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COMPETITIVE INVENTORY MODELS (*)

by Prakash P. SHENOY (2)

Abstract. – *This paper deals with the question of optimal inventory sizes in a competitive environment in which the demand for a product at a vendor depends on the inventory level of the product at that vendor relative to the inventories of the same product held by other competing vendors. The total demand for the product at all vendors is assumed to be fixed. This question is examined for two different replenishment policies: base stock policy where the vendors keep a constant level of inventory by ordering the product at the same rate as the observed demand; and continuous review policy where all vendors replenish their inventories periodically when the inventory level drops to a predetermined level.*

Keywords : Inventory Theory; EOQ Models; Game Theory.

Résumé. – *Cet article traite du niveau optimal d'un stock dans un contexte de concurrence : la demande d'un produit chez un vendeur dépend du niveau du stock du produit chez celui-ci, par rapport au niveau des stocks du même produit chez des concurrents. La demande totale du produit chez l'ensemble des vendeurs est supposée connue. On examine les cas de deux politiques différentes de réapprovisionnement :*

– *la politique dite du « stock de base » pour laquelle les vendeurs gardent constant le niveau du stock en commandant le produit au fur et à mesure des demandes;*

– *la politique de passation périodique de commandes, lorsque le niveau du stock descend au dessus d'un niveau fixé.*

I. INTRODUCTION

This paper deals with the management of inventories in a competitive environment in which the demand for a product at a vendor depends on the inventory level of the product at that vendor relative to the inventories of the same product held by other competing vendors.

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In the classical inventory models called the Economic Order Quantity (EOQ) models, a crucial assumption made is that the demand rate for a product is constant and that it is independent of the level of the inventory of the product. The demand rate in these models is exogenous to the model and is unaffected by the decision variable—the order (or production) quantity of the product. Such an assumption is plausible in a monopoly situation where the demand for the product can only be satisfied at a single vendor or in cases where the vendor has a contract to supply the product at a fixed rate. However, in a non-monopoly situation, such an assumption may not be valid. This probably is the reason for the failure of the classical models to explain the large inventories held by automobile dealers, liquor stores, etc., all of which exist in a very competitive environment. For example, the demand for cars at an automobile dealer may depend on how many cars the dealer has at his dealership (physically) relative to the number of cars held by other competing automobile dealers. Assuming that the other factors that affect demand rate such as price, location of dealership, advertising, etc., are the same, a dealer with a larger inventory of cars than his competitors is likely to sell proportionately more cars than his competitors simply because of his larger inventory. Other examples of products where the demand for the product is affected by the inventory level of the product are alcoholic beverages, electrical appliances, televisions, computers, stereo equipment, etc., all of which are sold usually in a competitive environment by several vendors.

One reason for this phenomenon is that the vendor with the larger inventory can stock the product with more combinations of attributes. For example, a car dealer with a larger inventory can have more cars with different combinations of attributes such as color, combination of options, type (sedan, coupe, station wagon, etc.), manufacturer, etc. A consumer who wishes to buy a car with a specific combination of attributes (color, options, etc.) is more likely to find such a car at a dealer who has a larger inventory than others. Hence, such a consumer is more likely to buy from that dealer than from any other dealers. Even a consumer who is undecided about what attributes (s)he wants will have more choices at a dealer with more cars and for that reason is more likely to buy from that dealer.

In this paper, we will assume that the demand for a product is directly proportional to the inventory level of the product relative to the total inventory of the product held by all competing vendors. We do not imply by this assumption that inventory creates demand. Instead we simply assume that inventory affects the distribution of the total demand. The total demand for the product at all vendors will be assumed to be fixed. The question that arises is what is the optimal order/production quantity for a vendor who is trying to maximize his market share and minimize his annual ordering cost and his annual holding cost. We will examine this question for two different replenishment policies: base stock policy where the vendors keep a constant level of inventory by ordering the product at the same rate as the demand; and continuous review policy where all the vendors replenish their inventory when the inventory level drops to a predetermined level. We will analyze this question assuming no cooperation between the vendors. In the next section we give a brief description of the classical EOQ model to compare our results with later. Then in section 3 the case of fixed inventory level (base stock policy) is analyzed. Finally, section 4 consists of the analysis of the case of continuous review policy. The proofs of all the theorems are stated in the appendix.

