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A LINEAR TIME ALGORITHM FOR THE COMPUTATION OF SOME DISTANCE FUNCTIONS BETWEEN CONVEX POLYGONS (*)

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Abstract. – *We present in this paper a linear time algorithm for the computation of some distance functions between convex polygons in* $\mathbb{R}^2$ *in the general case where they can intersect. The interest for this problem comes from applications in robot vision, pattern recognition and contour fitting,* see *Cox, Maitre, Minoux and Ribeiro* [1989]. *The algorithm is base on previous work of Atallah* [1983] *for the computation of the Hausdorff distance between disjoint convex polygons. Computational results are presented.*

Keywords : Computational geometry; convex polygons; distance computation; contour fitting; robot vision; algorithms.

Résumé. – *Nous présentons dans ce papier un algorithme de complexité linéaire pour le calcul de quelques fonctions de distance entre polygones convexes dans* $\mathbb{R}^2$, *dans le cas général où leur intersection est non-vide. L'intérêt pour ce problème provient d'applications en vision par ordinateur, reconnaissance des formes et recalage de contours* (voir *Cox, Maitre, Minoux et Ribeiro* [1989]). *L'algorithme est basé sur le travail d'Atallah* [1983] *pour le calcul de la distance de Hausdorff entre polygones convexes disjoints. Des résultats numériques sont présentés.*

Mots clés : Géométrie algorithmique; polygones convexes; calcul de distances; recalage de contours; vision par ordinateur; algorithmes.

1. INTRODUCTION

The computation of the projection of a point in $\mathbb{R}^2$ on a convex polygon is a basic problem in computational geometry. An extension of this problem consists in the computation of the distance between two convex polygons,

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which is based on the computation of the projection of each vertex of each polygon on the other one.

An application of this problem is described in Cox, Maitre, Minoux and Ribeiro [1989], consisting in the determination of the optimal matching of two given convex polygons (not necessarily having the same number of vertices, and with a non-empty intersection in the general case) representing, for instance, two distinct series of observations of the same contour or planar object. There it is shown that finding the projection of any vertex of one of the polygons on the other is the basic subproblem to be solved at each step of the algorithm which obtains their optimal matching.

In that reference, the authors use a trivial $O(n_1 n_2)$ algorithm for the computation of the distance between the two convex polygons, where n_1 and n_2 are their number of vertices. As logarithmic time algorithms are available for the computation of the projection of each vertex (see e. g. Edelsbrunner [1985]), they can be used for building an $O(n_1 \log n_2 + n_2 \log n_1)$ algorithm for the computation of the same distance.

We present in this paper a linear time algorithm running in time $O(n_1 + n_2)$ for the computation of some distance functions between convex polygons, among them the Hausdorff distance and that defined by Cox, Maitre, Minoux and Ribeiro [1989]. The algorithm is based on previous work of Atallah [1983] for the computation of the Hausdorff distance in the case where the two polygons do not intersect.

The paper is organized as follows. We present in Section 2 the problem statement, along with the basic notation. The algorithm of Atallah for disjoint polygons is studied in Section 3 with the necessary modifications to compute the distance defined by Cox, Maitre, Minoux and Ribeiro [1989]. Computational results are also presented, illustrating how better this algorithm behaves in practice, when compared to the trivial $O(n_1 n_2)$ one. Following, we give in Section 4 the detailed description of our distance computation algorithm for the general case where the polygons can intersect. Finally, computational results and conclusions are presented in Section 5.

