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FAILURE FREE WARRANTY POLICIES FOR NON-REPAIRABLE PRODUCTS: A REVIEW AND SOME EXTENSIONS (*)

by D. G. NGUYEN ([1]) and D. N. P. MURTHY ([2])

Abstract. — *This paper reviews failure free warranty policies for non-repairable products and examines some extensions and generalizations.*

Keywords : Non-repairable products; Warranty; Failure free policy.

Résumé. — *Ce papier présente une revue des polices de garanties, « Failure free », pour produits non réparables et examine quelques extensions et généralisations.*

1. INTRODUCTION

Individual consumers frequently purchase products which include a warranty of one kind or another. Most of these purchases are relatively small, and most consumers rarely pay much attention to the exact nature of the warranty, nor do they think about its actual cost. From the manufacturer's point of view, however, such considerations are very important since they can rather drastically affect profitability. In situations involving complex, costly products, and dealings with large organisations (corporations, government agencies, etc.) rather than individual consumers, the nature and cost of the warranty can become very important, indeed, both to the manufacturer and to the consumer.

Warranty cost modelling and analysis has recived a lot of attention in the literature. For non-repairable products the two major types of warranty

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policies that have been studied extensively are the failure free warranty and the rebate warranty. In this paper we focus our attention on failure free warranty policies. The aim of the paper is two-fold. Firstly, to review and secondly, to generalize and extend, the work done in the past. In the process we derive some of the earlier results but our method of derivation is totally different from those previously used.

The outline of the paper is as follows. In section 2 we discuss three different types of failure free warranty policies and review the work done in the past. In section 3 we carry out some analysis common to all the different types of failure free policies. Sections 4-6 discuss each policy separately. In each of these three sections we not only derive expression for the expected warranty costs, but also discuss approximation and bounds.

2. FAILURE FREE WARRANTY

The three different failure free warranties that we will be discussing in this paper are as follows:

(i) *Failure free warranty with fixed period* T(FWF):

This policy is the most commonly used, especially for complex and expensive products. In this policy, the manufacturer agrees to pay the repair or replacement cost for all failures during the warranty period $(0, T]$. In other words, the consumer is guaranteed that for the original price of the product he will have a functioning product for at least a time period of length T.

(ii) *Failure free warranty with renewed period* T(FWR):

This policy is commonly used for highly reliable and inexpensive products such as small electrical appliances. In this policy, when a product fails within the warranty period T the manufacturer not only repairs or replaces the product free of charge, but also give the consumer a new warranty of length T that supercedes the old one. In this way the consumer is guaranteed that for the price of the original product he will receive new or repaired products free of charge until one of the products functions for a time period longer than T.

(iii) *Combined failure free warranty with fixed period* T *and renewed period* W(FWC):

This policy is a generalization of the FWF policy and the FWR policy. It consists of a fixed period T followed by a renewed period W. In this policy, if a failure occurs in $(0, T]$, the product is repaired or replaced free of charge and the warranty continues unchanged. If a failure occurs in $(T, T+W]$, the product is repaired or replaced free of charge and a new warranty with renewed period W is issued. The policies $W=0$ and $T=0$ correspond to the FWF policy with length T and the FWR policy with length W respectively.

Early models for estimating the warranty cost for the failure free warranty with fixed period were developed by Karmarka (1978) and Balachandran *et al.* (1981). In these models, the effect of warranty on replacement purchases over the product life cycle has been ignored. Blischke and Scheuer (1975) have approached the problem of calculating the costs over the product life cycle for products sold under warranty. They assume that after a failure the owner of the product instantaneously purchases an identical replacement if the failure is not covered by a free warranty replacement. They obtain approximations for the expected total profit to the manufacturer, when the product has a relatively long life cycle. Further work by Blischke and Scheuer (1981), Biedenweg (1981) and Mamer (1982) extend Blischke and Scheuer (1975) to obtain exact expressions for these quantities. In all of these papers, excepting Biedenweg (1981), the warranty policies considered are the failure free with fixed period warranties.

