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TRADING AREAS OF FACILITIES WITH DIFFERENT SIZES (*)

by H. A. EISELT (1) and G. LAPORTE (2)

Abstract. — *In this paper we analyze the problem of finding the trading area for a facility on a linear market. Given the objective of maximizing profit, we first build a model with the facility sizes as variables. Then an algorithm is developed which determines the trading areas of all facilities for a given set of weights. Finally we parametrically change the weight of one of the given facilities and study the resulting changes in its trading area and thus find the optimal weight given the profit-maximizing objective.*

Keywords : Voronoi diagrams; trading areas; market models.

Résumé. — *Dans cet article, on étudie le problème consistant à déterminer les aires de marché d'établissements situés sur une droite. On envisage d'abord un problème de maximisation de profit dans le cas où les poids des établissements sont des variables. En deuxième lieu, on décrit un algorithme pour la déterminaison des aires de marché pour des poids donnés. On modifie ensuite les poids de façon paramétrique afin d'étudier leur effet sur les aires de marché et on en déduit les poids optimaux dans le contexte de maximisation de profit.*

Mots clés : diagrammes de Voronoi; aires de marché; modèles de marché.

I. INTRODUCTION

The concept of Voronoi diagrams has been known for a long time. The first to use these diagrams for practical problems was the geographer Theissen (1911) who applied the concept to a spatial missing data problem. Essentially, Voronoi diagrams can be described as follows. Given a space S (which may be some $\mathbb{R}^d$, any subset of it or a graph), a set of n given points $P_1, P_2, \ldots, P_n$ located in S and a metric, then the *Voronoi set* associated with point P_i is $V(P_i)$ which is defined as the set of points closer to P_i than to any P_j, $j \neq i$. The collection of Voronoi sets is called the *Voronoi diagram*. Most of the pertinent references deal with Voronoi diagrams in $\mathbb{R}^2$ with L_1, L_2, and L_∞

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metrics; here we usually refer to Voronoi areas rather than sets. For instance, an optimal algorithm for the construction of Voronoi diagrams in $\mathbb{R}^2$ with the Euclidean metric has been described by Shamos and Huey (1975). For a recent survey of a variety of problems related to Voronoi diagrams, see Eiselt and Pederzoli (1986).

The first to develop a model for a *locational game* was Hotelling (1929). Under rather restrictive assumptions he showed that the optimal locations for his two ice-cream vendors on the beach were at the center of the market with each of the two vendors capturing half of the market. Many extensions of this basic model have been discussed in the literature. For example, it was shown that the so-called social optimum has both vendors located one-quarter of the length of the market away from its edges. Recently, social optima were compared with individually optimized solutions, for details see Eiselt (1987). On the other hand, it was shown by Teitz (1968) that, as opposed to the two-vendor case, in the case of three ice-cream vendors there is no longer any equilibrium.

In this study we will combine the concepts of Voronoi diagrams and those of locational games. The paper is organized as follows. In the second section we describe the model which is the basis of our discussion. In the third section, we develop an algorithm which determines the Voronoi areas for a given set of points assuming that all weights are fixed and in the fourth section, we examine the effects of weight changes on the Voronoi areas.

II. THE MODEL

The space considered in this paper is a straight line segment, a so-called linear market. The n given points $P_1, P_2, \ldots, P_n$ have fixed locations. If no confusion can arise we use the expression P_i for the i-th given point as well as for its location on the line segment. For simplicity we refer to P_i as the i-th facility. The area served by this facility will be termed Voronoi area or *trading area*. It is assumed that all facilities offer a homogeneous service. Customers, who are interested in the service provided by these facilities, are distributed along the line segment. We suppose that the purchasing power represented by these customers is uniformly distributed along the market. In this short-to-medium run analysis we exclude new entries to the market as well as relocation of one or more of the facilities, the only decision parameter available to the decision-maker at the facilities are the sizes (or "*weights*") of the facilities. Here we will use the form weight since it is more general. The weight of a facility is a conglomerate measure of attractiveness of a facility;

