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SOME REMARKS ON TWO DEGREES OF ASYMMETRY IN THE TRAVELING SALESMAN PROBLEM (*)

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Abstract. — *The quality of a heuristic for the traveling salesman problem is determined by its bound. Heuristics where the performance bounds depend on the asymmetry of the distance matrix are discussed. Two types of asymmetry, its relations and the relationship between different performance bounds are investigated. An algorithm for attaining a "good" and "easily-computable" bound is described.*

Keywords : Traveling salesman problem; heuristic algorithm; performance bound; degree of asymmetry; subgradient algorithm.

Résumé. — *La qualité d'une heuristique pour le problème du voyageur de commerce est déterminée par ses bornes. Nous examinerons les heuristiques où les bornes de performance dépendent du degré d'asymétrie de la matrice des distances. Nous étudierons deux types d'asymétrie, leurs inter-relations et la relation entre différentes bornes de performance. Nous décrivons un algorithme pour atteindre une borne qui soit "bonne" et "facile à calculer".*

Mots clés : Problème du voyageur de commerce; algorithme heuristique; borne de performance; degré d'asymétrie; algorithme subgradiant.

1. INTRODUCTION

The traveling salesman problem (TSP) can be described as follows. Find a closed directed path (tour) T^* with

$$V(T^*) \leqq V(T) \quad \text{for all tours } T,$$

where $V(T)$ denotes the length of the tour T. This problem is defined by the distance matrix $C=(c_{ij})$, $i, j=1, \ldots, n$, where c_{ij} denotes the distance from town i to town j. There exist many algorithms for solving this problem

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(*cf.* e. g. [10]). We know two kinds of algorithms: the exact and the heuristic algorithm. In both kinds of algorithms the symmetry (or asymmetry) plays an important role.

Now we try to find an approximation tour T_a with:

$$V(T_a) \leqq t(C)\, V(T^*).$$

There are heuristic algorithms where the performance bound $t(C)$ depends on the degree $S(C)$ of asymmetry of C (*cf.* section 2). Otherwise, using a heuristic algorithm for the symmetric matrix $D := (C + C^T)$ the estimated performance bound contains the expression $(1 + S(C))/(1 + k)$ which thus also includes the degree of asymmetry of C [*cf.* relation (2)].

Another idea of assigning an asymmetric C to a symmetric matrix $\mathbf{C}$ is given in [8]. But the dimension increases, in the worst case, by 2 and, further, all matrices $\mathbf{C}$ do not fulfil the triangle inequality.

The quality of a heuristic algorithm is characterized by its performance bound, that means a small bound ensures *a priori* a good approximation tour T_a. In order to select a favourable heuristic method we shall therefore also investigate the degree of asymmetry of $C(u)$ where $c(u)_{ij} = c_{ij} - u_i + u_j$. We compute the best attainable degree $S(C(u^*))$ (*cf.* section 3, 4). In the proposed algorithm we choose as initial value the solution u'' of a least square problem (*cf.* section 2). In section 6 we introduce another proper degree of asymmetry, $ST(C)$. If the matrix C fulfils the triangle inequality then the inequality $ST(C)) \leqq S(C(u))$ holds for all u. Because the determination of $ST(C)$ is, in general, a very hard task we recommend to compute $S(C(u^*))$. At the end of the paper we define a socalled asymmetry gap and conjecture an inequality.

2. NOTATIONS

The matrix C fulfils the triangle inequality (TI) if the following inequalities are true:

$$\text{(TI)} \qquad c_{ij} \leqq c_{ik} + c_{kj} \quad \text{for all } i, j, k \text{ distinct.}$$

In case of C fulfils (TI) a in [3] presented heuristic algorithm, an extended version of an algorithm of Christofides [1], yields a tour T_a with:

$$V_C(T_a) \leqq 1.5\, S(C)\, V_C(T^*). \tag{1}$$

It seems useful to determine the best possible value of $S(C)$.

If T is a tour then the tour in the opposite direction is denoted by T'. Let C' be the transposed matrix C and we obtain $V_{C'}(T)=V_C(T')$. Let T^* be an optimal tour with respect to the matrix C and T^{**} be an optimal tour with respect to the symmetric matrix D. If we have calculated a tour T_a with

$$V_D(T_a) \leqq r(n)\, V_D(T^{**})$$

then we get

$$\min\{V_C(T_a),\ V_{C'}(T_a)\} \leqq (1+S(C))/(1+k)\cdot r(n)\, V_C(T^*) \tag{2}$$

where

$$k\cdot \min\{V_C(T_a),\ V_{C'}(T_a)\}=\max\{V_C(T_a),\ V_{C'}(T_a)\},$$

and $S(C)$ denotes the degree of asymmetry of the matrix C.

