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BLAS PELEGRIN

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HEURISTIC METHODS FOR THE p-CENTER PROBLEM (*)

by Blas PELEGRIN [1]

Abstract. — *This paper deals with heuristic methods for the p-center problem in any metric space. Due to the complexity of this problem, a variety of heuristic algorithms have been proposed in different cases, usually in Network and Planar Location. Most of them are related to some class of method that can be used for the problem in any metric space. A review of these methods is presented and a new method is also proposed which is a 2-approximation heuristic for some particular cases of the problem.*

Keywords : Location; Cluster Analysis; Heuristics.

Résumé. — *Cet article traite par des méthodes heuristiques le problème du p-centre dans n'importe quel espace métrique. A cause de la complexité de ce problème, une variété d'algorithmes heuristiques a été proposée dans différents cas, généralement en localisation en réseaux et dans le plan. La majorité d'entre eux est reliée à quelques classes de méthodes pouvant être utilisées pour ce problème dans n'importe quel système métrique. Ici, on présente l'ensemble de ces méthodes et on propose aussi une méthode nouvelle, qui est une 2-approximation heuristique pour quelques cas particuliers de ce problème.*

1. INTRODUCTION

Let X be any metric space, with metric $d(.,.)$, and $M=\{P_1, P_2, \ldots, P_m\}$ be any finite subset of X, with positive weights w_i for each $P_i \in \mathrm{M}$. In a general framework, the p-center problem is defined as to find p points $C_1, C_2, \ldots, C_p$ in X so that the maximal weighted distance between each point P_i and its closest point C_j, $j=1, 2, \ldots, p$, is minimized. The problem is formulated as:

$$\text{(F1)}\quad \underset{C_1, \ldots, C_p \in X}{\text{Minimize}}\ Z(C_1, C_2, \ldots, C_p) = \underset{1 \leq i \leq m}{\text{Max}} \left\{ w_i \underset{1 \leq j \leq p}{\text{Min}}\ d(P_i, C_j) \right\}.$$

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[1] Departamento de Matemática Aplicada y Estadística, Universidad de Murcia, Campus de Espinardo, Murcia (Spain).

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Alternatively, the problem can be seen as to find a partition $\alpha=\{M_1, M_2, \ldots, M_p\}$ of the set M into p disjoint subsets, so that the maximum among the maximal weighted distances between the best point (1-center) and the points in each subset is minimized. Then it is formulated as:

$$\text{(F2)} \quad \underset{\alpha \in P(M,\, p)}{\text{Minimize}}\, R_\alpha = \text{Max}\{R(M_1), R(M_2), \ldots, R(M_p)\}$$

where $P(M, p)$ denotes the set of all partitions of M into p disjoint subsets and $R(M_j)$ denotes the optimal value of the 1-center problem associated with the points in M_j, $j=1, \ldots, p$. Then optimal centers, $C_1, C_2, \ldots, C_p$, are optimal solutions to the 1-center problems associated with an optimal partition.

The problem arises typically in the field of Location Theory, where it has been studied mainly in two contexts: Network Location and Planar Location. The first, when X is a network, M is the set of nodes, and d is given by the shortest path (*see* [12 to 15, 18]). The second, when X is $\mathbb{R}^2$ and d is given by a norm function, usually the euclidean or the rectangular norm (*see* [1, 7, 23, 29, 30]).

The problem can also be found in Cluster Analysis when the aim is to create a number p of groups in a set of objects M, given by points in $\mathbb{R}^n$, so that the maximal dissimilarity, measured by a metric d, between each object P_i and the center C_j of its group is minimized (*see* [27]).

When the set of centers is constrained to be in a finite set, the problem is usually known as the *p-center problem*. Otherwise, it is also known as the *absolute p-center problem* in Network Location and as the *continuous p-center* problem in Planar Location and Cluster Analysis. In all these cases, it has been proved to be NP-hard, even to approximate the problem sufficiently closely (*see* [17, 19, 20]). Due to its complexity, a variety of heuristic algorithms have been proposed in the literature in different cases, usually in Network and Planar Location (*see* [6, 9, 10, 12, 16, 23, 25]).