For each of the three policies defined above, we derive the expected total cost to the consumer and the expected total profit to the manufacturer over the product life cycle. Bounds and approximations to these quantities are obtained. We also derive the expected cost and profit per unit time for an infinite life cycle (i. e. long run average cost and profit).

In this paper it is assumed that both warranty replacements and purchase replacements are instantaneous. Also, replacements are manufactured at the same cost and marketed at the same price during the product life cycle. In the derivation we have taken the approach of computing the total cost exclusive of the initial purchase cost. This approach has the advantege of allowing us to treat the situation in which the cost of the first item sold is different from the cost of subsequent items.

Some of our results for the FWF Policy and the FWR Policy have been derived by other researchers or can be obtained by analogy with counter models ([3]).

([3]) The counter models can be described as follows. Consider a counter for registering pulses. Pulses arrive at the counter according to a renewal process. Due to inertia the counter will not register some of these pulses. The time during which this inertia lasts is called deadtime. In a Type I counter a deadtime is produced only by a registered pulse. In a Type II counter each arriving pulse produces a deadtime. These counter models have been studied by Pyke (1958) and Smith (1957).

3. PRELIMINARY ANALYSIS

Let X_1, $X_2 \ldots$, be the lifetimes of successive items. It is assumed that X_1, $X_2 \ldots$, are independent and identically distributed random variables with distribution function $F(t)$. Let μ and σ^2 be the mean and variance of X, respectively, i. e. $\mu = E[X]$ and $\sigma^2 = \text{Var}[X]$. For any distribution function $F(t)$, we shall use $f(t) = F'(t)$ as the density function and $\bar{F}(t) = 1 - F(t)$ as the survival function.

In the failure free warranty, the manufacturer will replace an item at no cost to the consumer if it fails under warranty. The consumer purchases a new item only if the previous one fails after the warranty has expired. The interval between purchases is given by:

$$Y = \sum_{i=1}^{K(T)+1} X_i$$

where $K(T)$ is the number of free replacements per puchase for a warranty policy with length T.

Note that $K(T)+1$ is a stopping time ([4]) for X_1 $X_2 \ldots$ Thus by Wald's equation [*see* Ross 1970, p. 38)] we have:

$$E[Y] = E[X].E[K(T)+1] = \mu[1 + \kappa(T)] \tag{1}$$

where $\kappa(T) = E[K(T)]$.

Let $N(t)$ be the number of the failures in $(0, t]$. Since after a failure, the failed item is replaced immediately by a new one, the counting process $\{ N(t), t \geqq 0 \}$ is a renewal process. Thus $E[N(t)] = M(t)$, where $M(t)$ is the renewal function associated with F and given by:

$$M(t) = F(t) + \int_0^t M(t-x)\, dF(x). \tag{2}$$

Let $N_Y(t)$ be the number of purchases in $(0, t]$. The counting process $\{ N_Y(t), t \geqq 0 \}$ is also a renewal process. Thus, $E[N_Y(t)] = M_Y(t)$ is the renewal

([4]) An integer-valued positive random variable K is said to be a stopping time for the sequence X_1, $X_2 \ldots$ if the event $\{ K=n \}$ is independent of X_{n+1}, $X_{n+2} \ldots$.

function associated with $F_Y(t)$, the distribution function of Y, i. e.

$$M_Y(t)=F_Y(t)+\int_0^t M_Y(t-x)\,dF_Y(x). \tag{3}$$

We now obtain the cost to the consumer for using a product with life cycle length L. It is assumed that L is greater than the length of the warranty period. Let P be the unit price. Then