II. THE CLASSICAL EOQ MODEL

The classical EOQ model determines the optimal order quantity that minimizes the sum of the annual ordering costs and the annual holding costs. The main assumptions of this model are as follows.

A.1. The demand for the product is constant and independent of the quantity ordered.

A.2. Shortages cannot be backordered.

A.3. Replenishment of inventory takes place instantaneously when order is received.

A.4. The lead time for receiving an order is deterministic and exactly known.

Under these assumptions, the optimal policy is to order the quantity Q where

$$\begin{gathered} Q=(2\,C_0\,R/C_h)^{1/2}\ \text{units} \\ C_0=\text{fixed ordering cost, \$/order} \\ R=\text{demand rate, units/yr} \\ C_h=\text{annual unit holding cost, \$/unit/yr} \end{gathered} \tag{2.1}$$

and the optimal time between orders is Q/R years. A plot of the inventory level $s(t)$ versus time t is shown in figure 2.1.

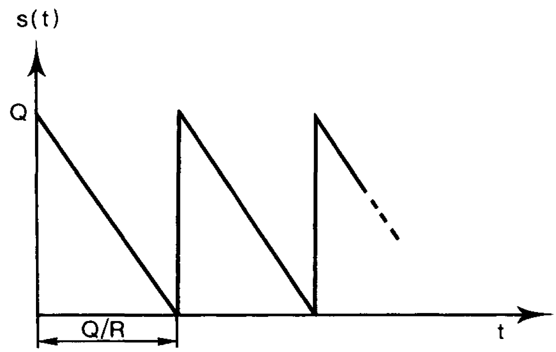

Figure 2.1. — A plot of inventory level vs time.

While assumptions A.2 to A.4 are easily relaxed, assumption A.1 is not so easy to drop. The case of nonsteady demand but still independent of the quantity ordered is dealt with using dynamic programming [*see* e. g., Wagner and Whitin (1958)]. The case of the demand being dependent on the inventory level has been examined only by Case (1979). Case limits his analysis to finding the optimal response if everyone else's inventory level is known. Our analysis and methods are different and we focus on the more general problem of finding an optimal policy for each vendor assuming only that the other vendors are rational decision makers.

III. THE BASE STOCK POLICY MODEL

In this model, there are n traders selling similar products at comparable prices. Furthermore, we assume that all factors (other than the inventory level) that determine the demand are the same at all vendors. In this particular model, we consider situations where all vendors maintain a fixed level of inventory by ordering the product at the same rate as the realized demand. This is typical in situations where the fixed ordering cost is insignificant relative to the holding cost.

Let $N \triangleq \{1, 2, \ldots, n\}$ represent the set of n traders labeled by the first n natural numbers. The main assumptions of this model are as follows.

B. 1. Demand rate at vendor i at time t, written as $R_i(t)$, is proportional to size of vendor i's inventory at time t expressed as a fraction of the total inventory held by all the vendors, i. e.,

$$R_i(t) \propto s_i(t) / \sum_{j=1}^{n} s_j(t), \qquad \text{for} \quad i=1, \ldots, n.$$