This paper deals with heuristic methods for the p-center problem in any metric space. Formulation (F2) is used and special attention is given to δ-approximate heuristics. Firstly, we review some methods, most proposed for particular cases of the problem, which can be used for any metric space. Secondly, we propose a new heuristic which is shown to be a 2-approximation algorithm in some particular cases. It is known that to produce partitions within δ times the optium is NP-hard for $\delta<2$ (*see* [16, 17]). Thus, in certain cases, the algorithm is "best possible". Finally, some general considerations are mentioned.

2. REVIEW OF HEURISTIC METHODS

Many heuristic algorithms, proposed for some particular cases of the p-center problem, are related to some class of method that can be used for the problem in the above general framework. In the following, we review heuristic methods of two types that can be extensively used.

Type 1: heuristics based on the 1-center problem

Most of heuristic proposed for other location-allocation problems, for instance the alternate location-allocation method and the exchange method (*see* [3, 4, 26, 28]), can be modified to be used for the p-center problem. Each of them iterates in the set of partitions $P(M, p)$ and contains a single center subroutine (SCS). This modified type of methods for the p-center problem is as follows: Start with any partition $\alpha = \{M_1, M_2, \ldots, M_p\}$. Use SCS to evaluate $R(M_j)$ and to obtain the center C_j associated with each subset M_j. Once R_α is evaluated, either a new partition α' with $R_{\alpha'} < R_\alpha$ is generated or the procedure is finished.

These methods can be used for the problem in any metric space whenever a SCS is given. If each center is constrained to be in M, which often happens in Cluster Analysis, the SCS is given by enumeration. In other cases, some SCS have been proposed for absolute and continuous centers under different measures of distance (*see* [2, 8, 11, 18, 21, 24]).

Type 2: δ-approximation heuristics

Other proposed heuristics generate output partitions without using any SCS. The most interesting are those that generate δ-approximate solutions, *i. e.*, some partition α such that $R_\alpha \leqq \delta R^*$ where R^* denotes the optimal value of (F2). To our knowledge, only one of these heuristics has been proposed for any metric space. It is as follows:

*Heuristic H*1

1. Choose $C_1 = P_k$ such that $w_k = \text{Max}\{w_i : i = 1, \ldots, m\}$. Set $j = 1$ and $d(P_i) = w_i d(P_i, C_1)$ for each $P_i \in M$.

2. While $j < p$ do:

$$C_{j+1} = P_t \quad \text{such that } d(P_t) = \text{Max}\{d(P_i) : i = 1, \ldots, m\}$$

$$d(P_i) = \text{Min}\{d(P_i),\ w_i\, d(P_i, C_{j+1})\} \quad \text{for each } P_i \in M.$$

3. Determine $\alpha = \{ M_1, M_2, \ldots, M_p \}$ where

$$M_j = \{ P_i : w_i d(P_i, C_j) \leqq w_i d(P_i, C_k),\ k = 1, \ldots, m \}$$

for $j = 1, \ldots, m$. If a point P_i belong to more than one set M_j assign it arbitrarily to one set.

H1 was given by Dyer and Frieze (9) for points in any metric space, and guarantees $\delta_0 = \mathrm{Min}\{3, 1 + \beta\}$, where β is the maximal ratio between the weights of points in M.

However, when a finite set L of possible values for R^* is known, as happens for $X = M$ and for some continuous problems, the following heuristic can also be used:

Heuristic H2

1. Arrange all the values in L into an increasing sequence $R_1 < R_2 < \ldots < R_q$.

2. Find the least value $R \in \{ R_1, R_2, \ldots, R_q \}$ for which the following subroutine yields an output S with $|S| \leqq p$

RANGE:

- Set $S = \varnothing$ and make unlabelled all points in M.
- While there is any unlabelled point in M do:

Choose an unlabelled point P_t of maximum weight, put $S = S \cup P_t$, and label P_t and every unlabelled point P_i such that $w_i d(P_i, P_t) \leqq 2R$.