(i) The expected total cost in $(0, L]$ is given by:

$$C(L, Y)=PM_Y(L).$$

(ii) The long run average cost is given by

$$\psi(T)=\lim_{L\to\infty} C(L, W)/L.$$

By using the elementary renewal theorem [*see* Ross (1970, p. 40)] and (1) we have:

$$\Psi(T)=\frac{P}{E[Y]}=\frac{P}{\mu[1+\kappa(T)]}.$$

The manufacturer's profit is simply the difference between his revenue (which is the cost to the consumer) and his cost. Let υ be the unit manufacturing cost. Then

(i) The expected profit per purchase is given by:

$$P-\upsilon[1+\kappa(T)] \tag{4}$$

(ii) The expected total profit in $(0, L]$ per consumer is given by:

$$C(L, T)-\upsilon M(L) \tag{5}$$

(iii) The long run average profit is given by:

$$\frac{P-\upsilon[1+\kappa(T)]}{\mu[1+\kappa(T)]}. \tag{6}$$

From the above results, the cost and profit are functions of $\kappa(T)$ and $M_Y(L)$. We obtain expressions for the above mentioned quantities for each of the three different policies defined in section 2. For each of the three policies we also examine the asymtotic behaviour of $M_Y(L)$. Bounds and approximation for $M_Y(L)$ are obtained for the case where F is new better than used (NBU).

A distribution F is NBU if

$$\bar{F}(x+y) \leqq \bar{F}(x)\ \bar{F}(y), \qquad x, y \geqq 0.$$

The above equation is equivalent to stating that the conditional survival probability $\bar{F}(x+y)/\bar{F}(x)$ of an item of age x is less than the corresponding survival probability $\bar{F}(y)$ of a new item. Equality in the above equation holds if and only if F is an exponential distribution. It is easily proved that if F has an increasing failure rate then F is also NBU.

4. FWF POLICY

In the FWF Policy with length T, the manufacturer will replace free of charge for all failures in $(0, T]$. Thus the number of free replacements per purchase, $K(T)$ follows a renewal process associated with F. It follows that:

$$\kappa(T) = M(T). \tag{7}$$

4.1. Analytical investigation of $M_Y(.)$

Under this policy, the consumer has to purchase a new item at the first failure after T. Thus $T = T + \gamma(T)$, where $\gamma(T)$ is the residual life of the item in service at time T. The distribution of $\gamma(T)$ is given by ross (1970) as:

$$F_\gamma(t) = F(T+t) - \int_0^T \bar{F}\,(T+t-x)\, dM(x) \tag{8}$$

F_Y is simply a translation of F_γ. Thus $F_Y(t) = 0$ for $t \leqq T$ and for $t > T$.

$$F_Y(t) = F(t) - \int_0^T \bar{F}(t-x)\, dM(x). \tag{9}$$

Taking the Laplace-Stieltjes transform of (9) yields ([5]).

$$\tilde{F}_Y(s) = \tilde{F}(s) - [1 - \tilde{F}(s)] \int_0^T e^{-sx}\, dM(x). \tag{10}$$

([5]) $\tilde{F}(s)$ denotes the Laplace-Stieltjes transform of the function $F(t)$, i. e.

$$\tilde{F}\,(s) = \int_0^\infty e^{-st}\, dF\,(t).$$

Note that $\{ N_Y(t),\ t \geqq 0 \}$ for the FWF Policy is a Type I counter with a constant deadtime T. Equations (9) and (10) agree with the results obtained by Pyke (1958) for this type of counter model.

From (10) we can obtain the moments of Y. In particular we have:

$$E[Y] = \mu [1 + M(T)] \tag{11}$$

$$\operatorname{Var}[Y] = [\sigma^2 - \mu^2 M(T)][1 + M(T)] + 2\mu \int_0^T x\, dM(x). \tag{12}$$

Equation (11) agrees with (1) and is a well known result. Equation (12) is the same as the expression for Var $[\gamma(T)]$ derived by Coleman (1982).