the components are its size, its relative price advantage, courteousness of staff, etc. Each customer is now attracted to every one of the given facilities. In the traditional (unweighted) Hotelling and Voronoi models, this attraction is exclusively based on the facility-customer distance. Here we will use an attraction function which is a blend of facility weight and facility-customer distance. In particular, define w_i, $i=1, \ldots, n$ as the weight of the i-th facility and let $d(P_i, x)$ denote the distance between P_i and a customer located at some point x. Then the degree to which a customer at x is attracted to the facility P_i is measured by the *attraction function* $\varphi(i, x)=w_i/d(P_i, x)$. Even though this attraction function is considerably simpler than those employed by Coelho and Wilson (1976) and other researchers, it still captures the essential behavioral features: the attraction of a customer to a facility increases with increasing facility weight and decreases with increasing facility-customer distance. A customer at some point x will then patronize the facility he is most attracted to. This is captured in the *service function* $\psi(x)=\max_i\{\varphi(i, x)\}$. Using this concept we can construct the Voronoi or trading areas $V(P_i)$. It can easily be shown [*see* for instance Eiselt, Pederzoli and Sandblom (1985)] that $V(P_i)$ is now no longer necessarily connected (or convex in two or more dimensions). On a linear market, this means that $V(P_i)$ may consist of a number of unconnected line segments. As an example, consider a linear market with P_1 being located at one end of the market, P_2 being one distance unit away from P_1, i.e. $d(P_1, P_2)=1$ and let $d(P_2, P_3)=d(P_3, P_4)=1$, and to the right of P_4 there are another two distance units without any other facility. Let the weights of the facilities be given as $w_1=20$, $w_2=6$, $w_3=2$, and $w_x=1$. Then the resulting Voronoi diagram can be visualized in figure 1. The points bordering the trading areas $V(P_i)$ are called *Voronoi points*. In other words, a customer located at, say 4 distance units away from P_1, (which is one unit to the right of P_4) will pass P_4, P_3, and P_2 in order to patronize P_1 since this is the facility he is most attracted to.

In order to simplify matters, one could assume that any customer located between two adjacent facilities P_i and P_{i+1}, will always patronize one of these two facilities. Clearly, the resulting trading areas will be connected making this case more tractable. Such a model has been used in an optimization process by Eiselt, Laporte and Pederzoli (1986). In general, the convex case could be applicable if the given facilities are widely dispersed. If they are densely clustered, any facility, no matter what its size, which is highly surrounded by other facilities, will have an almost non-existent trading area. This is not a realistic model.

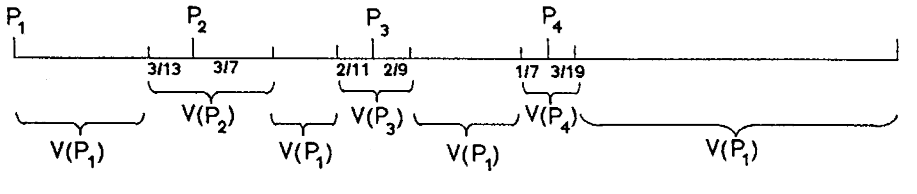

Figure 1. — **Voronoi diagram with unconnected trading areas.**

In the ensuing discussion we assume an underlying optimization model as follows. First assume that each facility operates independently, i. e. we address the case of decentralized decision-making. The cost at any facility is assumed to be a function of its weight. Finally, given uniformly distributed purchasing power, the revenue of a facility is proportional to its trading area. Here we will concentrate on the size of the trading area, i. e. the revenue, and incorporate the cost component later.

The problems addressed in the succeeding two sections are as follows: given a number of facilities with fixed locations and weights, what are the trading areas? and secondly, what happens with respect to the trading areas if the weight of one of the given facilities changes?

III. TRADING AREAS FOR FIXED FACILITY WEIGHTS

In this section we devise a procedure which enables us to determine the trading areas of a given set of facilities. As usual, let $P_1, P_2, \ldots, P_n$ denote the facilities as well as their fixed locations, let w_i be the weight of the i-th facility and denote by $d(P_i, P_j)$ the distance between the i-th and the j-th facility. Finally, let E_L and E_R symbolize the left and right end of the linear market, respectively. In order to develop a procedure it is useful to prove.

