RAIRO. RECHERCHE OPÉRATIONNELLE

W. V. GEHRLEIN

Frequency estimates for linear extension majority cycles on partial orders

RAIRO. Recherche opérationnelle, tome 25, nº 4 (1991), p. 359-364

<http://www.numdam.org/item?id=RO_1991__25_4_359_0>

Recherche opérationnelle/Operations Research
(vol. 25, n° 4, 1991, p. 359 à 364)

FREQUENCY ESTIMATES FOR LINEAR EXTENSION MAJORITY CYCLES ON PARTIAL ORDERS (*)

by W. V. GEHRLEIN [1]

Abstract. — *A linear order L on a set X is a linear extension of partial order P on X if* $P \subseteq L$. $\mathscr{L}(P)$ *denotes the set of all linear extensions of P, and* $\mathscr{L}(x, y)$ *denotes the subset of* $\mathscr{L}(P)$ *for which* $xL_i y$ *if* $L_i \in \mathscr{L}(x, y)$. *A linear extension majority (LEM) relation M is defined by xMy if* $\# \mathscr{L}(x, y) > \# \mathscr{L}(y, x)$. *A LEM cycle exists on P if xMy, yMz and zMx for some x, y,* $z \in X$. *Similarly, M' on X is defined by xM'y if* $\# \mathscr{L}(x, y) \geqq \# \mathscr{L}(y, x)$ *for* $x \neq y$.*A LEM quasi-cycle exists if xM'y, yM'z and zM'x for some x, y,* $z \in X$ *and the equality part of the M' definition holds for exactly one of the pairs in the triple. A Monte-Carlo simulation study is conducted to obtain estimates of the relative freqency with which LEM cycles and LEM quasi-cycles are observed.*

Keywords : Linear extensions; partial orders.

Résumé. — *Un ordre linéaire L sur un ensemble X est une extension linéaire d'un ordre partiel P sur X si* $P \subseteq L$. $\mathscr{L}(P)$ *dénote l'ensemble de toutes les extensions linéaires de P;* $\mathscr{L}(x, y)$ *dénote le sous-ensemble de* $\mathscr{L}(P)$ *pour lequel* $xL_i y$ *si* $L_i \in \mathscr{L}(x, y)$. *Un ensemble linéaire majoré (Linear Extension Majority ou LEM) M est défini par xMy si* $\# \mathscr{L}(x, y) > \# \mathscr{L}(y, x)$. *Une période LEM existe sur P si xMy, yMz et zMx pour certains x, y,* $z \in X$.

Parallèlement, M' sur X est défini par xM'y si $\# \mathscr{L}(x, y) \geqq \# \mathscr{L}(y, x)$ *pour* $x \neq y$. *Une quasi-période LEM existe si xM'y, yM'z et zM'x pour certains x, y,* $z \in X$ *et la partie d'égalité de la définition M' tient pour exactement une des paires du triple. Une étude de simulation de Monte Carlo et entreprise pour obtenir des évaluations de la fréquence relative avec lesquels on observe les péiodes LEM ainsi que les quasi périodes LEM.*

Mots clés : Extensions linéaires; ordres partiels.

(*) Received April 1990.
This research was supported through a fellowship from the Center for Advanced Study of the University of Delaware. Assistance from James Mulligan (Department of Economics, University of Delaware) and William Stewart (Départemente Economie, Groupe ESC Lyon) is also acknowledged.
([1]) Department of Business Administration, University of Delaware, Newark, DE 19716, U.S.A.

Recherche opérationnelle/Operations Research, 0399-0559/91/04 359 06/$ 2.60

1. INTRODUCTION

Let P denote a partial order on a set X of cardinality n. A partial order is a binary relation that is asymmetric ($xPy \Rightarrow \text{not}\, yPx$, $\forall x, y \in X$) and transitive (xPy and $yPz \Rightarrow xPz$, $\forall x, y, z \in X$). A linear order L is a binary relation that is asymmetric, transitive, and complete (xLy or yLx, $\forall x, y \in X$). L is a linear extension of P if $P \subseteqq L$. $\mathscr{L}(P)$ denotes the set of all linear extensions of P. For any $x, y \in X$, $\mathscr{L}(x, y)$ denotes the subset of $\mathscr{L}(P)$ such that if $L_i \in \mathscr{L}(x, y)$ then xL_iy, with $\mathscr{L}(x,y) \cap \mathscr{L}(y,x) = \varnothing$ and $\mathscr{L}(x,y) \cup \mathscr{L}(y,x) = \mathscr{L}(P)$. A linear extension majority (LEM) relation M on X is defined such that xMy for $x, y \in X$ if ${}^{\#}\mathscr{L}(x,y) > {}^{\#}\mathscr{L}(y,x)$. P's exist such that M on X can cycle with xMy, yMz, zMx for some x, y, $z \in X$. Cycles also exist on larger sets of elements, with $x_1 Mx_2M \ldots Mx_m Mx_1$. These are referred to as LEM cycles on m elements, or m-cycles. Examples of partial orders with LEM cycles are given in [1, 2, 3, 4, 5, 6, 8, 11, 12]. The purpose of this study is to obtain Monte-Carlo simulation estimates of the relative frequency with which LEM cycles exist. This study significantly extends results from a preliminary simulation study in [12].