Having obtained $F_Y(t)$, $M_Y(L)$ can be computed by solving (3). $M_Y(L)$ can also be obtained in terms of $F_\gamma(t)$ and is given by:

$$M_Y(L) = \begin{cases} 0 & \text{for } 0 \leqq L \leqq T \\ F_\gamma(L-T) & \text{for } T < L < 2T \\ F_\gamma(L-T) + \int_0^{L-2T} M_Y(L-T-x)\, dF_\gamma(x) & \text{for } L > 2T. \end{cases} \tag{13}$$

To use this method $F_\gamma(t)$ can be computed from (8) or by using an algorithm developed by Arnold and Groenveld (1981).

Example 1: Let F be an exponential distribution with parameter λ, i. e. $\bar{F}(t) = e^{-\lambda t}$. For this distribution the failure rate is constant and equal to λ. Thus, it is easily seen that $M(T) = \lambda T$.

Due to the memoryless property of the exponential distribution, $F_\gamma(t)$ is also an exponential distribution with parameter λ. Thus $F_Y(t)$ is simply a translated exponential. The renewal function for a translated exponential is well known [e. g. *see* Cinlar (1975, p. 288)] and given by:

$$M_Y(L) = \sum_{n+1}^{[L/T]} \{ 1 - e^{-\lambda(L-nT)} \sum_{k=0}^{n-1} \lambda (L-nT)^k/k! \} \tag{14}$$

where $[x]$ denotes the greatest integer contained in x.

4.2. Bounds and approximation for $M_Y(.)$

To obtain bounds for $M_Y(L)$ when F is NBU the following lemma is needed.

LEMMA 1: *If F is NBU then F_Y for the FWF Policy is also NBU.*

Proof: From (9), $\bar{F}_Y(t)=1$ for $t \leqq T$ and for $t > T$:

$$\bar{F}_Y(t)=\bar{F}(t)+\int_0^T \bar{F}(t-x)\,dM(x). \tag{15}$$

We need to prove that for all x, $y \geqq 0$:

$$\bar{F}_Y(x+y) \leqq \bar{F}_Y(x)\,\bar{F}_Y(y).$$

It is easily seen that the above inequality is always satisfied when either x or $y \leqq T$. When both x and $y > T$, we have

$$\bar{F}_Y(x+y)=\bar{F}(x+y)+\int_0^T \bar{F}(x+y-u)\,dM(u).$$

Since F is NBU

$$\bar{F}_Y(x+y) \leqq \bar{F}(x)\,\bar{F}(y)+\bar{F}(x)\int_0^T \bar{F}(y-u)\,dM(u) \leqq \bar{F}(x)\,\bar{F}_Y(y).$$

From (15) $\bar{F}(x) \leqq \bar{F}_Y(x)$. Thus $\bar{F}_Y(x+y) \leqq \bar{F}_Y(x)\,\bar{F}_Y(y)$. This completes the proof. ■

Bounds for the renewal function of a NBU distribution are given in Barlow and Proschan (1975, p. 171). From this and Lemma 1 we have the following theorem.

THEOREM 2: *If F is NBU then:*

$$\{L/E[Y]\}-1 \leqq M_Y(L) \leqq L/E[Y] \quad ■$$

As a consequence of Theorem 2 if F is NBU then $M_Y(L)$ can be approximated by :

$$\hat{M}_Y(L)=L/E[Y]-1/2. \tag{16}$$

The error in the approximation is $< 1/2$ uniformly for all $L > 0$.

For large L, $M_Y(L)$ can also be approximated by the asymtotic value $A_Y(L)$ given by [*see* cox (1962, p. 47)].

$$A_Y(L)=\frac{L}{E[Y]}+\frac{\mathrm{Var}[Y]}{2E^2[Y]}-\frac{1}{2}. \tag{17}$$

The above approximation does not require that F be NBU. However, it is not possible to estimate the maximum error of this approximation. Cox suggests that $A_Y(L)$ is a reasonable approximation of $M_Y(L)$ for $L>\bar{L}$, where $\bar{L}$ is given by

$$\bar{L}=E^3[Y]/\mathrm{Var}[Y] \tag{18}$$

Blischke and Scheuer (1981) compare $A_Y(L)$ with simulated values and Nguyen (1984) compares these with the approximation given by (16).