LEMMA 1: *Let P_j be two different facilities on the market with $w_i \leqq w_j$. Then $V(P_i)$ cannot embed any point $P \in V(P_j)$.*

Proof: Assume, without loss of generality that $P_i < P_j$. First note that the equation $\varphi(i, x) = \varphi(j, x)$ has two solutions given by

$$x' = (P_i w_j - P_j w_i)/(w_j - w_i) \tag{1}$$

and

$$x'' = (P_i w_j + P_j w_i)/(w_j + w_i). \tag{2}$$

These solutions satisfy $x' < P_i < x'' < P_j$. Furthermore $\varphi(i, x) > \varphi(j, x)$ if $x' < x < x''$ and $\varphi(i, x) < \varphi(j, x)$ if $x < x'$ or $x > x''$. Therefore, $V(P_i) \subseteqq [x', x'']$ and $V(P_j) \subseteqq [E_L, x'] \cup [x'', E_R]$. This proves the lemma. ■

An immediate consequence of lemma 1 is

COROLLARY 2: *If P_i is the facility with the smallest weight, then $V(P_i)$ is convex.*

This enables us to design a procedure for the determination of the Voronoi diagram. First consider only the facility with the smallest weight (ties are broken arbitrarily). Let this facility be P_i. According to corollary 2, $V(P_i)$ is convex and thus it is bordered by exactly two Voronoi points. Let those two points be v_l and v_r where v_l is located to the left and v_r is located to the right of P_i. Suppose that the facilities are consecutively numbered from left to right. The attraction of facility P_j, $j = 1, \ldots, n$ at point v_t can be expressed as $\varphi(j, v_l) = w_j/d(P_j, v_l)$ for all j or as $w_j/[d(P_j, P_i) - d(v_l, P_i)]$ for all $j < i$. Similarly, the points of equal attraction of P_i and P_j, $j > i$ are at $d^j(v_l, P_i) < [w_i/(w_j - w_i)]\, d(P_j, P_i)$ for all $j > i$. Clearly, the tightest of these bounds applies and thus the boundary of $V(P_i)$ is located at v_l at a distance from P_i of

$$\min \left\{ \min_{j<i} \left\{ \frac{w_i}{w_i + w_j} d(P_j, P_i) \right\}; \min_{j>i} \left\{ \frac{w_i}{w_j - w_i} d(P_j, P_i) \right\}; d(P_i, E_L) \right\}. \tag{3}$$

The right boundary point v_r of $V(P_i)$ can be calculated similarly. Then the Voronoi area of the facility with the smallest weight has been determined in linear time since no more than n boundary points have to be compared for each v_l and v_r, each such boundary point is evaluated in constant time. For convenience reorder now the points, so that $w_1 \leqq w_2 \leqq \ldots \leqq w_n$. Ties are again broken arbitrarily. Suppose now that the Voronoi areas $V(P_1)$, $V(P_2), \ldots, V(P_{i-1})$ are already known. Using lemma 1, the Voronoi area $V(P_i)$ can then be determined as follows. First delete all points P_1, $P_2, \ldots, P_{i-1}$ from the line. Note now that P_i is the facility with the smallest weight. Consequently, the above procedure with relation (5) is again applicable to P_i. Let its result be a set $S(P_i)$. Then the Voronoi area of P_i is $V(P_i) = S(P_i) \setminus \bigcup_{j=1}^{i-1} V(P_j)$. This procedure is repeated $(n-1)$ times, and the facility with the largest weight captures all territory not occupied by any other facility. Thus we obtain

LEMMA 3: *The weighted Voronoi diagram on the line can be found in* $O(n^2)$ *time.*

Also, as a byproduct we find that

COROLLARY 4: *The maximal number of Voronoi points is* $2n$.

and as a consequence of the construction of $V(P_i)$ from $S(P_i)$ we obtain

COROLLARY 5: *The Voronoi area of the facility with the* k-th *smallest weight has no more than k connected components.*

IV. INTRODUCTION OF A NEW FACILITY WITH VARIABLE WEIGHT

In this section we will study the effects of the parametric change of the weight of a single facility, say P_i. We proceed as follows: initially set $w_i \leftarrow 0$ and assume that the trading areas of all other facilities have already been determined, e. g. with the method developed in the previous section. Before analyzing the effects of an increase of w_i, consider the service function

$$\psi(x) = \max_k \{ \varphi(k, x) \} = \max_k \{ w_k/d(x, P_k) \}, \qquad \text{where} \quad k \in \{1, \ldots, n\}.$$

Attraction and service functions are displayed in figure 2 where the solid lines indicate the respective attraction functions and the shaded line represents the service function.