2. PROBLEM STATEMENT

The two convex polygons P_1 and P_2 are given by the following data:

(*a*) n_1 (resp. n_2), the number of vertices of polygon P_1 (resp. P_2);

(*b*) the coordinates (a_j^1, b_j^1) (resp. (a_j^2, b_j^2)) of the j-th vertice v_j^1 (resp. v_j^2) of P_1 (resp. P_2), for any $j \in \{0, 1, \ldots, n_1 - 1\}$ (resp. $j \in \{0, 1, \ldots, n_2 - 1\}$); and

(*c*) the numbering of the vertices of P_1 (resp. P_2) is such that for any $j \in \{0, 1, \ldots, n_1 - 1\}$ (resp. $j \in \{0, 1, \ldots, n_2 - 1\}$), then the segment joining vertices v_j^1 and $v_{j+1 \bmod n_1}^1$ (resp. v_j^2 and $v_{j+1 \bmod n_2}^2$) is the j-th edge of polygon P_1 (resp. P_2).

According to (*c*) above, suppose that the vertices of both P_1 and P_2 are numbered in counterclockwise cyclic order. For any point with coordinates (u, v), we define its distance $\delta_{P_l}(u, v)$ to polygon $P_l (l = 1, 2)$ as

$$\delta_{P_l}(u, v) = \begin{cases} \| (u, v) - Proj_{P_l}(u, v) \|, & \text{if } (u, v) \notin Interior\ (P_l), \\ 0, & \text{otherwise,} \end{cases}$$

where $Proj_{P_l}(u, v)$ denotes the point of P_l achieving the minimum euclidean distance to (u, v), *Interior* (X) denotes the interior of polygon X (*i.e.*, X minus its boundary), and $\| . \|$ is the L_2 norm.

With this notation, the distance defined by Cox, Maitre, Minoux and Ribeiro [1989] can be expressed as

$$D_{CMMR}(P_1, P_2) = D_{CMMR}(P_2, P_1) = \sum_{j=0}^{n_1 - 1} (\delta_{P_2}(a_j^1, b_j^1))^2 + \sum_{j=0}^{n_2 - 1} (\delta_{P_1}(a_j^2, b_j^2))^2,$$

i.e., the sum of the squares of the euclidean distances from each vertex of each polygon to its projection on the other polygon. The Hausdorff distance (see Grunbaum [1967]) considered in Atallah [1983] can also be expressed with this notation, as

$$D_H(P_1, P_2) = D_H(P_2, P_1) = \max_{l = 1, 2} \left\{ \max_{j = 0, 1, \ldots, n_l - 1} \delta_{P_{3-l}}(a_j^l, b_j^l) \right\}$$

$$= \max \left\{ \max_{j = 0, 1, \ldots, n_1 - 1} \delta_{P_2}(a_j^1, b_j^1), \max_{j = 0, 1, \ldots, n_2 - 1} \delta_{P_1}(a_j^2, b_j^2) \right\},$$

i.e., the maximum euclidean distance from any vertex of any polygon to its projection on the other one.

We consider in this paper the problem of the fast computation of $D_{CMMR}(P_1, P_2)$ by a linear time algorithm. The algorithm discussed in Section 3 below for the case of disjoint polygons is an adaptation of that presented in Atallah [1983] and can be easily extended to other distance functions. It will also be useful in the development of the algorithm described in Section 4 for the general case of non-disjoint polygons.

3. DISJOINT POLYGONS

Throughout this section we assume that P_1 and P_2 are disjoint (*i. e.*, they do not intersect and none of them contains the other). A supporting line r of a convex polygon P is a straight line passing through one of its vertices and such that the interior of P lies entirely on one side of r. The following lemma holds:

LEMMA 1 (Atallah [1983]): *Consider a vertex* $x=(u, v)$ *of* P_2 *and its projection* $y=Proj_{P_1}(u, v)$ *on* P_1. *Then, the perpendicular to* xy *at* y *is a supporting line* r *of* P_1, *and* x *and* P_1 *are on different sides relative to that line* (*see fig.* 1).