3. Augment S arbitrarily to a set of p points. Denote $C_1, C_2, \ldots, C_p$ these points.

4. Determine $\alpha = \{ M_1, M_2, \ldots, M_p \}$ where

$$M_j = \{ P_i : w_i d(P_i, C_j) \leqq w_i d(P_i, C_k),\ k = 1, \ldots, m \}$$

for $j = 1, \ldots, m$. If a point P_i belong to more than one set M_j assign it arbitrarily to one set.

H2 was given by Plesnik (25) for points in a network and guarantees $\delta = 2$ for both the p-center and the absolute p-center problems. However, as it is shown above, H2 can be used for any metric space, whenever a finite set of possible values for R^* is known. Then H2 is also a 2-approximation heuristic, since RANGE output S with $|S| \leqq p$ for a given R if there exists a partition α with $R_\alpha \leqq R$ (*see* [25]), which implies $R_\alpha \leqq 2R_0 \leqq 2R^*$ where R_0 is the least value found by RANGE.

3. A NEW HEURISTIC METHOD

We propose a new heuristic method for the problem in any metric space. The method is based on a lower bound of $R(M_j)$ given in Dearing and Francis (5) by:

$$B(M_j) = \mathrm{Max}\{ w_i w_k d(P_i, P_k)/(w_i + w_k) : P_i, P_k \in M_j \}$$
$$j = 1, \ldots, p.$$

First, let R_{ik} denotes the value $w_i w_k d(P_i, P_k)/(w_i + w_k)$ for each i, $k = 1, \ldots, m$, $i \neq k$.

Heuristic H3

1. Choose an initial partition $\alpha = \{M_1, M_2, \ldots, M_p\}$ and calculate $R_0 = \mathrm{Max}\{B(M_1), B(M_2), \ldots, B(M_p)\}$.

2. Make a list L arranging all the distinct values $R_{ik} \leqq R_0$ into an increasing sequence.

3. If $|L| > 1$ take any $R \in L$ such that $R < R_0$, make unlabelled all points in M and go to 4. Else, $L = \{R_0\}$, output R_0 and α.

4. Choose an unlabelled point P_t of maximum weight and set $M'_t = P_t \cup \{P_i : P_i$ is unlabelled and $R_{it} \leqq R\}$. Label all points in M'_t. If all points in M are labelled go to 5, else go to 4.

5. Set $\alpha' = \{M'_t : M'_t$ is generated in 4$\}$. If $|\alpha'| \leqq p$ reset $L = \{R_{ik} : R_{ik} \leqq R\}$, $\alpha = \alpha'$, and $R_0 = R$. Else reset $L = \{R_{ik} : R_{ik} > R\}$. Go to 3.

We now state some properties of H3.

PROPERTY 1. — The complexity of H3 is $O(m^2 \log m)$.

Proof. — Steps 1 and 2 can be performed in time $O(p \log m)$ and $O(m^2 \log m)$ respectively. Step 3 to step 5 is a binary search that find the minimum value in L for which the partition α' generated in step 4 verifies $|\alpha'| \leqq p$. As step 4 is $O(m^2)$ and the binary search is $O(\log m)$, it follows that the complexity of H3 is $O(m^2 \log m)$.

PROPERTY 2. — If $R^* \in L$ then H3 is a 2-approximation algorithm and the output value R_0 is a lower bound of R^*.

Proof. — Let C'_t denotes an optimal solution to the 1-center problem associated with M'_t for each M'_t generated in step 4. It is verified that

$$R(M'_t) = \mathrm{Max}\{ w_i d(P_i, C'_t) : P_i \in M'_t \} \leqq \mathrm{Max}\{ w_i d(P_i, P_t) : P_i \in M'_t \}.$$

As $R_{it} \leqq R$ and $w_i \leqq w_t$ then $w_i d(P_i, P_t) \leqq (w_i + w_t) R/w_t \leqq 2R$ for each $P_i \in M'_t$. Therefore $R(M'_t) \leqq 2R$ and $R_{\alpha'} = \text{Max}\{R(M'_t) : M'_t$ is generated in step $4\} \leqq 2R$.