B. 2 Total annual demand denoted by $R = \sum_{j=1}^{n} R_j(t)$ is constant at all times t.

B. 3 Shortages cannot be backordered.

B. 4 Lead time between ordering and delivery is deterministic and exactly known.

B. 5 Inventory is replenished continuously with rate of replenishment equal to rate of demand.

Assumption B. 5 is applicable in cases where the ordering cost C_0 is small in comparison to the unit holding cost C_h for each vendor. Assumption B. 5 implies that the inventory level of each vendor will be fixed with respect to time. In practice though, this will be impossible to achieve exactly. There will be minor fluctuations in the inventory level. However, if the inventory level is large, the small fluctuations in the inventory level will not affect the results very much. The decision variable for each vendor i will be the fixed level of inventory, denoted by Q_i, to maintain. By assumption B. 1 and B. 2, it follows that demand at vendor i will be at a rate independent of t given by

$$R_i = RQ_i/Q \tag{3.1}$$

where R_i denotes the annual demand at vendor i, and Q denotes the total inventory at all vendors, i. e., $Q = \sum_{j=1}^{n} Q_j$.

Throughout this paper we will assume that the total inventory at all vendors is strictly positive. Let $C_{h,i}$ denote vendor i's cost of holding one unit of the product in inventory for one year. This parameter will typically include interest expense, storage cost, insurance, depreciation, absolescence, theft, breakage and spoilage, etc. In general, all variable costs associated with holding inventory are included in $C_{h,i}$. Since we assume that the items are continuously replenished, this would require that an order be placed for each unit sold. Hence, in this case, the ordering cost will also be included in $C_{h,i}$. Let P_i denote the profit per unit realized by vendor i before taking into account the holding cost. Then vendor i's payoff function is his annual net profit given as follows:

$$\begin{aligned}\Pi_i(Q_1, \ldots, Q_n) &= P_i R_i - C_{h,i} Q_i \\ &= P_i R Q_i / Q - C_{h,i} Q_i \end{aligned} \tag{3.2}$$

Throughout the paper, we will assume that for all i, $C_{h,i} > 0$ and $P_i > 0$. Note that we have a n-person non-zero-sum game where the players are the vendors, the decision variable for each vendor i is the fixed level of inventory Q_i to maintain and the payoff function for each vendor i is Π_i which depends on the actions of all the vendors. The data required for the model are $C_{h,i}$ and P_i for all i, and R.

The question that arises is what level of inventory should vendor i maintain assuming that he is interested in maximizing his annual net profit as given above in expression (3.2). This of course depends on the actions (inventory levels) of the other vendors. First, we will answer this question assuming that vendor i can observe the inventory levels of all the other vendors and that the other vendors will not change their levels as a result of vendor i's actions. Of course, the latter assumption is not very realistic. However, after we complete this analysis, we will drop this unrealistic assumption and answer the more general question of what vendor i should do assuming only that the other vendors are, like himself, only trying to maximize their respective annual net profit.

We will start by computing an upper bound on the inventory level that vendor i will ever carry assuming that the inventory holding costs should be no more than the contribution from sales. The most demand that vendor i could achieve is to corner the entire market, i. e., R. Hence, denoting the upper bound by $Q_{i,\max}$, we have $C_{h,i} Q_{i,\max} = P_i R$, i. e.,

$$Q_{i,\max} = P_i R / C_{h,i} = R/\alpha_i, \qquad \text{where} \quad \alpha_i = C_{h,i}/P_i \tag{3.3}$$

$Q_{i,\max}$ represents the inventory level such that if vendor i held more inventory than this level (for whatever reasons), then he would realize a loss for certain. The parameter α_i represents the smallest frequency of demand necessary for vendor i to offset his cost of keeping one unit in inventory for one year. Vendors with relatively small values of α_j are of course more cost efficient than vendors with relatively large values of α_j.

Given $Q_1, \ldots, Q_{i-1}, Q_{i+1}, \ldots, Q_n$, the inventory levels of all other vendors, what is the optimal inventory level for vendor i? This is a classical optimization problem formally written as follows.

PROBLEM P1:

$$\text{maximize}\,\Pi_i(Q_i \mid Q_1, \ldots, Q_{i-1}, Q_{i+1}, \ldots, Q_n)$$

subject to $Q_i \geqq 0$. We have the following result.