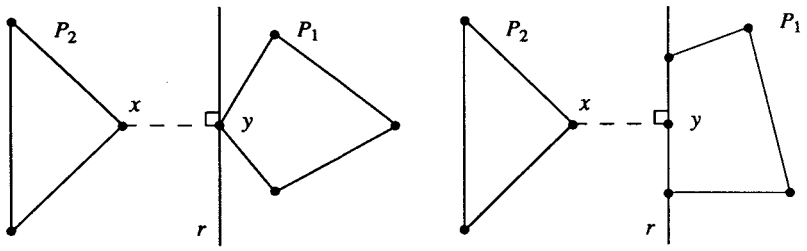


Figure 1. – Illustration of Lemma 1.

We address now the computation of the first part of $D_{CMMR}(P_1, P_2)$, *i.e.*, the sum of the squares of the euclidean distances from each vertex v_j^1 of P_1 to its projection on P_2. Let $d_j=\delta_{P_2}(a_j^1, b_j^1)$ be the distance from vertex v_j^1 to P_2. Computing every d_j separately may take $O(n_2)$ for each one, resulting in a total of $O(n_1 n_2)$ time. We show in the following how to compute all the d_j's (and hence D_{CMMR}) in $O(n_1+n_2)$ time.

For every j, let y_j be the point of P_2 for which $d_j=\|v_j^1-y_j\|$, *i.e.*, $y_j=Proj_{P_2}(a_j^1, b_j^1)$. Note that y_j is either a vertex of P_2 or the foot of the perpendicular from v_j^1 to one of the edges of P_2. Let σ denote the sequence $y_0, y_1, \ldots, y_{n_1-1}, y_0$. We say that σ is *moving counterclockwise* at y_j, if $v_{j+1}^1 \bmod n_1$ is to the left of $y_j v_j^1$. If $v_{j+1}^1 \bmod n_1$ is to the right of $y_j v_j^1$, then we say that σ is *moving clockwise* at y_j. If $v_{j+1}^1 \bmod n_1$ belongs to the line $y_j v_j$, then we say that σ is *stationary* at y_j.

LEMMA 2: v_{j+1}^1 *and* y_{j+1} *are on the same side relatively to the line* $y_j v_j^1$.

We say that σ *changes direction* at y_j (i) if it is moving clockwise at y_{j-1} and it is moving counterclockwise at y_j or (ii) if it is stationary or moving counterclockwise at y_{j-1} and it is moving clockwise at y_j. Then, it follows

that:

LEMMA 3 (Atallah [1983]): σ *changes direction at most twice.*

Lemma 3 above ensures the linear time complexity of the algorithm described next. The first part of the algorithm is given below and consists in the computation of the sum of the squares of the distances from each vertex of P_1 to its projection on P_2. Its second part is basically the same and is omitted here, consisting in the computation of the sum of the squares of the distances from each vertex of P_2 to its projection on P_1. The algorithm starts by the computation of $y_0 = Proj_{P_2}(a_0^1, b_0^1)$, which can be easily done in $O(n_2)$ time by a trivial algorithm. Following, Lemma 3 is used to compute y_{k+1} from y_k, for all $k=0, 1, \ldots, n_1-2$ in $O(n_1+n_2)$ time.

Step 1 (finding y_0):

Computed d_0^2 and find y_0 by a trivial linear time algorithm, set $D_{CMMR}(P_1, P_2) \leftarrow d_0$ and $k \leftarrow 0$.

Step 2 (finding y_{k+1}):

Case (i): v_{k+1}^1 is to the left of $y_k v_k^1$: Search for y_{k+1} by traversing polygon P_2 counterclockwise, starting at y_k, until one of the following conditions is satisfied (see *fig.* 2):

(*a*) The edge of P_2 being considered (call it $v_q^2 v_{q+1}^2$) is such that the foot (call it z) of the perpendicular from v_{k+1}^1 to $v_q^2 v_{q+1}^2$ is between v_q^2 and v_{q+1}^2. In this case, set $y_{k+1} \leftarrow z$.

(*b*) The vertex of P_2 being considered (call it v_q^2) is such that the perpendicular to $v_{k+1}^1 v_q^2$ at v_q^2 is a supporting line r of P_2. In this case, set $y_{k+1} \leftarrow v_q^2$.

