RAIRO. Recherche opérationnelle

T. Nakagawa

A summary of replacement models with changing failure distributions

RAIRO. Recherche opérationnelle, tome 23, nº 4 (1989), p. 343-353

<http://www.numdam.org/item?id=RO_1989__23_4_343_0>

Recherche opérationnelle/Operations Research
(vol. 23, n° 4, 1989, p. 343 à 353)

A SUMMARY OF REPLACEMENT MODELS WITH CHANGING FAILURE DISTRIBUTIONS (*)

by T. NAKAGAWA ([1])

Abstract. — *A unit has changing failure distributions for each duration time between events such as fault, use, preventive maintenance and shock. This paper formulates a general replacement model where a unit is replaced by a new unit at failure or after a specified number N of events. As special cases, each optimal number N* to minimize the expected cost rates of four models is discussed. These results could be applied to actual models as one approximation solution.*

Keywords : Replacement, changing distributions, expected cost, optimization.

Résumé. — *Une unité possède des lois de probabilité de panne qui changent pour chaque durée entre les événements tels que défaillance, usure, maintenance préventive et choc. Nous formulons un modèle général de renouvellement ou une unité est remplacée par une nouvelle unité en cas de panne ou après un nombre spécifié N d'événements. Comme cas spéciaux, nous examinons le nombre optimal N* qui minimise le coût moyen de quatre modèles. Ces résultats peuvent être appliqués à des modèles réels comme solution approximative.*

1. INTRODUCTION

The times to failures of some units might be measured by the following factors: The number of events of fault, use, preventive maintenance, and shock. As a typical example, Morimura [7] suggested the model where a unit is replaced at the N-th failure and the previous failures were corrected with minimal repair. These are called discrete replacement models and are summarized in Nakagawa [9]. These models are actually applied to the maintenance of a computer system: The system stops because of intermittent faults or transient failures and undergoes maintenance if more than N faults or failures have occurred.

Suppose that a unit is replaced at a specified number of events such as fault, use, preventive maintenance, and shock. Most replacement models have

(*) Received April 1988.
([1]) Department of Industrial Engineering, Aichi Institute of Technology, Toyota 470-03, Japan.

assumed that a unit after maintenance is as good as new or the failure rate is the same as before maintenance [2]. Actually, this assumption might not be true. Most maintenance improves a unit, and hence, its failure rate decreases but not to zero. Lie and Chun [5] introduced an improvement factor in failure rate after preventive maintenance, and Canfield [3] suggested a failure rate function where the shape is changing at preventive maintenance, however, it is monotone with time.

As one approximation method, we assume that for each duration time between events, a unit has changing probabilities to failure. Aroian [1] and Sreedharan [13] have studied life testing of a unit with changing failure rates in cycles. Under the assumptions, we formulate a general replacement model where a unit is replaced at number N of events or at failure, whichever occurs first. The expected cost rate is derived and the optimum policy to minimize it is discussed, when the expected cost rates between events increase. As special cases of a general model, a unit is replaced at number N of faults, uses, preventive maintenances and shocks. Then, an optimal number N^* of each model is given by a unique solution of an equation. We finally consider a replacement model where a unit undergoes minimal repair at failures and is replaced at number N of events.

2. GENERAL REPLACEMENT MODEL

Consider the time over an infinitely long event j ($j=1, 2, \ldots$) that the unit should be operating. Let X_j be a random variable denoting the duration time in the j-th period of events, and $F_j(t) \equiv \Pr\{X_j \leqq t\}$, and $\mu_j \equiv \int_0^\infty \bar{F}_j(t)\,dt$ where $F_j(0) \equiv \Pr\{X_j = 0\} < 1$ and $\bar{F}_j \equiv 1 - F_j$. It is assumed that X_j is independent with each other.

The probability that the unit fails during the j-th period of events is p_j and $\bar{p}_j \equiv 1 - p_j$ ($j=1, 2, \ldots$), $\bar{p}_0 \equiv 1$. Suppose that the unit begins to operate at time 0 and is replaced by a new unit at number N of events or at failure, whichever occurs first.