5. FWR POLICY

In the FWR Policy with length T, a failed item is replaced free of charge if it fails before age T. Thus $K(T)$ is given by a geometric probability distribution, i. e.

$$\Pr\{K(T)=n\}=\bar{F}(T)[F(T)]^n, \qquad n\geqq 0$$

so that

$$\kappa(T)=F(T)/\bar{F}(T). \tag{19}$$

We now derive an integral equation for $\bar{F}_Y(t)$. Clearly $\Pr\{Y>t\}=1$ for $t\leqq T$. Let X_1 be the first failure time after a purchase. Conditioning on this and using the fact that the warranty is renewed if $X_1\leqq T$, then for $t>T$:

$$\Pr\{Y>t\mid X_1=x\}=\begin{cases}\Pr\{Y>t-x\}, & x\leqq t\\ 0, & T<x\leqq t\\ 1, & x>t\end{cases}$$

Unconditioning, we obtain

$$\Pr\{Y>t\}=\int_0^T \Pr\{Y>t-x\}\,dF(x)+\int_t^\infty dF(x)$$

Thus $\bar{F}_Y(t)=1$ for $t\leqq T$ and for $t>T$:

$$\bar{F}_Y(t)=\bar{F}(t)+\int_0^T \bar{F}_Y(t-x)\,dF(x) \tag{20}$$

Taking the Laplace-Stieltjes transform of (20) yields

$$\tilde{\bar{F}}_Y(s)=\{\tilde{\bar{F}}(s)-\tilde{\varphi}(s)\}/\{1-\tilde{\varphi}(s)\} \tag{21}$$

where $\tilde{\varphi}(s)=\int_0^T e^{-st}\,dF(t)$

The moments of Y can be obtained from (21) and are given as follows:

$$E[Y]=\mu/\bar{F}(t) \tag{22}$$

$$\operatorname{Var}[Y]=\{\sigma^2\bar{F}(T)-\mu^2 F(T)+2\mu\int_0^T t\,dF(t)\}/\bar{F}^2(T). \tag{23}$$

The Laplace-Stieltjes transform of $M_Y(t)$ is given by:

$$\tilde{M}_Y(s)=\frac{\tilde{F}_Y(s)}{1-\tilde{F}_Y(s)}=\frac{\tilde{F}(s)-\tilde{\varphi}(s)}{1-\tilde{F}(s)}. \tag{24}$$

From the above equation we have;

$$\tilde{M}_Y(s)=-\tilde{\varphi}(s)+\tilde{M}_Y(s)\,\tilde{\varphi}(s)$$

Taking the inverse transform of the above equation yields:

$$M_Y(L)=F(L)-F(T)+\int_0^{L+T} M_Y(L-t)\,dF(t) \tag{25}$$

Equation (24) can also be rewritten as:

$$\tilde{M}_Y(s)=\frac{\tilde{F}(s)}{1-\tilde{F}(s)}-\Phi(s)\left\{1+\frac{\tilde{F}(s)}{1-\tilde{F}(s)}\right\}=\tilde{M}(s)-\tilde{\varphi}(s)-\tilde{M}(s)\,\tilde{\varphi}(s)$$

Taking the inverse transform of the above equation yields:

$$M_Y(L)=M(L)-F(T)-\int_0^T M(L-t)\,dF(t) \tag{26}$$

Thus $M_Y(L)$ can be computed from (25) or (26). An equation similar to (26) has been derived by Biedenweg (1981) by using renewal argument. Also note that $\{N_Y(t),\ t\geqq 0\}$ for the FWR Policy is a Type II counter with a constant deadtime T. Equation (26) agrees with the result obtained by Smith (1957) for this type of counter model.

