(y), c_{M(y)+1}(y))]]$.

For our model we have

$$Y_1^* = \begin{cases} Y_1, & \text{if } Y_1 \leq t \\ t + X_t, & \text{if } Y_1 \leq t, \quad 0 \leq X_t \leq T - t \\ T, & \text{if } Y_1 > t, \quad T - t < X_t \end{cases} \quad (5)$$

and

$$R_1^* = \begin{cases} c_u + \sum_{i=1}^{M(Y_1)} g(C(S_i),\, c_i(S_i)), & \text{if } Y_1 \le t \\ c_r + \sum_{i=1}^{M(t)} g(C(S_i),\, c_i(S_i)), & \text{if } Y_1 > t,\ 0 \le X_t \le T - t. \\ c_p + \sum_{i=1}^{M(t)} g(C(S_i),\, c_i(S_i)), & \text{if } Y_1 > t,\ T - t < X_t \end{cases} \tag{6}$$

We are now ready to derive the expressions for $E[Y_1^*]$ and $E[R_1^*]$. First note that

$$\begin{aligned} E[Y_1^*] &= \int_0^t y\, dF_p(y) + \bar{F}_p(t)\left(\int_0^{T-t} (t+x)\, dF_t(x) + \int_{T-t}^{\infty} T dF_t(x)\right) \\ &= \int_0^t \bar{F}_p(y)\, dy - t\bar{F}_p(t) + \bar{F}_p(t)\left(t + \int_0^{T-t} \bar{F}_t(x)\, dx\right) \\ &= \int_0^t \bar{F}_p(y)\, dy + \bar{F}_p(t) \int_0^{T-t} \bar{F}_t(x)\, dx \\ &= \int_0^t \bar{F}_p(y)\, dy + \bar{F}_p(t)\, U(t,\, T), \end{aligned} \tag{7}$$

where $U(t,\, T) = \int_0^{T-t} \bar{F}_t(x)\, dx = \int_0^{T-t} \bar{F}(t+x)\, dx / \bar{F}(t)$.

Using the Lemma 1 and the independence of Y_1 and $\{M(t),\, t \ge 0\}$, we can write

$$\begin{aligned} E[R_1^*] &= c_u F_p(t) + \int_0^t E\left[\sum_{i=1}^{M(y)} g(C(S_i),\, c_i(S_i))\right] dF_p(y) \\ &\quad + \bar{F}_p(t)\left\{E\left[\sum_{i=1}^{M(t)} g(C(S_i),\, c_i(S_i))\right] + c_r F_t(T-t) + c_p \bar{F}_t(T-t)\right\} \\ &= c_u F_p(t) + \int_0^t \int_0^y h(z)\, q(z)\, r(z)\, dz\, dF_p(y) \\ &\quad + \bar{F}_p(t) \int_0^t h(y)\, q(y)\, r(y)\, dy + \bar{F}_p(t)\left(c_r F_t(T-t) + c_p \bar{F}_t(T-t)\right), \end{aligned}$$

which on simplification is equal to

$$c_u F_p(t) + \int_0^t h(y) \bar{F}_p(y) q(y) r(y) dy + \bar{F}_p(t) \left(c_r F_t(T-t) + c_p \bar{F}_t(T-t)\right). \tag{8}$$

For the infinite-horizon case we want to find a pair (t, T) which minimizes the long-run expected cost per unit of the policy. Recall that

$$B(t, T) = \frac{\left\{ \begin{array}{c} c_u F_p(t) + \int_0^t h(y) \bar{F}_p(y) q(y) r(y) dy \\ + \bar{F}_p(t) \left(c_r F_t(T-t) + c_p \bar{F}_t(T-t)\right) \end{array} \right\}}{\int_0^t \bar{F}_p(y)\, dy + \bar{F}_p(t) U(t, T)} \tag{9}$$

Now, we shall attempt to minimize $B(t, T)$ with respect to (t, T). Our basic assumptions are:

A1: $r(t)$ is strictly increasing and $r(t) \to \infty$ as $t \to \infty$

A2: $p(t)$ and $h(t)$ are nondecreasing continuous fonction and $p(0)=0$

A3: $c_u \geq c_r > c_p > 0$, $(c_u - c_p) p(t) + h(t) q(t) > (c_r - c_p) q(t)$ and $c_r > h(t) > 0$ for $t \geq 0$.