The probability that the unit is replaced at number N of events before failure is

$$\prod_{j=1}^{N} \bar{p}_j, \tag{1}$$

and the probability that the unit is replaced at failure is

$$1-\prod_{j=1}^{N}\bar{p}_j. \tag{2}$$

Further, the expected number of events before replacement is

$$\sum_{j=1}^{N}(j-1)p_j\left[\prod_{i=1}^{j-1}\bar{p}_i\right]+(N-1)\left[\prod_{j=1}^{N}\bar{p}_j\right]=\sum_{j=1}^{N-1}\left[\prod_{i=1}^{j}\bar{p}_i\right]. \tag{3}$$

Note that this does not include the event at which the unit is replaced. The mean time to replacement is

$$\sum_{j=1}^{N}p_j\left[\prod_{i=0}^{j-1}\bar{p}_i\right]\left[\sum_{i=1}^{j}\mu_i\right]+\left[\prod_{j=1}^{N}\bar{p}_j\right]\left[\sum_{j=1}^{N}\mu_j\right]=\sum_{j=1}^{N}\mu_j\left[\prod_{i=0}^{j-1}\bar{p}_i\right]. \tag{4}$$

the expected cost rate is, from the renewal reward theorem [12],

$$C(N)=\frac{c_1\left[1-\prod_{j=1}^{N}\bar{p}_j\right]+c_2\left[\prod_{j=1}^{N}\bar{p}_j\right]+c_3\sum_{j=1}^{N-1}\left[\prod_{i=1}^{j}\bar{p}_i\right]}{\sum_{j=1}^{N}\mu_j\left[\prod_{i=0}^{j-1}\bar{p}_i\right]}, \tag{5}$$

where c_1 = cost of replacement at failure, c_2 = cost of replacement at number N of events with $c_2<c_1$, and c_3 = cost of one event.

We seek an optimal scheduled number N^* which minimizes $C(N)$ in (5). It is assumed that $[(c_1-c_2)p_j+c_3]/\mu_j$ $(j=1, 2, \ldots)$ are strictly increasing in j, *i. e.*, the expected cost rates in the j-th period of events, including no scheduled replacement cost, increase with the number of events.

A necessary condition that there exists a finite and unique N^* is that an N^* satisfies $C(N+1)\geqq C(N)$ and $C(N)<C(N-1)$. From these inequalities, we have

$$L(N)\geqq c_2 \quad \text{and} \quad L(N-1)<c_2, \tag{6}$$

where $L(0)\equiv 0$, and

$$L(N)\equiv(c_1-c_2)\left\{\frac{p_{N+1}}{\mu_{N+1}}\sum_{j=1}^{N}\mu_j\left[\prod_{i=0}^{j-1}\bar{p}_i\right]-\left[1-\prod_{j=1}^{N}\bar{p}_j\right]\right\}$$
$$+c_3\left\{\frac{1}{\mu_{N+1}}\sum_{j=1}^{N}\mu_j\left[\prod_{i=0}^{j-1}\bar{p}_i\right]-\sum_{j=1}^{N-1}\left[\prod_{i=1}^{j}\bar{p}_i\right]\right\}\qquad(N=1,2,\ldots).$$

It is evident that

$$L(N)-L(N-1)=\sum_{j=1}^{N-1}\mu_j\left[\prod_{i=0}^{j-1}\bar{p}_i\right]\left\{\frac{(c_1-c_2)p_{N+1}+c_3}{\mu_{N+1}}-\frac{(c_1-c_2)p_N+c_3}{\mu_N}\right\}>0.$$

Thus, if a solution N^* to (6) exists, it is unique. Further, if $L(\infty)\equiv\lim_{N\to\infty}L(N)>c_2$ then a finite solution exists. Hence, we may compute only a minimum N^* such that $L(N)\geqq c_2$, and the resulting minimum cost rate is

$$\frac{(c_1-c_2)p_N+c_3}{\mu_N}<C(N)\leqq\frac{(c_1-c_2)p_{N+1}+c_3}{\mu_{N+1}}.\qquad(7)$$

3. FOUR REPLACEMENT MODELS

3.1. Replacement after N faults

The unit stops operation because of faults due to noise, temperature, power supply variations and poor electric contacts. Then, consider the following two kinds of faults; The unit operates again instantly by the detection of faults with probability α $(0<\alpha\leqq 1)$, and conversely, it fails with probability $1-\alpha$. The mean times between faults are μ_j $(j=1,2,\ldots)$, which are decreasing in j.