Sketch of the proof:

$$\begin{aligned}(1+k)\cdot \min\{V_C(T_a),\ V_{C'}(T_a)\} &= V_D(T_a) \leqq r(n)\, V_D(T^{**})\\ &\leqq r(n)\, V_D(T^*) \leqq (1+S(C))\cdot r(n)\, V_C(T^*).\end{aligned}$$

We always have $k \geqq 1$, and therefore:

$$\min\{V_C(T_a),\ V_{C'}(T_a)\} \leqq \frac{1}{2}(1+S(C))\cdot r(n)\, V_C(T^*).$$

Relation (2) shows that the quality of T_a depends on the bound $(1+S(C))/(1+k)\cdot r(n)$ which should be as small as possible. Therefore we try to find a small value of the degree of asymmetry of C.

Now we consider the following degree of asymmetry:

$$S(C):=\begin{cases}+\infty & \text{if there exists an index pair } i, j \text{ with}\\ & \quad \operatorname{sign}(c_{ij}) \neq \operatorname{sign}(c_{ji})\\ \max\{c_{ij}/c_{ji}:\ \text{for all } i, j \text{ with } c_{ji}\neq 0\} & \text{else.}\end{cases}$$

In the following we investigate the degree of asymmetry under transformations of the type

$$c(u)_{ij}:=c_{ij}-u_i+u_j. \tag{3}$$

If C fulfils (TI), so does $C(u)$ for all vectors u.

Now we try to find a vector u^* with:

$$S(C(u^*)) \leqq S(C(u)) \quad \text{for all } u. \tag{4}$$

In [9] the following least square approach is discussed in order to obtain an approximation for u^*. Let

$$f(u) := \sum_{i,j} (c(u)_{ij} - c(u)_{ji})^2,$$

and determine a vector u'' with:

$$f(u'') \leqq f(u) \quad \text{for all } u. \tag{5}$$

Obviously, we have $S(C(u^*)) \leqq S(C(u''))$. If C fulfils (TI) then the matrix $C(u'')$ has the following properties (*cf.* [9]):

(*a*) $S(C(u'')) \leqq n-1$.

(*b*) $c(u'')_{ij} \geqq 0$ for all i, j with $i \neq j$.

(*c*) If there exists an index pair i, j $(i \neq j)$ with $c(u'')_{ij}=0$ then the original problem is equivalent to a problem with $n-1$ towns. We delete the row i and the column i (or the row j and column j).

(If $c(u'')_{ij}=0$ then $c(u'')_{ji}=0$ and we get $c(u'')_{ki}=c(u'')_{kj}$ for all $k \neq i$, $k \neq j$.)

Now we discuss an algorithm for determining a vector u^* as (4).

3. THE INITIAL PROCEDURE

Let C fulfil (TI). In the other case use the transformation from [5] and [7] in order to get a matrix which fulfils (TI). We compute

$$c_{kk} := \min_{i,j} \{c_{ik} + c_{kj} - c_{ij} \colon i, j, k \text{ distinct}\} \quad \text{for all } k$$

and after that

$$c_{ij} := c_{ij} - \frac{1}{2}(c_{ii} + c_{jj}) \quad \text{for all } i, j.$$

Obviously, the transformed matrix fulfils (TI).

Compute a vector u'' as (5). Delete all elements (rows and columns) with $c(u'')_{ij}=0$. Now we have a matrix $C(u'')$ with $c(u'')_{ij} > 0$. It is easy to show that then $c(u^*)_{ij} > 0$ follows.

4. THE SUBGRADIENT ALGORITHM

A solution u^* of (4) can be determined by means of the following algorithm. After the initial procedure we have to solve the optimization problem

$$S(C(u)) \to \min, \qquad u \in R^n.$$

We apply the subgradient technique from [2].

With $f_{ij}(u) := c(u)_{ij}/c(u)_{ji}$ we obtain:

$$\nabla f_{ij}(u) = (c(u)_{ji})^{-2}(c_{ij}+c_{ji})(0, \ldots, 0, \underset{i}{-1}, 0, \ldots, 0, \underset{j}{1}, 0, \ldots, 0)^T.$$

Let $I(u) := \{(i, j) : i \neq j,\ S(C(u)) = f_{ij}(u)\}$. Then we have

$$\partial S(C(u)) = \operatorname{conv}\{\nabla f_{ij}(u) : (i, j) \in I(u)\}$$

(*cf.* e. g. [2]).

Now we sketch the well-known subgradient algorithm:

ALGORITHM:

S0: Set $u^0 := u''$ [as (5)]; $k := 0$.

S1: Determine a vector $s^k \in \partial S(C(u^k))$.

S2: Calculate (choose) the step size $t_k \geqq 0$.

S3: $u^{k+1} := u^k - t_k s^k$; $k := k+1$; Go to S1.

But, in general, there is not a monotonous convergence of the function values $S(C(u^k))$. In order to obtain a "good" subgradient and therefore monotonous convergence, we choose in step S1 s^k as a vector in the direction of deepest descent.