The notion of a LEM quasi-cycles was introduced in [12]. Let M' on X be defined by $xM'y$ if ${}^{\#}\mathscr{L}(x,y) \geqq {}^{\#}\mathscr{L}(y,x)$ for $x \neq y$. A LEM quasi-cycle exists on P if there are $x, y, z \in X$ such that $xM'yM'zM'x$ and the equality part of the M' definition holds for exactly one pair in the triple. We also obtain Monte-Carlo simulation estimates of the relative frequency with which LEM quasi-cycles exist.

Examples of LEM cycles and LEM quasi-cycles are very limited. It has been shown in [11] that there are no partial orders with LEM cycles or LEM quasi-cycles for $n \leqq 8$, and that there are only five nonisomorphic partial orders with LEM cycles with $n=9$ and eight nonisomorphic partial orders with LEM quasi-cycles with $n=9$. Given that there are 183,231 nonisomorphic partial orders with $n=9$ [10], the likelihood that a randomly selected partial order has a LEM cycle or a LEM quasi-cycle is very small with $n=9$. Despite this observation, it has not been possible to identify significantly large subsets or specializations of partial orders that cannot have LEM cycles or LEM quasi-cycles. It has been shown [12] that semiorders cannot have LEM cycles or LEM quasi-cycles. Partial order P is a semiorder if it is also true that (xPy and $zPw \Rightarrow xPw$ or zPy, $\forall x, y, z, w \in X$) and that (xPy and $yPz \Rightarrow xPw$ or wPz, $\forall x, y, z, w \in X$). A limited Monte-Carlo simulation study [12] only produced two partial orders with LEM cycles for $n \geqq 10$ and three partial orders with LEM quasi-cycles for $n \geqq 10$. By conducting a large simulation study to identify a number of partial orders with LEM cycles and

LEM quasi-cycles, some conjectures regarding the existence of LEM cycles and LEM quasi-cycles can be tested.

2. SIMULATION FORMAT

Random partial orders were generated for this Monte-Carlo simulation study. The procedure for generating random partial orders started by generating a random diagram. A diagram S of a partial order P on X consists of all ordered pairs in P that are not implied by transitivity. That is, if xPy and yPz, then xPz would not be included in S. Once a random diagram has been generated, a random partial order is obtained from it by taking the transitive closure. The transitive closure S^t of S is initially given by S, and is then modified by checking all relevant x, y and z triples in X to determine if $xS^t y$ and $yS^t z$. If this is true, then $xS^t z$ is added whether it is in S^t or not. This checking continues until all necessary S^t relations have been added to make S^t transitive, and then we set $P = S^t$.

The specific procedure starts by generating an initial diagram S^1 on $X = \{x_1, x_2, \ldots, x_n\}$ that is included in $L = x_1 L x_2 L \ldots L x_n$ by a process developed in [7, 9, 12, 14]. We begin by generating a random permutation s on $\{1, 2, \ldots, n\}$ and, for each $j = 2, 3, \ldots, n$, we select a y_i at random from $\{x_{s(1)}, x_{s(2)}, \ldots, x_{s(j-1)}\}$. Then $S^1 = L \cap \{[y_2, x_{s(2)}], [x_{s(2)}, y_2], \ldots, [y_n, x_{s(n)}], [x_{s(n)}, y_n]\}$ so that S^1 contains $n-1$ ordered pairs. Let P^1 be the partial order that results from taking the transitive closure of S^1 and let I^1 denote the symmetric complement of P^1. This generation procedure requires that S^1 and P^1 are connected. If P^1 is connected then X cannot be partitioned into subsets A and B such that $xI^1 y$ for all $x \in A$ and for all $y \in B$. An algorithm due to Hoffmann [13] is then used to determine if P^1 has a LEM cycle or LEM quasi-cycle, and P^1 for any such cycle is printed out. The next step of the generation process attempts to form another diagram by adding a relation to S^1. Each pair in I^1 is examined to determine if the addition of an S^1 relation on that pair, consistent with L, would result in another diagram. That is, to determine if the relation on that pair cannot be obtained by transitivity from P^1. Let T^1 denote the subset of pairs in I^1 that are consistent with L and would result in a diagram when added to S^1. If $T^1 = \varnothing$ then the generation process returns to generating a new S^1. If $T^1 \neq \varnothing$ then a pair from T^1 is selected at random and is added to S^1 to form the diagram S^2. Then P^2 is checked for a LEM cycle or a LEM quasi-cycle. We then form another diagram by adding a relation to S^2 in the same fashion as S^2 was obtained from S^1. Apoint is eventually reached when $T^i = \varnothing$ and a new S^1