Example 2: Let $\bar{F}(t)=e^{-\lambda t}$. Then from (26) we have :

$$M_Y(L)=\lambda(L-T)\,e^{-\lambda T} \tag{27}$$

$F_Y(t)$ can be obtained from (20). Another form of $F_Y(t)$ has been derived by Pyke (1958). From his result $F_Y(t)=0$ for $t\leqq T$ and for $t>T$.

$$F_Y(t)=\bar{F}(T)\sum_{n=1}^{\infty}[F(T)]^{(n-1)}F_U^{(n-1)}*F_V(t) \tag{28}$$

where $*$ denotes Stieltjes convolution, $F^{(n)}(t)$ denotes the n-fold Stieltjes convolution of $F(t)$ with itself, and

$$F_U(t)=F(t)/F(T), \qquad 0\leqq t\leqq T$$
$$F_V(t)=[F(t)-F(T)]/\bar{F}(T), \qquad t\geqq T.$$

As for FWF Policy, F_Y for the FWR Policy is NBU if F is NBU. To prove this, we need the following Lemma from A-Hameed and Proschan (1975).

LEMMA 3: *Let $H(t)$ be a distribution function with $\bar{H}(t)$ given by:*

$$\bar{H}(t)=\sum_{n=0}^{\infty}\bar{P}_n \Pr\{N(t)=n\}$$

where $N(t)$ is a general counting process having independent interarrival times X_n, $n=1, 2\ldots$ with distribution $F_n(t)$. Then H is NBU if:

(i) $\bar{P}_j\bar{P}_k\geqq\bar{P}_{j+k}$ *for $j, k=0, 1, \ldots$ and*

(ii) *For all $t\geqq 0$, $F_n(t)$ is nondecreasing in n, and*

(iii) *$F_n(t)$, $n=1, 2, \ldots$ are NBU.* ■

We have the following result.

LEMMA 4: *If F is NBU then F_Y for the FWR Policy is also NBU.*

Proof: Let $G^{(n)}(t)=F_V * F_U^{(n-1)}(t)$, with $G^{(0)}(t)=1$ and $G^{(1)}(t)=F_V(t)$.

It is easily seen that $G^{(n)}(t)$ is the distribution of $Y=V+U_1+\ldots+U_{n-1}$, where V and U have distribution F_V and F_U respectively. From (28) we have for $t>T$

$$\bar{F}_Y(t)=1-\bar{F}(T)\sum_{n=1}^{\infty}[F(T)]^{n-1}G^{(n)}(t)$$
$$=\sum_{n=0}^{\infty}[F(T)]^n[G^{(n)}(t)-G^{(n+1)}(t)]$$
$$=\sum_{n=0}^{\infty}[F(T)]^n \Pr\{N(t)=n\}$$

where $N(t)$ is the modified renewal process with the first interarrival time having distribution F_V and subsequent interarrival times having distribution F_U.

Thus $\bar{F}_Y(t)$ has the same form as $\bar{H}(t)$ in Lemma 3, with $\bar{P}_n=[F(T)]^n$. To prove Lemma 4, we need to show that (i), (ii), and (iii) of Lemma 3 are satisfied. Conditions (i) and (ii) are easily seen to be satisfied. Also if F os NBU then

$$(a)\quad \bar{F}_V(x)\,\bar{F}_V(y)=\bar{F}(x)\,\bar{F}(y)/\bar{F}^2(T)\geqq \bar{F}(x+y)/\bar{F}(T)=\bar{F}_V(x+y)$$

$$\begin{aligned}(b)\quad \bar{F}_U(x)\,\bar{F}_U(y) &= [1-F(x)/F(T)]\,[1-F(y)/F(T)]\\ &\geqq 1-[F(x)+F(y)-F(x)\,F(y)]/F(T)\\ &= 1-[1-\bar{F}(x)\,\bar{F}(y)]/F(T)\geqq 1-[-\bar{F}(x+y)]/F(T)=F_U(x+y).\end{aligned}$$

From (a) and (b), F_V and F_U are NBU. Thus condition (iii) is also safisfied. This completes the proof. ■

From Lemma 4, bounds and approximation for $M_Y(L)$ when F is NBU can be obtained, in a manner similar to those for the FWF Policy.