Suppose that the unit is replaced at number N of faults or at failure, whichever occurs first. Then, putting $p_j=1-\alpha$ in (5), the expected cost rate is

$$C_1(N)=\frac{c_1(1-\alpha^N)+c_2\alpha^N+c_3\sum_{j=1}^{N-1}\alpha^j}{\sum_{j=1}^{N}\alpha^{j-1}\mu_j} \tag{8}$$

Using the results of the previous section, an optimal number N^* satisfies the two inequalities

$$L_1(N)\geqq\frac{c_2-c_3}{(1-\alpha)(c_1-c_2)+c_3}$$

and

$$L_1(N-1)<\frac{c_2-c_3}{(1-\alpha)(c_1-c_2)+c_3}, \tag{9}$$

where $L_1(0)\equiv 0$,

$$L_1(N)\equiv\frac{1}{\mu_{N+1}}\sum_{j=1}^{N}\alpha^{j-1}\mu_j-\frac{1-\alpha^N}{1-\alpha}\qquad (N=1, 2, \ldots).$$

Since $\mu_j>\mu_{j+1}$, $L_1(N)$ is strictly increasing in N. Hence, if a solution N^* to (9) exists, it is unique, and the resulting cost rate is

$$\frac{(1-\alpha)(c_1-c_2)+c_3}{\mu_{N^*}}<C_1(N^*)\leqq\frac{(1-\alpha)(c_1-c_2)+c_3}{\mu_{N^*+1}}. \tag{10}$$

We put formally $\alpha=1$, and $\mu_{j+1}=\int_0^\infty\{[R(t)]^j/j!\}e^{-R(t)}dt$ $(j=0, 1, 2, \ldots)$ which represents the mean times between failures in a non-homogeneous Poisson process with an intensity function $R(t)$. Then, this corresponds to Policy III of Morimura [7] where a unit is replaced at the N-th failure and undergoes only minimal repair at failures.

In particular, when $\bar{F}_j(t)=\exp\{-[\lambda+(j-1)\lambda_0]t\}$ $(j=1, 2, \ldots)$, *i.e.*, the failure rate increases by λ_0 with the number of failures, there exists a finite and unique minimum N^* such that

$$\sum_{j=1}^{N}\frac{1+N(\lambda_0/\lambda)}{1+(j-1)(\lambda_0/\lambda)}-(N-1)\geqq\frac{c_2}{c_3}.$$

3.2. Model 2: Replacement after N uses

The unit is intermittently used and neither fails nor deteriorates while it is not used, *i. e.*, the time is measured only by the time of uses [6]. Suppose that each duration time of a usage period has an exponential distribution $(1-e^{-\theta t})$. The failure times of the unit have distributions $F_j(t)$ $(j=1, 2, \ldots)$ in the j-th usage period, which are increasing in j, *i. e.*, $F_j(t)<F_{j+1}(t)$ for any $t>0$. Then, the probability that the unit fails during the j-th period is

$$\int_0^{\infty} e^{-\theta t}\, dF_j(t)=F_j^*(\theta),$$

where $F_j^*(.)$ represents the Laplace-Stieltjes transform of $F_j(t)$ and $\bar{F}_j^* \equiv 1-F_j^*$ $(j=1, 2, \ldots)$, $\bar{F}_0^* \equiv 1$.