THEOREM 3.1: *Problem P1 has a unique optimal solution, denoted by Q_i^*, given as follows.*

$$Q_i^* = \begin{cases} (Q_{i,\max}\bar{Q}_i)^{1/2} - \bar{Q}_i & \text{if } 0 < \bar{Q}_i \leqq Q_{i,\max} \\ 0 & \text{if } \bar{Q}_i > Q_{i,\max} \end{cases}$$

where $\bar{Q}_i = Q_1 + \ldots + Q_{i-1} + Q_{i+1} + \ldots + Q_n$, *and*

$$Q_{i,\max} = R/\alpha_i.$$

First note that vendor i's optimal response is a function only of $\bar{Q}_i$, the total inventory held by the other vendors regardless of how this is distributed among the vendors. This observation is of course a direct consequence of our assumption that a vendor's demand depends only on the vendor's share of total inventory and not on its distribution. A graph of Q_i^* versus $\bar{Q}_i$ is shown in figure 3.1.

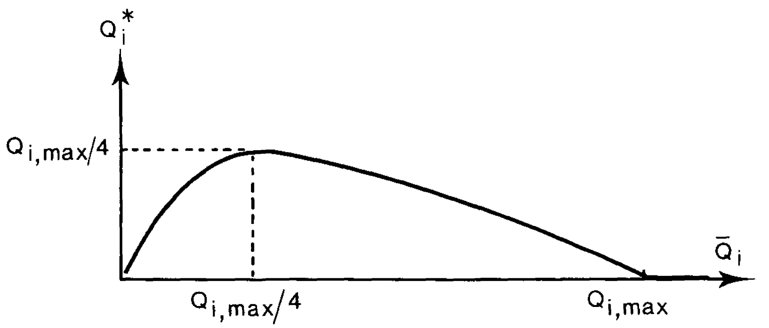

Figure 3.1. — A graph of Q_i^* versus $\bar{Q}_i$.

Second, if vendor i limits his objective to short term profit maximizing, then he should not carry any inventory if his competitors are carrying "too much" inventory, i.e., more than $Q_{i,\max}$. This is because the cost of maintaining any level of inventory will exceed the contribution from sales generated by the same inventory. Note that what is "too much" for vendor i need not be "too much" for the other vendors particularly if their cost efficiency factor α_j is smaller than that of vendor i. Vendors with relatively small values of α_j have a competitive advantage in this respect. Another way to view this result is that if vendor i has the resources to absorb short term losses and furthermore perceives his competitors to be short term profit maximizers, then one possible strategy to eliminate his competition is to carry a large inventory long enough to cause his competitors to stop carrying the product. This strategy is particularly effective for those vendors who have cost advantage (relatively small values of α_j) relative to other vendors. However, this strategy is a double-edged sword because if the other vendors decide to respond to this strategy by also carrying large inventories, then everyone loses in this situation.

Finally, if vendor i limits his actions to optimal responses, then the maximum inventory he will ever carry is $Q_{i,\max}/4$ and furthermore, this maximum is achieved when the competitors are also carrying (in total) the same amount, i. e., $Q_{i,\max}/4$. Therefore, when vendor i's competitors inventories $\bar{Q}_i$ is in the range $(0, Q_{i,\max}/4]$, an optimal response by vendor i to an increase in inventory by the competitors is to increase his own inventory. However, when $\bar{Q}_i$ is in the range $[Q_{i,\max}/4, Q_{i,\max}]$, then the optimal response to an increase in inventory by the other vendors is for vendor i to decrease his own inventory.

We will illustrate these results by an example.