Let $S(C(u^k)) = \operatorname{conv}\{a_1, \ldots, a_m\}$ and $B := (b_{pq})$ with $b_{pq} := a_p^T a_q$, $p, q = 1, \ldots, m$. Thus we solve

$$\begin{aligned} &h(v) := v^T B v \to \min \\ &\text{subject to} \quad v \geqq 0, \quad \sum_{j=1}^{m} v_j = 1. \end{aligned} \tag{6}$$

Let v^* be a solution of this problem. Then set $s^k := Av^*$ with $A = (a_1, \ldots, a_m)$. Because it is in general very difficult to solve (6) exactly we solve in our case (6) approximately.

5. NUMERICAL PROBLEMS

A subgradient algorithm for determining the best attainable degree of asymmetry is presented. If we know the degree then we can decide by inequality (2) whether it is better to consider the asymmetric matrix C and to use methods for the asymmetric case or it is better to calculate $D=C+C^T$ and to use methods for the symmetric problem.

A program is written in FORTRAN for solving the problem (4). The numerical experiments show: at almost all iteration points u^k the function $S(C(.))$ was differentiable, i. e. we have, in practice, a gradient method.

Now we consider the following example. Let

$$\begin{aligned} c_{ij}&=0 \quad \text{for all } i, j \quad \text{with} \quad i>j \quad \text{and} \quad j \leqq k \\ c_{ij}&=1 \quad \text{for the remaining pairs with } i \neq j, \\ k&=1, \ldots, n-1. \end{aligned} \tag{7}$$

We obtain $S(C(u''))=1$ for $k=1$ and $S(C(u''))=n-1$ for $k=2, \ldots, n-1$. Further, $S(C(u^*))=k$ for $k=1, \ldots, n-1$.

This example demonstrates that it is useful to determine the value $S(C(u^*))$ in order to get sharp inequalitities (1) and (2).

For $n=20$ and $k=2$ we have obtained the following values $S(C(u^0))=19$, $S(C(u^1))=3.67$ and $S(C(u^2))=2.06$.

Randomly generated problems show the following. The values $S(C(u^k))$ decrease very fast in the first phase of the algorithm. Later we have the bad convergence of the (sub-)gradient method. Only in the last phase we need subgradient techniques.

6. ANOTHER DEFINITION OF THE DEGREE OF ASYMMETRY

We introduce now the following definition of the degree of asymmetry:

$$ST(C):=\begin{cases} +\infty & \text{if there exists a tour } T \text{ with} \\ & \quad \operatorname{sign}(V_C(T)) \neq \operatorname{sign}(V_C(T')) \\ \max\{V_C(T)/V_C(T'):\ V_C(T') \neq 0\} & \text{else.} \end{cases}$$

Under transformations as (3) we obtain $ST(C(u))=ST(C)$ for all vectors u.

It is easy to determine the value $S(C)$ but it is, in general, very difficult to get $ST(C)$.

For example (7) we obtain $ST(C)=(n-1)/(n-k)$ for $k=1, \ldots, n-1$.

If C fulfils the triangle inequality then we have $V_C(T) \geqq 0$ and if there exists a tour T'' with $V_C(T'')=0$ then $V_C(T)=0$ holds for all tours T (*cf.* [4]). Hence, $ST(C) \leqq S(C)$ holds for all matrices C with (TI).

An interesting proposition is: $ST(C)=1$ iff $C(u'')$ is symmetric (*cf.* [11]). In this case we obtain

$$1=ST(C)=S(C(u^*))=S(C(u'')).$$

For the example (7) $(k=n-1)$ we get

$$n-1=ST(C)=S(C(u^*))=S(C(u'')).$$

Therefore, if C fulfils (TI) we obtain the following inequality:

$$1 \leqq ST(C) \leqq S(C(u^*)) \leqq S(C(u'')) \leqq n-1.$$

7. ON THE ASYMMETRY GAP

We consider only the case where C fulfils (TI). We have

$$ST(C) \leqq n-1$$

and example (7) $(k=n-1)$ demonstrates that this inequality is sharp. Further, we obtain

$$S(C(u'')) \leqq (n-1)\,ST(C),$$

and example (7) $(k=2)$ shows that this inequality is asymptotically sharp.

The value of d with

$$S(C(u^*))=d(C)\,ST(C)$$

is called the *asymmetry gap*.

If the asymmetry gap is large the estimations (1) and (2) are bad. It is possible in both cases to use the value $ST(C)$ but we can calculate only the value $S(C(u^*))$ easily.

For example (7) we obtain $d(C)=k(n-k)/(n-1)$.

Further, we conjecture that the following inequality is true:

$$1 \leqq d(C) \leqq \begin{cases} n^2/(4(n-1)) & \text{if } n \text{ is even,} \\ (n+1)/4 & \text{if } n \text{ is odd} \end{cases}$$

for all matrices C with (TI).

Finally we have

$$V_D(T^{**}) \leqq n\, V_C(T^*),$$

and example (7) ($k=n-1$) shows that this inequality is tight.

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