6. FWC POLICY

In the FWC Policy with warranty lengths T and W, an item is replaced free of charge if either it fails in $(0, T+W]$ or its lifetime is less than W. Thus the FWC Policy consists of a FWF Policy with length T followed by a FWR Policy with length W.

Let $\kappa(T, W)$ be the expected number of free replacements per purchase. Note that additional free replacements after time T (due to the second part of the policy) are provided only if there is at least one failure occuring in $(T, T+W]$.

Recall that the residual life of the item in service at time T is $\gamma(T)$. Thus from (7) and (19) we have:

$$E[K(T, W)\,|\,\gamma(T)>W]=M(T)$$

$$E[K(T, W)\,|\,\gamma(T)\leqq W]=M(T)+1+F(W)/\bar{F}(W)$$

By the law of total probability:

$$\kappa(T, W)=M(T)+F_\gamma(W)/\bar{F}(W) \tag{29}$$

Let Y_F be the time between purchases for the FWF Policy with length T and Y_R be the time between purchases for the FWC Policy with length W. The distribution F_{Y_F} and F_{Y_R} have been derived in the previous sections. We now derive $\bar{F}_Y$ for the FWC Policy in terms of $\bar{F}_{Y_F}$ and $\bar{F}_{Y_R}$. Clearly $\bar{F}_Y(t)=1$ for $t\leqq T+W$. Conditioning on the time of the first failure after T we have for $t > T + W$

$$\Pr\{Y > t \mid Y_F = x\} = \begin{cases} \Pr\{Y_R \geqq t - x\}, & T < x \leqq T + W \\ 0, & T + W < x < t \\ 1, & x > t \end{cases}$$

Unconditioning we obtain:

$$\Pr\{Y > t\} = \int_T^{T+W} \Pr\{Y_R > t - x\}\, dF_{Y_F}(x) + \int_t^{\infty} dF_{Y_F}(x)$$

Thus $\bar{F}_Y(t) = 1$ for $t \leqq T + W$ and for $t > T + W$

$$\bar{F}_Y(t) = \bar{F}_{Y_F}(t) + \int_T^{T+W} \bar{F}_{Y_R}(t - x)\, dF_{Y_F}(x) \tag{30}$$

Taking the Laplace-Stieltjes transform of (30) yields:

$$\tilde{F}_Y(s) = \tilde{F}_{Y_F}(s) - \{1 - \tilde{F}_{Y_R}(s)\} \int_T^{T+W} e^{-sx}\, dF_{Y_F}(x).$$

By noting that $F_{Y_F}(t - T) = F_\gamma(t)$ for $t \geqq T$, the above equation can be rewritten as

$$\tilde{F}_Y(s) = \tilde{F}_{Y_F}(s) - (1 - \tilde{F}_{Y_R}(s)\}\, e^{-sT} \int_0^W e^{-st}\, dF_\gamma(t) \tag{31}$$

Using (31) and the results for the moments of Y_F and Y_R (derived in the previous sections), it can be shown that:

$$E[Y] = \mu\{1 + M(T) + F_\gamma(W)\, \bar{F}^{-1}(W)\} \tag{32}$$

$$\mathrm{Var}[Y] = \mathrm{Var}[Y_F] + F_\gamma(W)\, \bar{F}^{-1}(W)\{\sigma^2 - \mu^2 - 2\mu^2 M(T) + 2\mu T\}$$

$$+ 2\mu \bar{F}^{-1}(W) \int_0^W t\, dF_\gamma(t) + F_\gamma(W)\, \bar{F}^{-2}(W) \left\{2\mu \int_0^W t\, dF(t) - \mu^2 F_\gamma(W)\right\} \tag{33}$$

with $\mathrm{var}[Y_F]$ given by (12).