is generated. Since each S^a is connected and we only add relations to reach S^i in generating new diagrams, every partial order that is generated is connected.

3. RESULTS

The results of this simulation study are shown in Table I. Connected partial orders were generated for $n=9$, 10, 11, 12, 13. The number of initial orders refers to the number of S^1 starting diagrams that were generated,

TABLE I

Simulation results on LEM cycles and LEM quasi-cycles.

No.	Initial orders	Total orders	LEM cycles	LEM quasi-cycles	Orders per cycle
9	45,000	338,187	4	11	22,546
10	15,000	140,497	6	5	12,772
11	4,100	46,505	8	4	3,875
12	1,000	12,923	2	2	3,231
13	300	4,819	0	0	–

while the number of total orders refers to the total number of diagrams over all S^1 starting diagrams and all augmented diagrams (S^i for $i \geqq 2$).

Each of the 27 partial orders with $n \geqq 10$ that had LEM cycles or LEM quasi-cycles from this simulation are listed in the appendix. Based on the output of this analysis, there is additional information to support a conjecture about LEM cycles in [12]. An examination of the number of total orders that were generated for each LEM cycle or LEM quasi-cycle that was observed generally supports the notion that the likelihood that a randomly generated partial order will have a LEM cycle or LEM quasi-cycle increases as n increases. The number of orders per cycle clearly decreases from $n=9$ to $n=12$, and the increase for the $n=13$ case is not of major significance due to the rarity of observations of LEM cycles and LEM quasi-cycles for $n \geqq 12$. Excessive computation time prohibits a significant extension of this simulation study for $n \geqq 13$.

An interval order Q is a partial order with the additonal restriction that xQy and $wQz \Rightarrow xQz$ or wQy, $\forall w, x, y, z \in X$. None of the 27 partial orders with LEM cycles or LEM quasi-cycles is an interval order. This would support the conjecture from [12] that interval orders cannot have LEM cycles or LEM quasi-cycles. However, an example of a LEM cycle on an interval order with $n=25$ has recently been discovered [2]. The list of partial orders with LEM cycles or LEM quasi-cycles in the appendix might be useful to determine some insight regarding specific structural aspects of partial orders that are required to produce LEM cycles or LEM quasi-cycles.

APPENDIX

Diagrams of Partial Orders With LEM Cycles or LEM Quasi-cycles*

10 Elements - Strict Cycles

(1>4,5,8),(2>3,4),(3>5),(5>6),(6>9),(7>8,9),(8>10),(9>10)	[4,6,8]
(1>2,4,7,8,9),(3>5,10),(5>6),(6>8)	[2,6,10]
(1>3,4),(2>9,10),(3>6,7,10),(4>8),(5>7),(8>10)	[3,4,5]
(1>5),(2>8),(3>6),(4>5,9),(5>7),(6>10),(7>8),(8>10)	[6,7,9]
(1>3,5,9),(2>5,8,9,10),(3>4,6,8),(6>7),(7>10)	[4,5,7]
(1>5),(2>4,9),(3>7),(4>5,10),(5>8),(6>8),(7>9),(9>10)	[4,6,7]

10 Elements - Quasi Cycles

(1>4,7,9),(2>3,5,10),(4>8),(5>6),(6>9),(8>10)	[3,6,7]
(1>4,9,10),(2>4,7),(3>6,8),(4>5),(6>9),(7>9),(8>9)	[4,6,7]
(1>9),(2>4),(3>8,9),(4>5),(5>8),(6>8,9),(7>8),(8>10)	[1,4,7]
(1>6,8,10),(2>3,7),(3>4),(4>6,8,10),(5>7,9,10),(6>9),(8>9)	[1,3,5]
(1>2,4,9),(2>3,6),(3>5),(5>7),(8>9,10)	[5,6,10]