Example 3.1: Consider two automobile dealers selling similar cars at comparable prices. Suppose that the total annual demand is 1,000 cars. Suppose that for vendor 1, the annual unit holding cost (in dollars) is 1,000, and that his unit profit before holding costs is 1,000. For vendor 2, let the annual unit holding cost be 1,000 and the unit profit before holding costs be 900. For vendor 1, $\alpha_1 = 1$ per year whereas for vendor 2, $\alpha_2 = 1.11$ per year. This means that for every car that vendor 1 holds in stock for a year, he has to realize a frequency of demand equal to 1 per year to offset his holding costs whereas for vendor 2 the corresponding parameter is 1.11 per year. Clearly, vendor 1 is more cost efficient than vendor 2. Suppose that both vendors have a constant inventory level of 100 cars. Then they split the demand equality and the resulting net annual profits are respectively $ 400,000 and $ 350,000. Suppose that vendor 1 decides to respond optimally to vendor 2's inventory level (assuming of course that vendor 2 would not subsequently change his inventory level). Then as indicated by Theorem 3.1, vendor 1 increases his inventory to 216 cars. Subsequently, his share of the demand increases to 684 cars whereas vendor 2's demand decreases to 316 cars. The resulting net annual profits are respectively \$648,000 and \$184,000. If contrary to vendor 1's expectation, vendor 2 decides to also change his stock level so as to be at an optimal level corresponding to vendor 1's new stock level, then he will increase his inventory level to 225 cars and the resulting net annual profits are \$274,000 and \$234,000 respectively for vendor 1 and 2. If this process of sequential optimization continues, the results are shown in table 3.1.

TABLE 3.1

Q_1 cars	Q_2 cars	Π_1 \$/yr	Π_2 \$/yr
100	100	400,000	350,000
216*	100	468,000	184,000
216	225*	274,000	234,000
249*	225	276,000	202,000
249*	224*	277,000	202,000

* denotes optimal integral response given competitor's inventory level.

In this example, sequential optimization eventually leads to the inventory levels such that neither vendor has any incentive to change his level unilaterally since either increasing or decreasing the stock level leads to a decrease in that vendor's profit. We call such a situation a "Nash Equilibrium" (NE). More formally, we say that $(Q_1^*, Q_2^*, \ldots, Q_n^*)$ is a Nash Equilibrium if for each vendor i, $i=1, \ldots, n$, Q_i^* is an optimal solution to the problem:

$$\text{(P2)} \qquad \text{maximize}\, \Pi_i(Q_i \mid Q_1^*, \ldots, Q_{i-1}^*, Q_{i+1}^*, \ldots, Q_n^*)$$

subject to: $Q_i \geqq 0$.

In the example above, at the Nash Equilibrium, both dealers are worse off than they were at the beginning when they both maintained an inventory level of 100 cars. In this respect, the inventory game resembles the classical prisoners' dilemma game: the Nash Equilibrium payoffs are not pareto optimal [*see* Luce and Raiffa (1957) for a discussion of the prisoners' dilemma game].

Suppose that the n vendors are labeled in increasing order of cost inefficiency, i. e., $\alpha_1 \leqq \alpha_2 \leqq \ldots \leqq \alpha_n$. Then the NE inventory levels are given by Theorem 3.2 below.

THEOREM 3.2: *Given the labeling of the n vendors as described above, suppose that* $\alpha_n \leqq \alpha/n-1$, *where* $\alpha = \sum_{j=1}^{n} \alpha_j$ *and* $\alpha_j = C_{h,j}/P_j$, *i. e., no vendor is too cost* *inefficient relative to other vendors. Then the NE inventory levels* Q_i^*, $i=1, \ldots, n$, *are given by*

$$Q_i^* = Q^* f_i \tag{3.4}$$

where
$$Q^* = \sum_{j=1}^{n} Q_j^* = R(n-1)/\alpha,$$

and

$$f_i = [\alpha - (n-1)\,\alpha_i]/\alpha$$

First we will comment on the condition on the cost inefficiency parameters. If the number of vendors is just 2, then the condition reduces to $\alpha_2 \leqq \alpha_1 + \alpha_2$ which is always true. As the number of vendors increases, the condition becomes more and more stringent requiring in the limit as $n \to \infty$ that all vendors have the same value for the cost inefficiency factor α_j. Essentially, this condition rules out the existence of vendors who are relatively "too inefficient." Note that the condition can be rewritten as $\alpha_n \leqq (\alpha_1 + \ldots + \alpha_{n-1})/(n-2)$.