The function $\psi(x)$ increases to infinity near the given facilities and it has break points at all Voronoi points. It should be pointed out that $\psi(x)$ has minima at only those Voronoi points where the attraction of a facility to its left equals the attraction of a facility to its right and their attraction of the Voronoi point is larger than that of any other facility. In figure 2, v_2 is such a Voronoi point. On the other hand, if the attractions of two facilities on one side of the Voronoi point are equal and larger than those of any other facility at a Voronoi point (such as v_1 in figure 2), then this point does not constitute a minimum of the service function.

Consider now increases of w_i. If w_i is positive but sufficiently small, then P_i is the facility with the smallest weight and according to corollary 2 its trading area is connected. Actually, a small area around P_i will develop as $V(P_i)$ as w_i increases. In general, for any positive weight $w_i > 0$, the attraction function $\varphi(i, x)$ consists of two branches of a hyperbola around P_i (as usual) which move upwards as w_i increases. If w_i is large enough, $\varphi(i, x)$ will be

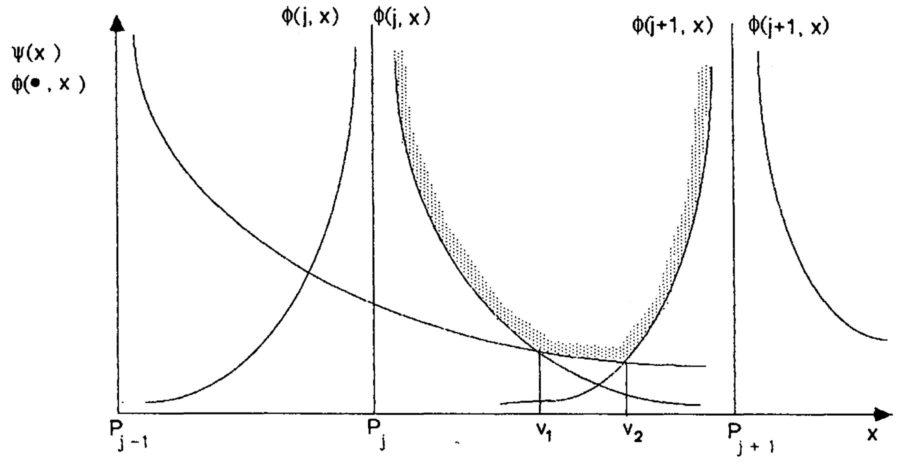

Figure 2. — **Attraction and service functions.**

higher than $\psi(x)$ at various places and wherever that occurs, a new piece of $V(P_i)$ is created. It is easy to show that these new pieces of $V(P_i)$ form around the Voronoi points. Suppose that this is not the case. Then there must be a weight w_i, for which $\varphi(i, x)$ equals $\psi(x)$ at a point x which is a linear convex combination of two adjacent Voronoi points v_j and v_{j+1}, i. e. $x=\lambda v_j+(1-\lambda) v_{j+1}$ with $\lambda \in]0; 1[$. In other words, $\varphi(i, x)>\psi(x)$ but $\varphi(i, v_j)<\psi(v_j)$ and $\varphi(i, v_{j+1})<\psi(v_{j+1})$. This would require $\varphi(i, x)$ to twice intersect $\varphi(k, x)$ which forms the piece of $\psi(x)$ between v_j and v_{j+1}; this is impossible since parts of branches of attraction functions intersect only once. Thus

LEMMA 6: *For increasing values of w_i, new pieces of $V(P_i)$ form around the Voronoi points.*

This lemma suggests a procedure for finding the entire trading area of facility P_i for all weights $w_i \in [0, \infty[$. First determine the service levels at all Voronoi points, i. e. find $\psi(v_1)$, $\psi(v_2)$, . . . , $\psi(v_V)$ where V denotes the number of Voronoi points. Then determine the weights at which P_i achieves the same attraction at those points. These "critical weights" are

$$w_i/d(P_i, v_k)=\psi(v_k) \qquad \text{or} \qquad w_i=\psi(v_k)\, d(P_i, v_k), \qquad k=1, \ldots, V.$$

These ratios are now reordered, so that

$$\psi(v_1)\,d(P_i, v_1) \leqq \psi(v_2)\,d(P_i, v_2) \leqq \ldots \leqq \psi(v_V)\,d(P_i, v_V).$$