11 Elements - Strict Cycles

(1>8),(2>6,11),(3>7,10),(4>5,9),(5>11),(6>10),(7>9),(8>9,10,11)	[2,3,4]
(1>5),(2>3,6),(3>4,9),(5>6),(6>7,8),(8>9,10),(9>11)	[4,7,9]
(1>7,8),(2>4,5),(3>9,11),(4>10),(5>8,9,10),(6>8),(7>9),(10>11)	[3,5,6]
(1>10,11),(2>4,10),(3>8),(4>9),(5>11),(6>8),(7>10),(8>10),(9>11)	[4,5,7]
(1>6,10),(2>3,9),(3>10),(4>7),(5>7,8),(6>8),(7>9,11)	[3,6,7]
(1>2,4,6,8),(2>5,9,11),(3>6,8,11),(4>7),(5>10),(6>7),(7>9),(9>10)	[5,7,8]
(1>6),(2>6,7,8,11),(3>4,7),(4>5,6,9),(9>10),(10>11)	[5,7,10]
(1>4,11),(2>5,7,8),(3>8),(4>6),(5>9),(6>7),(9>10)	[6,9,11]

11 Elements - Quasi Cycles

(1>4,7,10),(2>6),(3>10),(4>9),(5>9),(6>8),(7>9,11),(8>10)	[3,6,7]
(1>2,3,10),(2>8),(3>6,8,9),(4>8,9,10),(5>9,10),(7>8,10),(8>11),(9>11),(10>11)	[6,9,10]
(1>3,11),(2>11),(3>4),(4>9),(5>9),(6>7),(7>8,9),(8>11),(10>11)	[2,3,5]
(1>3,8,10),(2>10),(3>4,9),(4>6),(5>8,11),(6>7)	[6,9,11]

12 Elements - Strict Cycles

(1>6),(2>5,11),(3>6,7,10),(4>6,8),(5>7),(6>9),(7>12),(8>12),(9>11)	[2,3,4]
(1>10),(2>3,4),(3>7,11),(4>6,12),(5>10),(7>8,9),(8>10),(10>12)	[1,4,7]

12 Elements - Quasi Cycles

(1>4),(2>3),(3>6),(4>7,8,9,12),(5>8),(6>11),(10>11),(11>12)	[3,4,10]
(1>4,6,7,12),(2>6),(3>5,7,9,11),(4>5,9,11),(6>9),(7>10),(8>12),(9>10)	[5,9,12]

*(a>b,c,d) denotes aSb, aSc, and aSd in diagram S. [a,b,c] denotes a LEM cycle on pairs (a,b), (b,c), and (c,a).

REFERENCES

1. M. Aigner, Combinatorial Search, *Wiley-Teubner*, Chichester, UK, 1988.
2. G. Brightwell, P. C. Fishburn and P. Winkler, Interval Ordered Linear Extension Cycles, Mimeograph, 1990.
3. K. Ewacha, P. C. Fishburn and W. V. Gehrlein, A Note on Linear Extensions on Height-1 Orders, *Order*, 1990, *6*, pp. 313-318.
4. P. C. Fishburn, On the Family of Linear Extensions of a Partial Order, *J. Combin. Theory*, 1974, *17*, pp. 240-243.
5. P. C. Fishburn, On Linear Extension Majority Graphs of Partial Orders, *J. Combin. Theory*, 1976, *21*, pp. 65-70.
6. P. C. Fishburn, Proportional Transitivity in Linear Extensions of Ordered Sets, *J. Combin. Theory*, 1986, *41*, pp. 48-60.
7. P. C. Fishburn and W. V. Gehrlein, A Comparative Analysis of Methods for Constructing Weak Orders from Partial Orders, *J. Math. Sociol.*, 1975, *4*, pp. 93-102.
8. B. Ganter, G. Häfner and W. Poguntke, On the Linear Extensions of Ordered Sets with a Symmetry, *Discrete Math.*, 1987, *63*, pp. 153-156.
9. W. V. Gehrlein, On Methods for Generating Random Partial Ordering Relations, *Oper. Res. Lett.*, 1986, *5*, pp. 285-291.
10. W. V. Gehrlein, An Algorithm for generating the Complete set of Nonisomorphic Partial Orders, Mimeograph, 1989.
11. W. V. Gehrlein and P. C. Fishburn, Linear Extension Majority Cycles on Small ($n \leqq 9$) Partial Orders, *Comput. Math. Appl.*, 1990, *20*, pp. 41-44.
12. W. V. Gehrlein and P. C. Fishburn, Linear Extension Majority Cycles for Partial Orders, *Ann. Oper. Res.*, 1990, *23*, pp. 311-322.
13. T. R. Hoffmann, An Algorithm for Topological Sorting Without Duplication, Mimeograph, 1986.
14. F. S. Roberts, Applied Combinatorics, *Prentice Hall*, Englewood Cliffs, NJ, 1984.