Case (ii): v_{k+1}^1 is to the right of $y_k v_k^1$: This case is similar to case (i), except that the search for y_{k+1} is done by traversing polygon P_2 clockwise, starting at y_k.

Case (iii): v_{k+1}^1 is on $y_k v_k^1$: Set $y_{k+1} \leftarrow y_k$.

Step 3 (next iteration):

Set $D_{CMMR}(P_1, P_2) \leftarrow D_{CMMR} + \| v_{k+1}^1 - y_{k+1} \|^2$ and $k \leftarrow k+1$. If $k = n_1 - 1$, then stop. Otherwise, return to step 1.

The correctness proof of the above algorithm is straightforward, since the distances from all vertices are taken into account. Lemma 3 guarantees that polygon P_2 will not be traversed more than twice. Then:

LEMMA 4 (Atallah [1983]): The algorithm presented above has time complexity $O(n_1+n_2)$.

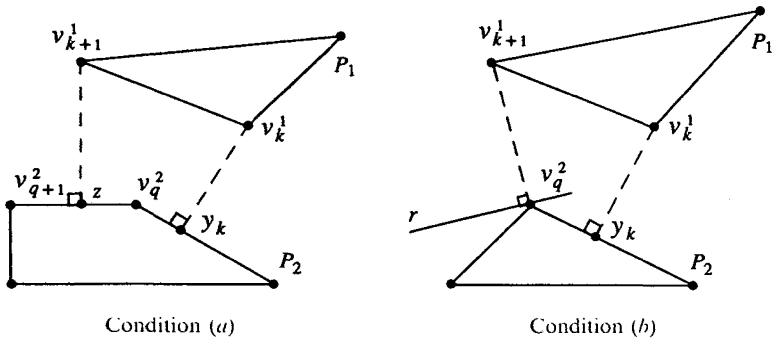


Figure 2. – Illustration of case (i).

We present in the following computational results obtained on randomly generated problems. For each value of $n \in \{5, 10, 15, 20, 25, 30, 35\}$, ten test problems are considered, each of which defined by two convex n-gons. The vertices of P_1 are uniformly distributed in the unit square with diagonal ((0, 0), (1, 1)), while those of P_2 are in that with diagonal ((0, 0), (1, −1)). The vertices of each n-gon are generated one-at-a-time, in order to ensure convexity. Table I gives the average computational times observed for each value of n on a 12 MHz IBM PC/AT microcomputer with an 80287 arithmetic processor, for both trivial $O(n_1 n_2)$ and linear time algorithms.

TABLE I

Computational times for the case of disjoint polygons (in seconds).

n	Alg. $O(n_1 n_2)$	Alg. $O(n_1 + n_2)$
5	0.203	0.099
10	0.763	0.226
15	1.688	0.345
20	2.989	0.490
25	4.632	0.615
30	6.675	0.714
35	9.081	0.855

4. A LINEAR TIME ALGORITHM FOR NON-DISJOINT POLYGONS

We address in this section the computation of $D_{CMMR}(P_1, P_2)$ in the general case where P_1 and P_2 are not necessarily disjoint. We assume that neither polygon is entirely contained in the other, the problem is trivial

otherwise. (In this case, suppose, without loss of generality, that $P_1 \subset P_2$. Then, the distance from every point of P_1 to P_2 is zero. To compute the distances from all vertices of P_2 to P_1, it suffices to traverse clockwise the vertices of P_1 and P_2.) Given P_1 and P_2, let