In fact, we can see that if (t^*, T) is an optimal pair minimizing $B(t, T)$, then

$$0 < t^* < T^* < \infty, \tag{10}$$

of which proof will be presented at the end of this section.

Differentiating $B(t, T)$ with respect to t and T, respectively, we have

$$\frac{dB(t, T)}{dt} = \frac{r(t) \bar{F}_p(t)}{\left[\int_0^t \bar{F}_p(y)\, dt + \bar{F}_p(t) U(t, T)\right]^2} W(t, T) \tag{11}$$

where

$$
\begin{aligned}
W(t, T) = {} & \{(c_u - c_r)\, p(t) - (c_r - c_p)\, q(t)\, \bar{F}_t(T-t) + h(t)\, q(t)\} \\
& \times\left\{\int_0^t \bar{F}_p(y)\, dy + \bar{F}_p(t)\, U(t, T)\right\} \\
& -\left\{c_u\, F_p(t) + \int_0^t h(y)\, \bar{F}_p(y)\, q(y)\, r(y)\, dy\right. \\
& \left. + \bar{F}_p(t)\, [(c_r - c_p)\, F_t(T-t) + c_p]\right\} q(t)\, U(t, T),
\end{aligned}
\tag{12}
$$

and

$$
\frac{dB(t, T)}{dT} = \frac{\bar{F}_t(T-t)\, \bar{F}_p(t)}{\left[\int_0^t \bar{F}_p(y)\, dy + \bar{F}_p(t)\, U(t, T)\right]^2} V(t, T), \tag{13}
$$

where

$$
\begin{aligned}
V(t, T) = {} & (c_r - c_p)\, r(T)\left\{\int_0^t \bar{F}_p(y)\, dy + \bar{F}_p(t)\, U(t, T)\right\} \\
& - \left\{c_u\, F_p(t) + \int_0^t h(y)\, \bar{F}_p(y)\, q(y)\, r(y)\, dy\right. \\
& \left. + \bar{F}_p(t)\, [(c_r - c_p)\, F_t(T-t) + c_p]\right\}
\end{aligned}
\tag{14}
$$

By virtue of (10), a necessary condition that a pair (t^*, T^*) minimizes $B(t, T)$ is that it satifies $dB(t, T)/dt = dB(t, T)/dt = 0$, or equivalently, $W(t, T) = V(t, T) = 0$, from which it follows that

$$
r(T)\, U(t, T) + \bar{F}_t(T-t) - \frac{(c_u - c_r)\, p(t)}{(c_r - c_p)\, q(t)} - \frac{h(t)}{(c_r - c_p)} = 0 \tag{15}
$$

and

$$
r(T) - \frac{\left\{\begin{array}{l} c_u - ((c_u - c_r)\, \bar{F}_p(t)/q(t)) \\ - h(t)\, \bar{F}_p(t) + \displaystyle\int_0^t h(y) \bar{F}_p(y)\, q(y)\, r(y)\, dy \end{array}\right\}}{(c_r - c_p) \displaystyle\int_0^t \bar{F}_p(y)\, dy} = 0, \tag{16}
$$

and moreover, from $V(t, T)=0$, we get

$$B(t, T) = (c_r - c_p)\, r(T). \tag{17}$$

If a pair (t^*, T^*) is an optimal solution, then (t^*, T^*) is a solution of (15) and (16), and the resulting minimum value of $B(t, T)$ is $(c_r - c_p) r(T^*)$. Therefore, (t^*, T^*) must have the minimum T among all (t, T)'s that satisfy conditions (15) and (16), and consequently, T^* is unique. On the other hand, denoting the left hand side of (15) by $A(t, T)$, it follows that