By letting $W = 0$ ($T = 0$) in the above results we obtain the results for the FWFPolicy with length T (FWR Policy with length W), as derived in the previous sections.

Example 3: Let $\bar{F}(t)=e^{-\lambda t}$. Recall that for this distribution $F_Y(t)=F(t)$. Also, $\tilde{F}(s)=\lambda/(s+\lambda)$. Thus from (10), (21) and (31) we have:

$$\tilde{F}_Y(s)=\frac{\lambda_e^{-sT}e^{-(s+\lambda)W}}{s+\lambda e^{-(s+\lambda)W}}$$

The above equation can be expanded in power series as

$$\tilde{F}_Y(s)=e^{-sT}\sum_{n=1}^{\infty}(-1)^{n-1}[\lambda ae^{-(s+\lambda)W}/s]^n$$

Taking the inverse transforms yields:

$$F_Y(t)=\sum_{n=1}^{\infty}(-1)^{n-1}[\lambda e^{-\lambda W}(t-T-nW)]U(t-T-nW)/n!$$

where $U(.)$ is the Heaviside unit step function.

Also, from (32) and (33) we have

$$E[Y]=T+\lambda^{-1}e^{\lambda W}$$

$$\mathrm{Var}[Y]=\lambda^{-2}e^{2\lambda W}-2W\lambda^{-1}e^{\lambda T}. \quad \blacksquare$$

LEMMA 5: *If F is NBU then F_Y for the FWC Policy is also NBU.*

Proof: We need to prove that for all $x,\ y\leqq 0$:

$$\bar{F}_Y(x+y)\leqq\bar{F}_Y(x)\bar{F}_Y(Y).$$

It is easily seen that the above inequality is always satisfied when either x or $y\leqq T+W$. When both x and $y>T+W$, we have:

$$\bar{F}_Y(x+y)=\bar{F}_{Y_F}(x+y)+\int_T^{T+W}\bar{F}_{Y_R}(x+y-u)\,dF_{Y_F}(u)$$

$$\leqq\bar{F}_{Y_F}(x)\bar{F}_{Y_F}(Y)+\bar{F}_{Y_R}(x)\int_T^{T+W}\bar{F}_{Y_R}(y-u)\,dF_{Y_F}(u)$$

since by Lemmas 1 and 3, F_{Y_F} and F_{Y_R} are NBU.

Also, from (30), it can be shown that $\bar{F}_{Y_F}(t)\leqq\bar{F}_Y(t)$ and

$$\bar{F}_{Y_R}(t)\leqq\bar{F}_Y(t)\quad\text{for all } t\leqq 0.$$

Thus:

$$\bar{F}_Y(x+y) < \bar{E}_Y(x)\,[\bar{F}_{Y_F}(y) + \int_T^{T+W} \bar{F}_{Y_R}(y-u)\,dF_{Y_F}(u)] = \bar{F}_Y(x)\,\bar{F}_Y(y)$$

and the proof is completed.

By Lemma 5, bounds and approximations for $M_Y(L)$ when F is NBU can be obtained, in a manner similar to those for the FWF Policy.

7. CONCLUSION

In this paper we have reviewed failure free warranty policies for non-repairable products and examined some generalizations and extensions. The failure free warranty policy for repairable product involves additional factors such as type of repair, repair *vs* replacement decision etc. For more on this and other types of policies for repairable products, *see* Nguyen and Murthy (1984 *a*, 1984 *b*) and the references cited therein.

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