$$P = (P_1 \cup P_2) - \textit{Interior}\ (P_1 \cap P_2),$$

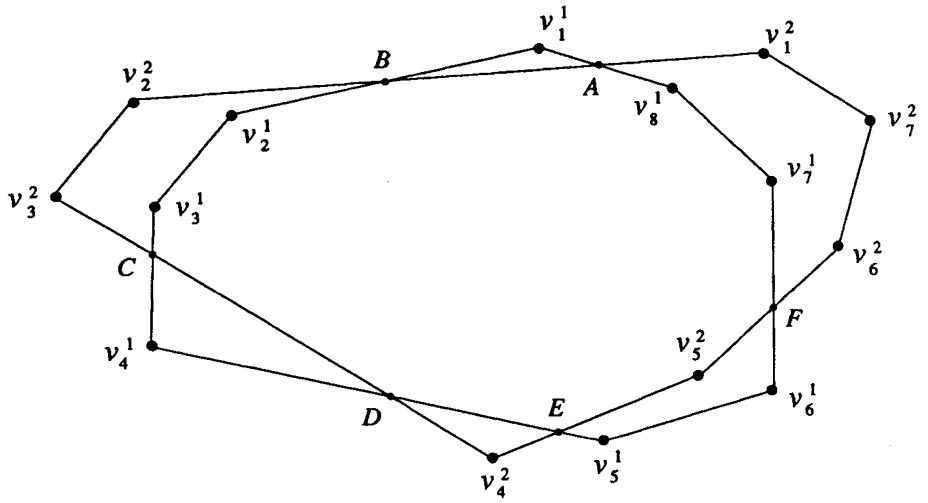
i.e., the region of the plane in P_1 or P_2 but not in the interior of their intersection. As both the union and the intersection of two convex polygons can be computed in linear time (see e.g. O'Rourke, Chien, Olson and Naddor [1982]), a description of region P can also be computed in linear time.

Region P consists of a number $m = O(n)$ of pieces, where $n = \max\{n_1, n_2\}$, each of which is typically non-convex. The boundary of each piece consists of two polygonal chains: one coming from the boundary of P_1 (which we call a P_1-chain), the other coming from that of P_2 (which we call a P_2-chain); see *fig.* 3 for an example. Suppose that the pieces appearing in the piece-decomposition of P are numbered counterclockwise, in the same way as the vertices of the polygons. Let us denote by $P_1\text{-}chain_k$ (resp. $P_2\text{-}chain_k$) the P_1-chain (resp. P_2-chain) in the k-th piece appearing in the piece-decomposition of P and by n_1^k (resp. n_2^k) its number of vertices. The *predecessor* and the *successor* pieces of the k-th piece are, respectively, the $k-1 \bmod m$ and the $k+1 \bmod m$ pieces of the piece-decomposition of P.

Given the k-th piece, suppose that $P_1\text{-}chain_k$ (resp. $P_2\text{-}chain_k$) is the external (resp. internal) chain, *i.e.*, all vertices of $P_2\text{-}chain_k$ are in the interior or in the boundary of P_1. Hence, the distances from all vertices of $P_2\text{-}chain_k$ (*i.e.*, the internal chain in the general case) to P_1 are null and do not need to be taken into acount in the computation of $D_{CMMR}(P_1, P_2)$. It follows from convexity that, for any vertex $v_j^1 \in P_1\text{-}chain_k$, the point y_j of P_2 closest to v_j^1 (*i.e.*, $y_j = Proj_{P_2}(a_j^1, b_j^1)$) is in either one the P_2-chains indexed by k (which is an internal chain), $k-1 \bmod m$ or $k+1 \bmod m$ (which are external chains):

THEOREM 1: *Let* $v_j^1 \in P_1\text{-}chain_k$ *and* $y_j = Proj_{P_2}(a_j^1, b_j^1)$ *be the point of* P_2 *closest to* v_j^1. *Then, either* (*a*) $y_j \in P_2\text{-}chain_k$ *or* (*b*) $y_j \in P_2\text{-}chain_{k-1 \bmod m}$ *or* $y_j \in P_2\text{-}chain_{k+1 \bmod m}$.