$$\begin{aligned}
\frac{dA(t,T)}{dt} &= r(T)\,[r(t)\,U(t, T) - 1] + r(t)\,\bar{F}_t(T-t) \\
&\quad - \left[\frac{(c_u - c_r)\,(p'(t)\,q(t) - p(t)\,q'(t))}{(c_r - c_p)\,q^2(t)} + \frac{h'(t)}{(c_r - c_p)}\right] \\
&\leq r(T)\left[\frac{\displaystyle\int_t^T r(y)\,\bar{F}(y)\,dy}{\bar{F}(t)} - 1\right] + r(t)\,\bar{F}_t(T-t) \\
&\quad - \left[\frac{(c_u - c_r)\,(p'(t)\,q(t) - p(t)\,q'(t))}{(c_r - c_p)\,q^2(t)} + \frac{h'(t)}{(c_r - c_p)}\right] \\
&= -[r(T) - r(t)]\bar{F}_t(T-t) \\
&\quad - \left[\frac{(c_u - c_r)\,(p'(t)\,q(t) - p(t)\,q'(t))}{(c_r - c_p)\,q^2(t)} + \frac{h'(t)}{(c_r - c_p)}\right] < 0 \\
&\qquad \text{for} \quad 0 \leq t < T
\end{aligned} \tag{18}$$

and

$$\frac{dA(t, T)}{dt} = r'(T)\,U(t, T) > 0 \qquad \text{for} \quad T > t. \tag{19}$$

Further, $A(t, t) = 1 - ((c_u - c_r)(p(t) + h(t)\, q(t))/((c_r - c_p)\, q(t)) < 0$ by assumption A3 and $A(t, \infty) = \infty$ for all $t \geq 0$. Hence, if we let $T(t)$ be T that satisfies (15) for each t, then $T(t)$ is a strictly increasing function of t, by the implicit function theorem.

One can conclude that the optimal pair (t^*, T^*) is uniquely determined; t^* is the solution of $K(t)=0$, where

$$K(t) = r(T)U(t, T) + \bar{F}_t(T-t) - \frac{(c_u - c_r)p(t) + h(t)q(t)}{(c_r - c_p)q(t)}, \quad (20)$$

$$T = r^{-1}\frac{\left\{\begin{array}{l} c_u - ((c_u - c_r)\bar{F}_p(t)/q(t)) \\ \quad - h(t)\bar{F}_p(t) + \int_0^t h(y)\bar{F}_p(y)q(y)r(y)\,dy \end{array}\right\}}{(c_r - c_p)\int_0^t \bar{F}_p(y)\,dy}, \quad (21)$$

and T^* is given by (21) with $t=t^*$. Moreover, the resulting minimum expected cost per unit time is given by

$$B(t^*, T^*) = (c_r - c_p)r(T^*). \quad (22)$$

Since $K(t)$ is a function of t only, it will be relatively easy to compute the solution of $K(t)=0$. By (21), we have $T \to \infty$ as $t \to 0$, so that $K(t)$ is positive to the left of where $K(t)$ crosses zero for the first time.

It remains to prove (10). Since

$$W(0, T) = -U(0, T)(c_r - h(0)q(0) - c_u p(0)) < 0, \qquad T > 0, \quad (23)$$

by assumption $p(0)=0$, $q(0)=1$ and $h(y) < c_r$ for $y \geq 0$. It follows from (11) that $dB(t, T)/dt < 0$ for sufficiently small $t>0$. Hence, we have $t^*>0$. We next prove $T^* < \infty$. Since

$$\frac{dV(t, T)}{dT} = (c_r - c_p)r'(T)\left(\int_0^t \bar{F}_p(y)\,dy + \bar{F}_p(t)U(t, T)\right) > 0. \quad (24)$$

Since $V(t, \infty) = \infty$, if follows from (13) that $dB(t, T)/dT > 0$ for sufficiently large T, so we have $T^* < \infty$, if $t^* < \infty$. Further using (9), one can easily see that

$$B(t, T) \geq \frac{\int_0^t h(y)\bar{F}_p(y)q(y)r(y)dy}{\int_0^t \bar{F}_p(y)\,dy + U(0, \infty)} \to \infty \quad \text{as } t \to \infty \quad (25)$$

which implies $t^* < \infty$, Therefore, $T^* < \infty$. Finally, if $t^* = T^*$, then from equation (14) and $V(T^*, T^*) = 0$, we have

$$V(T^*, T^*) = (c_r - c_p)\, r(T^*) \int_0^{T^*} \bar{F}_p(y)\, dy - \Big[c_u F_p(T^*) + \int_0^{T^*} h(y)\, \bar{F}_p(y)\, q(y)\, r(y) dy + c_p \bar{F}_p(T^*)\Big] = 0. \quad (26)$$