The unit is replaced at failure or after the number N of uses is completed. Then, putting $p_j=F_j^*(\theta)$ and $\mu_j=1/\theta$ in (5), the expected cost rate is

$$C_2(N)=\frac{c_1\left[1-\prod_{j=1}^{N}\bar{F}_j^*(\theta)\right]+c_2\left[\prod_{j=1}^{N}\bar{F}_j^*(\theta)\right]+c_3\sum_{j=1}^{N-1}\left[\prod_{i=1}^{j}\bar{F}_i^*(\theta)\right]}{(1/\theta)\sum_{j=1}^{N}\left[\prod_{i=0}^{j-1}\bar{F}_i^*(\theta)\right]} \tag{11}$$

where c_1 = cost of replacement at failure, c_2 = cost of replacement after number N of uses with $c_2<c_1$, and c_3 = cost of one use.

From the assumption that $F_j(t)<F_{j+1}(t)$, we can easily prove that $F_j^*(\theta)<F_{j+1}^*(\theta)$ $(j=1, 2, \ldots)$, *i. e.*, the probabilities of failures in a usage period increase with the number of uses. An optimal N^* satisfies uniquely the two inequalities

$$L_2(N) \geqq \frac{c_2-c_3}{c_1-c_2} \quad \text{and} \quad L_2(N-1)<\frac{c_2-c_3}{c_1-c_2}, \tag{12}$$

where $L_2(0) \equiv 0$ and

$$L_2(N) \equiv F_{N+1}^*(\theta)\sum_{j=1}^{N}\left[\prod_{i=0}^{j-1}\bar{F}_i^*(\theta)\right]-\left[1-\prod_{j=1}^{N}\bar{F}_j^*(\theta)\right] \qquad (N=1, 2, \ldots).$$

If each duration time of a usage period is not exponential and has a general distribution $G(t)$ with mean $1/\theta$ then the expected cost rate in (11) is rewritten as

$$C_2(N)=\frac{c_1\left[1-\prod_{j=1}^{N}F_j^+(G)\right]+c_2\left[\prod_{j=1}^{N}F_j^+(G)\right]+c_3\sum_{j=1}^{N-1}\left[\prod_{i=1}^{j}F_i^+(G)\right]}{(1/\theta)\sum_{j=1}^{N}\left[\prod_{i=0}^{j-1}F_i^+(G)\right]}, \tag{13}$$

where $F_j^+(G)\equiv\int_0^\infty G(t)\,dF_j(t)$.

3.3. Model 3: Replacement after N preventive maintenances

The unit is maintained preventively at scheduled times jT $(j=1,2,\ldots)$ where T is previously specified, and has a failure time distribution $F_j(t)$ after the $(j-1)$ th completion of preventive maintenance. It is assumed that $F_j(t)<F_{j+1}(t)$ for any $t>0$, *i.e.*, the probabilities of failures increase with the number of preventive maintenances.

The unit is replaced at failure or at time NT, whichever occurs first. Then, the mean time to replacement is

$$\sum_{j=1}^{N}\left[\prod_{i=0}^{j-1}\bar{F}_i(T)\right]\int_0^T[t+(j-1)T]\,dF_j(t)+(NT)\left[\prod_{j=1}^{N}\bar{F}_j(T)\right]$$
$$=\sum_{j=1}^{N}\left[\prod_{i=0}^{j-1}\bar{F}_i(T)\right]\int_0^T\bar{F}_j(t)\,dt,$$

where $\bar{F}_j\equiv 1-F_j$ and $\bar{F}_0\equiv 1$. Thus, putting $p_j=F_j(T)$ and $\mu_j=\int_0^T\bar{F}_j(t)\,dt$ in (5), the expected cost rate is

$$C_3(N)=\frac{c_1\left[1-\prod_{j=1}^{N}\bar{F}_j(T)\right]+c_2\left[\prod_{j=1}^{N}\bar{F}_j(T)\right]+c_3\sum_{j=1}^{N-1}\left[\prod_{i=1}^{j}\bar{F}_i(T)\right]}{\sum_{j=1}^{N}\left[\prod_{i=0}^{j-1}\bar{F}_i(T)\right]\int_0^T\bar{F}_j(t)\,dt}, \tag{14}$$

where $c_1=$cost of replacement at failure, $c_2=$cost of replacement at number N of preventive maintenances with $c_2<c_1$, and $c_3=$cost of one preventive maintenance.