Proof: Case (*a*): $y_j \in \textit{Interior}\,(P_1)$. Suppose that $y_j \notin P_2\text{-}chain_k$. Then, the line segment $v_j^1 y_j$ intersects $P_2\text{-}chain_k$. Hence, there is a point of $P_2\text{-}chain_k$ which is closer to v_j^1 than y_j. Therefore, if y_j lies in $\textit{Interior}\,(P_1)$, it is necessarily a point of $P_2\text{-}chain_k$.



$n_1 = 8 \quad n_2 = 7 \quad m = 6$

$P_1\text{-}chain_1 = \{A, v_1^1, B\}$	$P_2\text{-}chain_1 = \{A, B\}$	$n_1^1 = 1$	$n_2^1 = 0$
$P_1\text{-}chain_2 = \{B, v_2^1, v_3^1, C\}$	$P_2\text{-}chain_2 = \{B, v_2^2, v_3^2, C\}$	$n_1^2 = 2$	$n_2^2 = 2$
$P_1\text{-}chain_3 = \{C, v_4^1, D\}$	$P_2\text{-}chain_3 = \{C, D\}$	$n_1^3 = 1$	$n_2^3 = 0$
$P_1\text{-}chain_4 = \{D, E\}$	$P_2\text{-}chain_4 = \{D, v_4^2, E\}$	$n_1^4 = 0$	$n_2^4 = 1$
$P_1\text{-}chain_5 = \{E, v_5^1, v_6^1, F\}$	$P_2\text{-}chain_5 = \{E, v_5^2, F\}$	$n_1^5 = 2$	$n_2^5 = 1$
$P_1\text{-}chain_6 = \{F, v_7^1, v_8^1, A\}$	$P_2\text{-}chain_6 = \{F, v_6^2, v_7^2, v_1^2, A\}$	$n_1^6 = 2$	$n_2^6 = 3$
$external_1 \equiv P_1 - chain_1$	$internal_1 \equiv P_2 - chain_1$		

Figure 3. – Piece-decomposition of region P.

Case (*b*): $y_j \notin Interior(P_1)$. Suppose that y_j belongs to an external P_2-chain other than $P_2\text{-}chain_{k-1 \bmod m}$ or $P_2\text{-}chain_{k+1 \bmod m}$. Then, again, the line segment $v_j^1 y_j$ intersects $P_2\text{-}chain_k$. Hence, there is a point of $P_2\text{-}chain_k$ which is closer to v_j^1 than y_j. Therefore, if y_j does not lie in $Interior(P_1)$, it is necessarily a point of either $P_2\text{-}chain_{k-1 \bmod m}$ or $P_2\text{-}chain_{k+1 \bmod m}$. ■

In the general case, Theorem 1 amounts to say that, given the vertices of any of the external chains appearing in the piece-decomposition of P, their contributions to $D_{CMMR}(P_1, P_2)$ can be evaluated just by looking at their distances to the intenal chain of this same piece and to the external chains of their predecessor and successor pieces.

We show now how to compute these distances efficiently. Whitout loss of generality, suppose that we want to compute the distance from some vertex v_j^1 of the external chain $P_1\text{-}chain_k$ to P_2 (*i.e.*, to chains $P_2\text{-}chain_k$, $P_2\text{-}chain_{k-1 \bmod m}$ and $P_2\text{-}chain_{k+1 \bmod m}$). We consider first the computation

of its distance to any of the external chains P_2-$chain_{k-1 \bmod m}$ or P_2-$chain_{k+1 \bmod m}$, say the latter. To do so, suppose that we transform P_2-$chain_{k+1 \bmod m}$ into a convex polygon by creating an artificial edge joining its first and last vertices. Again from convexity, the distance we want to compute is the same distance from v_j^1 to the polygon so formed. Suppose now that we proceed in the same way with P_1-$chain_k$. Since P_1-$chain_k$ and P_2-$chain_{k+1 \bmod m}$ are disjoint, except for the last vertex of the former and the first one of the latter (which is a point of the intersection of the polygonal lines associated with P_1 and P_2, hence not necessarily a vertex of either P_1 or P_2), the linear time algorithm given in Section 3 can now be used to compute in time $O(n_1^k + n_2^{k+1 \bmod m})$ the distances from all vertices of P_1-$chain_k$ to P_2-$chain_{k+1 \bmod m}$, and not only the distance from v_j^1. The same idea holds for the computation of the distances from all vertices of P_1-$chain_k$ to P_2-$chain_{k-1 \bmod m}$ in time $O(n_1^k + n_2^{k-1 \bmod m})$.