It also follows that

$$[(c_u - c_p)\, p(T^*)\, r(T^*) + h(T^*)\, q(T^*)\, r(T^*)] \int_0^{T^*} \bar{F}_p(y)\, dy - \left[c_u F_p(T^*) + \int_0^{T^*} h(y)\, \bar{F}_p(y)\, q(y)\, r(y) dy + c_p \bar{F}_p(T^*)\right] = 0, \quad (27)$$

since T^* must minimize $B(T, T)$, *i. e.*,

$$B(t^*, T^*) = B(T^*, T^*) \le B(T, T) = \frac{c_u F_p(T) + \int_0^T h(y)\, \bar{F}_p(y) q(y)\, r(y)\, dy + c_p \bar{F}_p(T)}{\int_0^T \bar{F}_p(y)\, dy}, \quad (28)$$

or equivalently T^* satisfies $dB(T, T)/dT = 0$. Recalling that

$$(c_u - c_r)\, p(t) + h(t) q(t) > (c_r - c_p)\, q(t)$$

which is equivalent to $(c_u - c_p)\, p(t) + h(t) q(t) > (c_r - c_p)$ and comparing (26) and (27), we have a contradiction. Hence, $t^* < T^*$ must be true. The proof of (10) is complete.

3. SPECIAL CASES

Case 1 ($p(y) = 0$, $g(C(y), c_i(y)) = c$): This is the policy considered by Tahara and Nishida [15]. In this case, if we put $p(y) = 0$, $q(y) = 1$, and $h(y) = c$ in (9), then we have the expression for the expected cost per unit time as

$$B(t, T) = \frac{c\, R(t) + c_r F_t(T - t) + c_p \bar{F}_t(T - t)}{t + U(t, T)}, \quad (29)$$

which agrees with equation (23) in Tahara and Nishida [15].

Case 2 ($t=0$): This is the classical age replacement policy considered by Barlow and Hunter [2]. In this case, if we put $t=0$ in (9), then we get the usual

$$B(0, T) = \frac{c_r\, F(T) + c_p\, \bar{F}(T)}{\displaystyle\int_0^T \bar{F}(y)\, dy}, \tag{30}$$

Case 3 ($t=T$, $p(y)=0$, $g(C(y), c_i(y))=c$): This is the policy II considered by Barlow and Hunter [2]. The problem reduces to the classical periodic replacement problem with minimal repair at failure. In this case, if we put $t=T$, $q(y)=1$, $p(y)=0$ and $h(y)=c$ in (9), then we get the usual

$$B(T, T) = \frac{c\, R(T) + c_p}{T}, \tag{31}$$

Case 4 ($t=T$, $p(y)=0$, $g(C(y), c_i(y))=c(y)$): This is the case considered by Boland [7].

Case 5 ($t=T$, $p(y)=0$, $g(C(y), c_i(y))=c_i$): Boland and Proschan [8] investigated this case. In particular they considered the cost structure $c_i=a+ic$.

Case 6 ($t=T$, $p(y)=p$, $0 \le p \le 1$ and $g(C(y), c_i(y))=C$): This is the case considered by Cléroux, Dubuc and Tilquin [9].

Case 7 ($t=T$, $g(C(y), c_i(y))=C(y)$): This is the case considered by Berg, Bienvenu and Cléroux [4].

Case 8 ($t=T$, $g(C(y), c_i(y))=c_i(y)$): This is the case considered by Block, Borges and Savits [5].

Case 9 ($t=T$): This is the case considered by Sheu [14].