Next we remark about the nature of the Nash Equilibrium. In expression (3.4), each f_j is a fraction between 0 and 1 such that $\sum_{j=1}^{n} f_j = 1$. Since $\alpha_1 \leqq \alpha_2 \leqq \ldots \leqq \alpha_n$, we can see that $f_1 \geqq f_2 \geqq \ldots \geqq f_n$. Therefore, we see that at the NE, $Q_1^* \geqq \ldots \geqq Q_n^*$, i.e., more cost efficient vendors carry more stock than less cost efficient vendors. Also note that the total inventory is directly proportional to the total demand, and inversely proportional to the total cost inefficiency factor of all the vendors. If all the vendors have the same cost inefficiency factor α_j, then we can write $Q^* = R(n-1)/n\,\alpha_j$. Hence the total inventory at all vendors is an increasing function of the number of vendors bounded from above by $R/\alpha_j = Q_{j,\,\max}$. This last observation implies that as the number of vendors increases, the total profit that is made by all vendors decreases. In the case of 2 vendors (with the same parameters α_j and P_j), the total inventory cost of the two vendors is equal to half the total market potential $P_i R$ leaving each vendor with a profit of 1/4 of the total market potential. In the case of 3 vendors, the corresponding total inventory cost is (2/3) $P_i R$ leaving each vendor with a profit of (1/9) P_iR and so on. This fact acts as a natural barrier to the entry of new vendors when there are already some established vendors in the market. Also, this explains the high inventories observed in a competitive market that cannot be accounted for by classical EOQ models.

The Nash equilibrium inventory levels is a normative solution concept. If vendor i is rational, i. e., prefers more profit to less, and assumes that all other vendors are also rational, then the level of inventory he should carry is as prescribed by the Nash Equilibrium. If all the other vendors also do likewise, then the Nash Equilibrium is attained and each vendor makes a profit given as in (3.6). If some vendor does not carry inventory as prescribed by the Nash Equilibrium but the others do, then this vendor makes less profit than he would have if he had carried inventory as per the NE. This is a compelling reason to follow the NE prescription.

All of the foregoing discussion assumes no cooperation between the vendors. If the vendors do cooperate and can make binding agreements to carry a uniformly lower level of inventory, then all vendors can do better than they would have if they had followed the NE prescription. This is typical of prisoners' dilemma type of situations.

If there do exist vendors who are "too inefficient" then the NE inventory levels are described in theorem 3.3 below.

THEOREM 3.3: *Let m be the largest integer $2 \leqq m \leqq n$ such that $\alpha_m \leqq (\alpha_1 + \ldots + \alpha_{m-1})/m-2$. Then the Nash Equilibrium inventory levels $(Q_1^*, \ldots, Q_n^*)$ are given as follows.*

$$Q_i^* = \begin{cases} Q^* f_i & \text{for } i=1, \ldots, m \\ 0 & \text{for } i=m+1, \ldots, n \end{cases}$$

where

$$f_i = [\alpha - (m-1)\,\alpha_i]/\alpha$$

$$\alpha = \sum_{j=1}^{m} \alpha_j,$$

and

$$Q^* = \sum_{j=1}^{m} Q_j^* = R\,(m-1)/\alpha. \tag{3.5}$$

Theorem 3.3 is similar to theorem 3.2 except that all vendors who are "too inefficient" carry no inventory at equilibrium. Hence, the problem essentially reduces to the case where there are only m vendors (those that are relatively cost efficient) and all results are exactly the same as before.