It is evident that $[(c_1-c_2)F_j(T)+c_3]/\int_0^T \bar{F}_j(t)\,dt$ is increasing in j since $F_j(T)<F_{j+1}(T)$. Hence, from (6), we may compute a unique minimum such that

$$(c_1-c_2)\left\{\frac{F_{N+1}(T)}{\int_0^T \bar{F}_{N+1}(t)\,dt}\sum_{j=1}^{N}\left[\prod_{i=0}^{j-1}\bar{F}_i(T)\right]\int_0^T \bar{F}_j(t)\,dt-\left[1-\prod_{j=1}^{N}\bar{F}_j(T)\right]\right\}$$
$$+c_3\left\{\frac{1}{\int_0^T \bar{F}_{N+1}(t)\,dt}\sum_{j=1}^{N}\left[\prod_{i=0}^{j-1}\bar{F}_i(T)\right]\right.$$
$$\left.\times\int_0^T \bar{F}_j(t)\,dt-\sum_{j=1}^{N-1}\left[\prod_{i=1}^{j}\bar{F}_i(T)\right]\right\}\geqq c_2. \quad (15)$$

3.4. Model 4: Replacement after N shocks

Consider the unit which fails because of damage due to shocks [4, 8]: Shocks occur in a renewal process with mean interval $(1/\theta)$. The probabilities that the unit fails at shock j is p_j $(j=1, 2, \ldots)$, which increase with the number of shocks, *i. e.*, $p_j<p_{j+1}$.

The unit is replaced at failure or shock N, whichever occurs first. Then, putting $\mu_j=1/\theta$ in (5), the expected cost rate is

$$C_4(N)=\frac{c_1\left[1-\prod_{j=1}^{N}\bar{p}_j\right]+c_2\left[\prod_{j=1}^{N}\bar{p}_j\right]+c_3\sum_{j=1}^{N-1}\left[\prod_{i=1}^{j}\bar{p}_i\right]}{(1/\theta)\sum_{j=1}^{N}\left[\prod_{i=0}^{j-1}\bar{p}_i\right]}, \quad (16)$$

where c_1 = cost of replacement at failure, c_2 = cost of replacement at number N of shocks with $c_2<c_1$, and c_3 = cost of one shock.

An optimal number N^* satisfies uniquely the two inequalities

$$L_4(N)\geqq\frac{c_2-c_3}{c_1-c_2} \quad\text{and}\quad L_4(N-1)<\frac{c_2-c_3}{c_1-c_2}, \quad (17)$$

where $L_4(0) \equiv 0$ and

$$L_4(N) \equiv p_{N+1} \sum_{j=1}^{N} \left[\prod_{i=0}^{j-1} \bar{p}_i\right] - \left[1 - \prod_{j=1}^{N} \bar{p}_j\right] \qquad (N=1, 2, \ldots).$$

In particular, suppose that $\lim_{j \to \infty} p_j = 1$, *i. e.*, the unit fails certainly at finite number of shocks. Then, if $\sum_{j=1}^{\infty} \left[\prod_{i=1}^{j} \bar{p}_i\right] > [(c_2 - c_3)/(c_1 - c_2)]$ then there exists a finite and unique N^* which satisfies (17).

4. CONCLUDING REMARKS

We have discussed the optimal policies for four discrete replacement models where a unit is replaced at number N of faults, uses, preventive maintenances and shocks. To analyze these models, we have assumed that a unit has changing failure distributions for each duration time between events. This assumption may seem severe but certainly would be applied to actual models as one approximation method. For example, suppose that $\bar{F}_j(t) = \exp\{-[\lambda + (j-1)\lambda_0]t\}$ $(j=1, 2, \ldots)$, *i. e.*, the failure rates are increasing slowly by λ_0 with the number of events. It is easy to estimate λ and λ_0 from actual data by life testing [1, 13] and to compute the optimal policy using these results. In this case, it is of interest that there exists a finite and unique N^* for all policies.