Case 10 ($T=\infty$, $p(y)=0$, $g(C(y), c_i(y))=c$): This is the case considered by Muth [10]. In this case, if we put $T=\infty$, $p(y)=0$, $q(y)=1$ and $h(y)=c$, in (9), then we get the following results as Muth [10] obtained

$$B(t, \infty) = \frac{c\, R(t) + c_r}{t + U(t, \infty)}. \tag{32}$$

Case 11 ($T=\infty$, $p(y)=p$, $q(y)=q=1-p$, $c_u=c_r$, $g(C(y), c_i(y))=C+(c/q)$): This is the case considered by Bai and Yun [1].

4. A NUMERICAL EXAMPLE

In the numerical analysis here we shall consider the system with the Weibull distribution which is one of the most common in reliability studies. The p.d.f. of the Weibull distribution with shape parameter β and scale parameter θ is given by

$$f(x) = \frac{\beta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} \exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\}, \qquad x > 0, \quad \beta, \theta > 0, \tag{33}$$

and the parameter of the distribution will be chosen as $\beta=2$ and $\theta=1{,}012.2$. Suppose that $g(C(y), c_i(y))=C(y)+c(y)$. Here we discuss a model where, at failure, one replaces the system or repairs it depending on the random cost C or repair. Let c_∞ be the constant cost. A replacement (type II failure) upon failure at age $y \leq t$ takes place if $C > \delta(y)\, c_\infty$, if $C \leq \delta(y)\, c_\infty$, then one proceeds a minimal repair (type I failure). The parameter $\delta(y)$ can be interpreted as a fraction of the constant cost c_∞ at age y, and $0 \leq \delta(y) \leq 1$. Here we consider the following parametric form of the repair cost limit function $\delta(y)=\delta e^{-ay}$ with $\delta \geq 0$ and $a \geq 0$. Suppose that the random repair cost C has a distribution $L(.)$ and density $l(.)$, with mean 700 and standard deviation 200 (the probability of a negative cost is negligible). If an operating system fails at age $y \leq t$, it is either replaced with a new system with probability

$$p(y) = 1 - \int_0^{\delta(y)\, c_\infty} l(x)dx. \tag{34}$$

or it undergoes minimal repair with probability

$$q(y) = \int_0^{\delta(y)\, c_\infty} l(x)dx. \tag{35}$$

Then the random part $C(y)$ of minimal repair cost at age $y \leq t$ has density

$$l_y(x) = \frac{l(x)}{q(y)} \qquad \text{for} \quad 0 \leq x \leq \delta(y)\, c_\infty$$

and

$$\begin{aligned} h(y) &= E[g(C(y), c_i(y))] = E[C(y) + c(y)] \\ &= \int_0^{\delta(y)\, c_\infty} x l_y(x)\, dx + c(y) = \frac{1}{q(y)} \int_0^{\delta(y)\, c_\infty} x l(x)\, dx + c(y). \end{aligned} \tag{36}$$

TABLE

Extended optimal age replacement policy for electron tubes with the Weibull distribution.
$c_u = 1{,}200,\ c_r = 1{,}200,\ c_p = 1{,}000,\ c_\infty = 1{,}100,\ \delta = 2,\ \theta = 1{,}012.2,\ C \sim N(700{,}200^2)$

δ	$c(y)=0$					$c(y)=0.1\,y$				
	a	$q(y)$	t^*	T^*	$B(t^*, T^*)$	a	$q(y)$	t^*	T^*	$B(t^*, T^*)$
1	0	.977	554	3,322	1.296,8	0	.977	479	3,343	1.305,0
1	0.000,5	*	820	3,289	1.284,1	0.000,8	*	1,072	3,299	1.287,9
10/11	0	.933	582	3,314	1.293,8	0	.933	501	3,337	1.302,9
10/11	0.000,5	*	1,285	3,266	1.275,2	0.000,8	*	1,439	3,292	1.285,3
9/11	0	.841	636	3,303	1.289,7	0	.841	542	3,329	1.299,7
9/11	0.000,3	*	880	3,277	1.279,6	0.000,8	*	1,714	3,300	1.288,3
8/11	0	.691	727	3,292	1.285,3	0	.691	610	3,322	1.296,8
8/11	0.000,3	*	1,170	3,269	1.276,4	0.000,5	*	1,071	3,306	1.290,8
7/11	0	.500	869	3,289	1.283,9	0	.500	709	3,321	1.296,5
7/11	0.000,3	*	1,882	3,282	1.281,5	0.000,3	*	968	3,319	1.295,8
5/11	0	.158	1,416	3,333	1.301,2	0	.158	1,026	3,357	1.310,6
3/11	0	.002,3	2,822	3,402	1.328,4	0	.002,3	1,531	3,408	1.330,5