We consider now the computation of the distance from v_j^1 to the internal chain P_2-$chain_k$. Suppose that instead of computing just the distance from v_j^1, we compute the distances from all vertices of P_1-$chain_k$ to chain P_2-$chain_k$, in the same way as in the previous paragraph. The idea here is to walk along P_1-$chain_k$ and P_2-$chain_k$ (like the way two sorted lists are merged), computing as we go along for each vertex of P_1-$chain_k$ that we encounter the point of P_2-$chain_k$ closest to it. What makes it work is the fact that, as we move monotonically along P_1-$chain_k$, we also move monotonically along P_2-$chain_k$. Computing the distances from all vertices of P_1-$chain_k$ to P_2-$chain_k$ takes hence $O(n_1^k + n_2^k)$. The algorithm follows:

Begin
Compute the union and the intersection of P_1 and P_2 using any linear time algorithm.
Compute and store the piece-decomposition of P.
Let m be the number of pieces in the piece-decomposition of P and n_1^k (resp. n_2^k) be the number of vertices of P_1 (resp. P_2) in chain P_1-$chain_k$ (resp. P_2-$chain_k$), for $k = 1, 2, \ldots, m$.
Let $external_k$ (resp. $internal_k$) be the external (resp. internal) chain of piece k, for $k = 1, 2, \ldots, m$. Set $n_{ext}^k \leftarrow n_1^k$ (resp. $n_{int}^k \leftarrow n_2^k$) if P_1-$chain_k$ is an external chain; $n_{ext}^k \leftarrow n_2^k$ (resp. $n_{int}^k \leftarrow n_1^k$) otherwise.
Set $D_{CMMR} \leftarrow 0$ and $k \leftarrow 1$.
While $k \leqq m$ **do**
  For each node of $external_k$ **do**
    Compute its distances $d^{k-1 \bmod m}$, d^k and $d^{k+1 \bmod m}$, respectively to chains $external_{k-1 \bmod m}$, $internal_k$ and $external_{k+1 \bmod m}$.
    Set $D_{CMMR} \leftarrow D_{CMMR} + (\min\{d^{k-1 \bmod m}, d^k, d^{k+1 \bmod m}\})^2$.
  end for each
  Set $k \leftarrow k+1$.
  end while
end begin

The vertices of the two polygons are stored in two doubly chained circular lists. The pieces of the piece-decomposition of P are also stored in a doubly chained circular list, in such a way that whenever we need to compute the distances from all vertices of $external_k$ to all those of $external_{k-1 \bmod m}$ and $external_{k+1 \bmod m}$, the latter can both be easily accessed. The elements of this doubly chained circular list are themselves linear lists, whose first element corresponds to an intersection of two edges of P_1 and P_2 and the others are the vertices of the associated external chain originating at this intersection (see *fig.* 4). In fact, the internal chains do not need to be stored, since their vertices can easily be obtained by difference from their predecessor and successor external chains.