Finally, suppose that the unit undergoes minimal repair at failures [2], and is replaced at number N of events. Let m_j be the expected number of minimal repairs during the j-th period of events. Then, the expected cost rate of a general replacement model can be expressed by

$$B(N) = \frac{b_1 \sum_{j=1}^{N} m_j + b_2 + (N-1) b_3}{\sum_{j=1}^{N} \mu_j}, \tag{18}$$

where b_1 = cost of minimal repair, b_2 = cost of replacement at number N of events, and b_3 = cost of one event with $b_3 < b_2$.

We can obtain expected cost rates and discuss optimum policies for the above four models, by the method similar to Section 3. For example, let $r_j(t)$ be the failure rate of $F_j(t)$, *i. e.*, $r_j(t) \equiv f_j(t)/\bar{F}_j(t)$ where f_j is a density of F_j.

Then, the expected cost rate of Model 3 where the unit is replaced at number N of preventive maintenances is, from [10, 11],

$$B_3(N)=\frac{b_1 \sum_{j=1}^{N} \int_0^T r_j(t)\,dt+b_2+(N-1)\,b_3}{NT} \tag{19}$$

If $r_j(t)<r_{j+1}(t)$ for any $t>0$ then an optimal number N^* satisfies uniquely the two inequalities

$$Q_3(N) \geqq \frac{b_2-b_3}{b_1} \quad \text{and} \quad Q_3(N-1)<\frac{b_2-b_3}{b_1}, \tag{20}$$

where $Q_3(0)\equiv 0$,

$$Q_3(N)\equiv N \int_0^T r_{N+1}(t)\,dt-\sum_{j=1}^{N} \int_0^T r_j(t)\,dt \qquad (N=1, 2, \ldots).$$

In particular case of $r_j(t)=\lambda+(j-1)\,\lambda_0$ $(j=1, 2, \ldots)$, an optimal number N^* is

$$N^*=\left[\frac{1}{2}+\left(\frac{1}{4}+\frac{2(b_2-b_3)}{\lambda_0\,T b_1}\right)^{1/2}\right]$$

where $[x]$ denotes the greatest integer contained in x.

REFERENCES

1. L. A. Aroian, *Sequential Life Tests for the Exponential Distribution with Changing Parameter*, Technometrics, Vol. 8, 1966, pp. 217-227.
2. R. E. Barlow and F. Proschan, *Mathematical Theory of Reliability*, John Wiley and Sons Inc., 1965.
3. R. V. Canfield, *Cost Optimization of Periodic Preventive Maintenance*, I.E.E.E. Trans. Reliability, Vol. R-35, 1986, pp. 78-81.
4. J. D. Esary, A. W. Marshall and F. Proschan, *Shock Models and Wear Processes*, Annals of Probability, Vol. 1, 1973, pp. 627-649.
5. C. H. Lie and Y. H. Chun, *An Algorithm for Preventive Maintenance Policy*, I.E.E.E. Trans. Reliability, Vol. R-35, 1986, pp. 71-75.
6. H. Mine, H. Kawai and Y. Fukushima, *Preventive Replacement of an Intermittently-used System*, I.E.E.E. Trans. Reliability, Vol. R-30, 1981, pp. 391-392.
7. H. Morimura, *On Some Preventive Maintenance Policies for IFR*, J, Operations Res. of Japan, Vol. 12, 1970, pp. 94-124.

8. T. NAKAGAWA, *On a Replacement Problem of a Cumulative Damage Model*, Operational Res. Q., Vol. 27, 1976, pp. 895-900.
9. T. NAKAGAWA, *A Summary of Discrete Replacement Policies*, European J., Operational Res., Vol. 17, 1984, pp. 382-392.
10. T. NAKAGAWA, *Periodic and Sequential Preventive Maintenance Policies*, J. Appl. Probability, Vol. 23, 1986, pp. 536-542.
11. D. G. NGUYEN and D. N. P. MURTHY, *Optimal Preventive Maintenance Policies for Repairable Systems*, Operations Res., Vol. 29, 1981, pp. 1181-1194.
12. S. ROSS, *Applied Probability Models with Optimization Applications*, Holden-Day, 1970.
13. K. E. SREEDHARAN, *Estimation of Periodically Changing Failure Rate*, I.E.E.E. Trans. Reliability, Vol. R-28, 1979, pp. 32-34.