Then for $w_i \in [0; \psi(v_1)\,d(P_i, v_1)[$, the function $\varphi(i, x)$ is higher than $\psi(x)$ only in the vicinity of P_i, so the trading area is a connected piece around P_i. For $w_i \in [\psi(v_1)\,d(P_i, v_1);\ \psi(v_2)\,d(P_i, v_2)[$, the trading area consists of a connected piece around P_i as well as a connected piece around v_2. In general, for

$$w_i \in [\psi(v_k)\,d(P_i, v_k);\ \psi(v_{k+1})\,d(P_i, v_{k+1})[,$$

the entire trading area of P_i consists of pieces around P_i and all v_j, $j=1, \ldots, k$. Note that it may happen that some of these pieces have grown together. This occurs if the next Voronoi point to be considered, say v_r, is adjacent to either P_i or to any Voronoi point v_j, $j<r$.

Rather than introducing the heavy machinery of a formally exact description of the procedure, we will explain the method by means of a small numerical example. Consider five given facilities $P_1, \ldots, P_5$ with weights $w_1=16$, $w_2=2$, $w_3=8$, w_4 variable and $w_5=12$. The distances between the facilities are $d(P_1, P_2)=12$, $d(P_2, P_3)=3$, $d(P_3, P_4)=3$ and $d(P_4, P_5)=7$. This situation together with the Voronoi points as well as the trading areas (tentatively assuming that $w_4=0$) is depicted in figure 3.

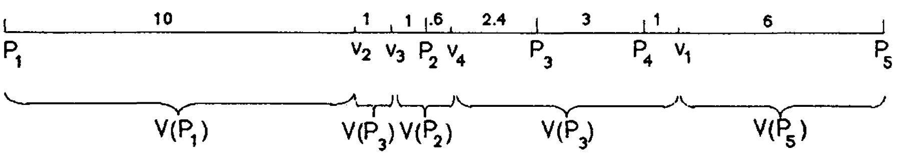

Figure 3. — Trading areas of the facilities in the example.

Calculating the service level at the Voronoi points, we obtain $\psi(v_1)=2$, $\psi(v_2)=1.6$, $\psi(v_3)=2$ and $\psi(v_4)=3.33$. Thus the critical weights are $w_4=2$, $w_4=12.8$, $w_4=14$ and $w_4=18$. For weights within the interval $[0{:}\ 2[$, $V(P_4)$ consists of a piece around P_4. To the left it extends to d_l distance units at a point which the attractions of P_4 and P_3 are equal, i. e.

$$w_4/d_l = w_s/[d(P_3, P_4) - d_l] = 8/(3 - d_l)$$

or

$$d_l = 3\,w_4/(8 + w_4).$$

Similarly, $V(P_4)$ extends d_r units to the right, to a point where $w_4/d_r = 8/3 + d_r$ or $d_r = 3\,w_4/(8 - w_4)$. In other words the size $|V(P_4)|$ of $V(P_4)$ is

$$|V(P_4)| = \frac{3\,w_4}{8 + w_4} + \frac{3\,w_4}{8 - w_4} \qquad \text{for} \quad w_4 \in [0;\,2[\tag{4}$$

If $w_4 > 2$, then a small piece around v_1, will form. Since, however, v_1 is adjacent to P_4, for $w_4 = 2$ the right part of the area around P_4 reaches v_1 and for $w_4 > 2$ it reaches beyond v_1, so that no unconnected pieces develop. In particular, $V(P_4)$ reaches d_r units to the right to a point where $w_4/d_r = w_5/(7 - d_r)$ [since all points to the right of v_1 belong to $V(P_5)$]. This yields

$$|V(P_4)| = \frac{3\,w_4}{8 + w_4} + \frac{7\,w_4}{12 + w_4} \qquad \text{for} \quad w_4 \in [2;\,12.8[\tag{5}$$

Increasing w_4 to and beyond 12.8 will create a new piece of $V(P_4)$ around v_2. Since v_2 is adjacent to P_1 and v_3, this piece will be unconnected to the current area. Its left border is at a distance of d_l from v_2 at a point where the attractions of P_4 and P_1 are equal i. e.