Let R be such that there exists a partition $\alpha = \{M_1, M_2, \ldots, M_p\}$ with $R_\alpha \leqq R$. In each iteration of step 4 the chosen point P_t belong to M_j for some j. If $P_i \in M_j$ and P_i is unlabelled when P_t is chosen, as $R_{it} \leqq B(M_j) \leqq R(M_j) \leqq R_\alpha \leqq R$, it follows that $P_i \in M'_t$. Therefore $|\alpha'| \leqq p$, where $\alpha' = \{M'_t : M'_t$ is generated in step $4\}$. Then, H3 iterates while there is some R in L such that $R^* \leqq R$.

From the above results, if $R^* \in L$ the output value R_0 and the output partition α will verify that $R_0 \leqq R^*$ and $R_\alpha \leqq 2R_0 \leqq 2R^*$.

PROPERTY 3. — If $R^* \in L$ and $w_i = w$, $i = 1, \ldots, m$, then H2 and H3 are equivalents. Besides, H1, H2 and H3 are 2-approximation heuristics.

Proof. — As $w_i = w$, $i = 1, \ldots, m$, $w_i d(P_i, \mathrm{P}_t) \leqq 2R$ is equivalent to $R_{it} \leqq R$ for any $R \in L$. Then, each iteration of RANGE in H2 generates the same set of labelled points that the corresponding iteration of step 4 in H3. Therefore, starting with the set L, H2 and H3 generate the same output partition. Since $\delta_0 = 2$, it follows that H1, H2, and H3 are 2-approximation heuristics.

From property 2 it follows that H3 is a 2-approximation algorithm when $B(M_j) = R(M_j)$, $j = 1, \ldots, p$. Necessary and sufficient conditions for this are given in (5) and (22). For instance, it happens in the following particular cases (*see* [5, 11, 22, 24]):

(*a*) X is a tree in Network Location.

(*b*) $X = \mathbb{R}^n$ and M is any set of collinear points.

(*c*) $X = \mathbb{R}^2$ and d is given by the Rectilinear norm.

(*d*) $X = \mathbb{R}^n$ and d is given by any weighted Tchebycheff norm.

In such cases, an interesting aspect of H3, in comparison with H1 and H2, is that H3 generates an output partition α for which is not necessary to use any SCS to evaluate R_α. Furthermore, H3 gives a better approximation than H1 for weighted problems since then $\delta_0 > 2$.

In any metric space, in spite of H3 can be used, there is no guarantee that it generates a partition within 2 times R^*. A sufficient condition for this is that $R_0 \leqq R^*$.

4. GENERAL CONSIDERATIONS

We have shown different heuristic methods for the p-center problem in any metric space. Heuristic methods of Type 1 can be used whenever a SCS is given. Usually, these methods are time consuming, so we recommend to use them after someone heuristic of Type 2. Besides, the quality of an output partition given by an heuristic of Type 2 can be improved by an heuristic of Type 1, taking that partition as an initial partition.

Concerning to heuristics of Type 2, H2 or H3 can be normally used for case (a) to case (d) given in section 3, and H2 for p-center problems on a Network. For all these cases, output partitions within 2 times R^* can be generated, and an upper bound of the related error $(R_\alpha - R^*)/R^*$ is given by $(R_\alpha - R_0)/R_0$. Furthermore, SCS have been given and then heuristics of Type 1 can be used to improve the quality of their output partitions. For problems in any metric space, H2 or H3 can also be used, but 2-approximation is not guaranteed unless a finite set of possible values for R^* is known or $R^* \in \{ R_{ik} : i, k = 1, \ldots, m, i \neq k \}$; however, H1 can always be used as a δ_0-approximation.

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