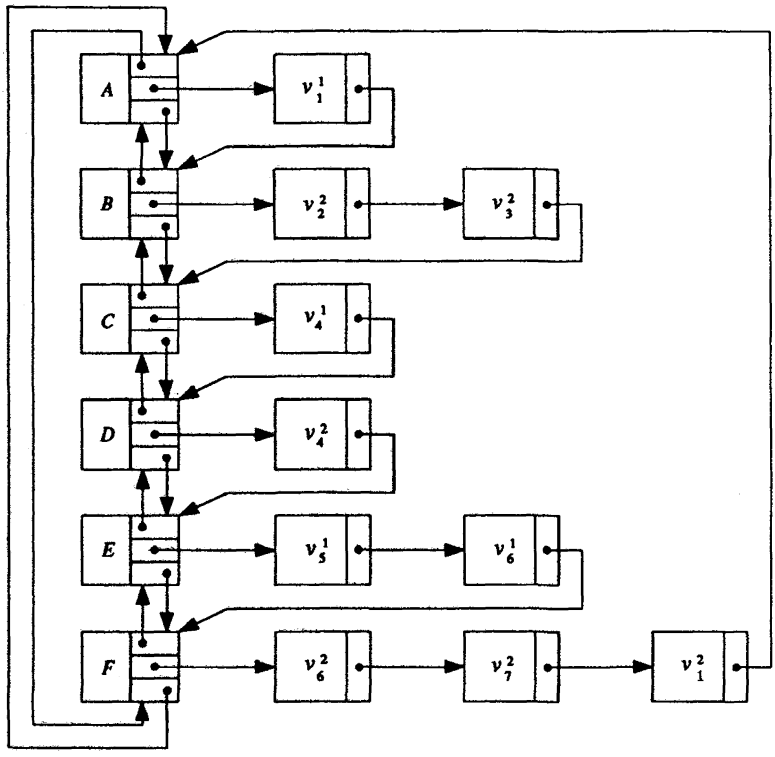


Figure 4. – Data structure for the example depicted in Figure 3.

THEOREM 2: *The algorithm described above has time complexity* $O(n)$, *where* $n = \max\{n_1, n_2\}$.

Proof: Denote by n_{ext}^k (resp. n_{int}^k) the number of vertices of $external_k$ (resp. $internal_k$). The computation of the distances from all vertices of $external_k$ to the other polygon takes $O(n_{\text{ext}}^k + n_{\text{int}}^k + n_{\text{ext}}^{k-1 \bmod m} + n_{\text{ext}}^{k+1 \bmod m})$. Hence, the overall complexity of the algorithm is

$$\sum_{k=1}^{m} O(n_{\text{ext}}^k + n_{\text{int}}^k + n_{\text{ext}}^{k-1 \bmod m} + n_{\text{ext}}^{k+1 \bmod m})$$

$$= \sum_{k=1}^{m} O(n_{\text{ext}}^k + n_{\text{int}}^k) + \sum_{k=1}^{m} O(n_{\text{ext}}^{k-1 \bmod m}) + \sum_{k=1}^{m} O(n_{\text{ext}}^{k+1 \bmod m})$$

$$= O(n_1 + n_2) + O(n) + O(n) = O(n). \quad \blacksquare$$

5. COMPUTATIONAL RESULTS AND CONCLUSIONS

The algorithm presented in Section 4 was applied to the same test problems described in Section 3, with the only modification that both polygons are now in the same unit square. The average computational times observed for each value of n (over ten test problems) on the same 12 MHz IBM PC/AT microcomputer with an 80287 arithmetic processor are presented in Table II, for a trivial $O(n_1 n_2)$ algorithm and for the linear time algorithm presented in Section 4.

TABLE II

Computational times for the case of non-disjoint polygons (in seconds).

n	Alg. $O(n_1 n_2)$	Alg. $O(n)$
5	0.252	0.254
10	0.786	0.468
15	1.436	0.615
20	2.591	0.853
25	3.587	0.951
30	5.174	1.121
35	6.532	1.593

It was noticed during the computational experiences reported in Cox, Maitre, Minoux and Ribeiro [1989] that most of the computational time observed for the solution of the problem of optimal matching of convex polygons corresponds to computations of the distance function between them. The linear time algorithm described here can then be used to considerably speed up the approach proposed in that paper. A very useful result would

be the extension of the algorithm presented in Section 4 for the computation of the distance between two convex polyhedra in $\mathbb{R}^3$.

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