* age-dependent

In our computations we consider the following two different cost cases:

Case *a*. $c_u = 1{,}200$, $c_r = 1{,}200$, $c_p = 1{,}000$, $c_\infty = 1{,}100$, $c(y) = 0$;

Case *b*. $c_u = 1{,}200$, $c_r = 1{,}200$, $c_p = 1{,}000$, $c_\infty = 1{,}100$, $c(y) = 0.1\,y$.

The parameters δ and a were varied to take the different values in order to see their influence on the optimal solution. The results are given in Table.

In this paper, the repair cost limit function $\delta(y) = \delta\, e^{-ay}$ is chosen for the purpose of easy computation. From the numerical results, we can derive the following remarks;

(i) some improvements can be made in the minimum cost per unit time if one allows for minimal repair at failure;

(ii) from Table, the minimum cost per unit time will be reduced when the probability of minimal repairing is age-dependent;

(iii) it can be seen that the present model is a generalization on previously known age replacement policies.

REFERENCES

1. D. S. Bai and W. Y. Yun, An Age Replacement Policy with Minimal Repair Cost Limit, *IEEE Trans. Reliab., R-35,* 1986, pp. 452-454.
2. R. E. Barlow and L. C. Hunter, Optimum Preventive Maintenance Policies, *Operations Research, 8,* 1960, pp. 90-100.
3. R. E. Barlow and F. Proschan, *Mathematical Theory of Reliability,* John Wiley, New York, 1965.
4. M. Berg, M. Bienvenu and R. Cleroux, Age replacement policy with age dependent minimal repair, *INFOR, 24,* 1986, pp. 26-32.
5. H. W. Block, W. S. Borges and T. H. Savits, Age-dependent Minimal Repair, *Journal of Applied Probability, 22,* 1985, pp. 370-385.
6. H. W. Block, W. S. Borges and T. H. Savits, A general Age Replacement Model with Minimal Repair, *Naval Research Logistics, 35,* 1988, pp. 365-372.
7. P. J. Boland, Periodic Replacement when Minimal Repair Costs Vary with Time, *Naval Research Logistics, 29,* 1982, pp. 541-546.
8. P. J. Boland and F. Proschan, Periodic Replacement with Increasing Minimal Repair Costs at Failure, *Operations Research, 30,* 1982, pp. 1183-1189.
9. R. Cleroux, S. Dubuc and C. Tilquin, The Age Replacement Problem with Minimal Repair and random Repair Costs, *Operations Research, 27,* 1979, pp. 1158-1167.
10. E. J. Muth, An Optimal Decision Rule for Repairs vs Replacement, *IEEE Trans. Reliab. R-26,* 1977, pp. 179-181.
11. S. M. Ross, *Applied Probability Models with Optimization Applications,* Holden-Day San Francisco, 1970.
12. T. H. Savits, Some Multivariate Distributions Derived from a Non-Fatal Shock Model, *J. Appl. Prob., 25,* 1988, pp. 383-390.

13. S. H. Sheu, A Generalized Block Replacement Policy with Minimal Repair and General Random Repair Costs for a Multi-unit System, *J. Opl. Res., 42,* 1991, pp. 331-341.
14. S. H. Sheu, A General Replacement Model with Minimal Repair and General Random Repair Costs, *Microelectron. Reliab., 31,* 1991, pp. 1009-1017.
15. A. Tahara and T. Nishida, Optimal Replacement Policy for Minimal Repair Model, *J. Operations Research. Soc. of Japan, 18,* 1975, pp. 113-124.