$$w_4/d_l = w_1/[d(P_1, v_2) - d_l] = 16/(10 - d_l)$$

or

$$d_l = (10\,w_4 - 128)/(16 + w_4).$$

Similarly, $V(P_4)$ extends d_r units to the right of v_2 to a point where P_4 and P_3 are equally attractive, i. e.

$$w_4/d_r = w_3/[d(P_3, v_2) - d_r] = 8/(5 - d_r)$$

or

$$d_r = (5\,w_4 - 64)/(w_4 - 8).$$

The part around P_4 grows in the same way as before, so that

$$|V(P_4)| = \frac{3w_4}{8+w_4} + \frac{7w_4}{12+w_4} + \frac{10w_4-128}{16+w_4} + \frac{5w_4-64}{w_4-8} \quad \text{for} \quad w_4 \in [12.8; 14[\tag{6}$$

Increasing w_4 beyond 14 does not create a new unconnected piece since the next Voronoi point to be considered is v_3 which is adjacent to v_2. Thus the only change is that the area between v_2 and v_3 is now completely in $V(P_4)$ which extends d_r units to a point right of v_3, so that $w_4/(7-d_r) = w_2/[d(P_2, v_3) - d_r] = 2/(1-d_r)$ or $d_r = (w_4-14)/(w_4-2)$. Thus

$$|V(P_4)| = \frac{3w_4}{8+w_4} + \frac{7w_4}{12+w_4} + \frac{10w_4-128}{16+w_4} + 1 + \frac{w_r-14}{w_4-2} \quad \text{for} \quad w_4 \in [14; 18[\tag{7}$$

where the first two terms describe the left and right pieces around P_4, the thrid term measures the area left to v_2, the fourth term denotes the area between v_2 and v_3 and the last term measures the area right of v_3.

Finally, if $w_4 \geqq 18$, then yet another unconnected piece develops, this time around v_4. Its left border extends d_l units to a point where

$$w_4/(5.4+d_l) = w_2/[d(P_2, P_4) - d_l] = 2/(.6-d_l)$$

or

$$d_l = (.6w_4 - 10.8)/(2+w_4).$$

On the right, the new piece extends d_r units to a point where

$$w_4/(5.4-d_r) = w_s/[d(P_3, v_4) - d_r] = 8/(2.4-d_r)$$

or

$$d_r = (2.4w_4 - 43.2)/(w_4-8).$$

Thus we obtain

$$|V(P_4)| = \frac{3w_4}{8+w_4} + \frac{7w_4}{12+w_4} + \frac{10w_4-128}{16+w_4} + 1 + \frac{w_4-14}{w_4-2} + \frac{.6w_4-10.8}{2+w_4} + \frac{2.4w_4-43.2}{w_4-8} \quad \text{for} \quad w_4 \in [18; \infty[\tag{8}$$

The size of the trading area $V(P_4)$ in relation to the weight w_4 is displayed in figure 4.

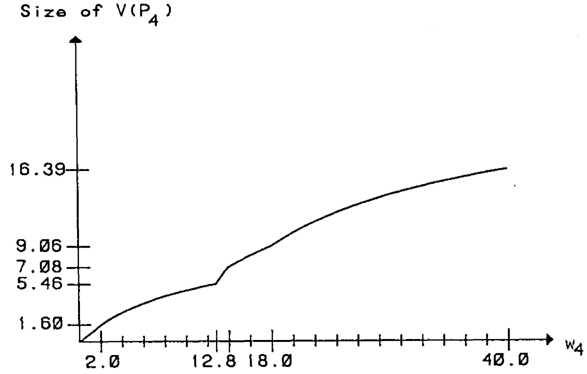

Figure 4. – Sizes of trading areas of P_4 for variable weights.

Recall that under the given assumptions, the size of a trading area is proportional to the revenue achieved for that facility. This means that the function in figure 4 is proportional to the revenue and by incorporating a cost curve in that figure (the costs were assumed to be a function of the weight of the facility), the profit function could be determined. This will enable the decision maker at the facility in question to choose its weight optimally.

CONCLUSION

In this paper we have introduced a spatial model based on the concept of Voronoi diagrams. Attraction and service level functions were introduced and a method was developed which determines the trading areas of a set of facilities with given weights. Finally it was shown how the trading area of an individual facility changes if its weight is altered.

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