IV. THE CONTINUOUS REVIEW MODEL

In the base stock policy model, we assumed that all vendors replenished their stock continuously maintaining a fixed inventory level. However, if ordering cost is not insignificant, then maintaining a fixed inventory level by ordering continuously can be an expensive proposition. For such situations, we will formulate a continuous review model where each vendor orders periodically like in the classical EOQ model. However, unlike the classical EOQ model, each vendor has to consider not only the annual ordering and holding costs but also the contribution from the demand that results from the inventory level. In the model described below we assume a condition of steady state i. e., each vendor i orders the same quantity Q_i of the product per order and that the inventory is replenished at periodic intervals when the inventory level reduces to zero.

The principal assumptions of this model are as follows.

C. 1 Total annual demand at all vendors is a constant.

C. 2 Demand rate at vendor i at time t is proportional to vendor i's inventory level relative to the total inventory (at all vendors) at the same time t.

C. 3 Each vendor i replenishes his inventory periodically to the same maximum level Q_i when the inventory level drops to zero.

C. 4 Replenishment is instantaneous.

Let $s_i(t)$ and $R_i(t)$ denote vendor i's stock level and demand rate respectively at time t. Then by assumptions C. 1 and C. 2, we can conclude that:

$$R_i(t) = R\, s_i(t)/s(t) \tag{4.1}$$

where

$$s(t) = \sum_{j=1}^{n} s_j(t) = \text{total inventory (at all vendors) at time } t,$$

and

$$R = \sum_{j=1}^{n} R_j(t) = \text{total constant demand.}$$

Without loss of generality, we assume that at time 0, all vendors have replenished their inventories to their respective maximum levels, i. e., $s_i(0) = Q_i$ for $i = 1, \ldots, n$. Before the next replenishment by any vendor, the inventory level of vendor i at any time t can be described as follows:

$$s_i(t) = s_i(0) - \int_0^t R_i(t)\,dt \quad \text{for all } i$$

$$= Q_i - \int_0^t (R\,s_i(t)/s(t)]\,dt \quad \text{for all } i. \tag{4.2}$$

Equations (4.2) constitute a system of n recursive integral equations the solution to which is described in theorem 4.1 below.

THEOREM4.1: *The solutions to the systems of equations described in* (4.1) *and* (4.2) *are given by:*

$$R_i(t) = RQ_i/Q \quad \textit{for all } i \tag{4.3}$$

$$s_i(t) = Q_i - RQ_i\,t/Q \quad \textit{for all } i \tag{4.4}$$

where $Q = \sum_{j=1}^{n} Q_j$.

Before replenishment by any vendor, the demand rate at any vendor as described by equation (4.3) is independent of t, and is given by the demand rate at time 0. Hence, for any vendor i, the stock level decreases linearly with time, Let vendor i be the first vendor to deplete completely his inventory. Then by assumption C.3, the time T_i between replenishments for vendor i is given by $s_i(T_i) = 0$, i.e., $Q_i - RQ_i\,T_i/Q = 0$, i.e.,

$$T_i = Q/R \quad \text{for all } i. \tag{4.5}$$

Note that the time between replenishments T_i is independent of i, i.e., T_i is the same for all vendors. All vendors will replenish their inventories at the same point in time. A graph of the stock levels of the n vendors versus time will be as shown in figure 4.1.

Suppose that vendor i's cost (in dollars) of holding one unit in inventory for one year is denoted by $C_{h,i}$ and the fixed cost (i.e., independent of the size of the order) in dollars of placing an order is denoted by $C_{0,i}$. Furthermore, let P_i denote vendor i's contribution to profit from selling one unit of the product before taking into account the holding and the ordering costs. Then vendor i's annual net profit is given as follows.

$$\Pi_i(Q_1, \ldots, Q_n) = \begin{cases} PRQ_i/Q - C_{0,i}R/Q - C_{h,i}Q_i/2 & \text{if } Q_i > 0 \\ 0 & \text{if } Q_i = 0. \end{cases} \tag{4.6}$$

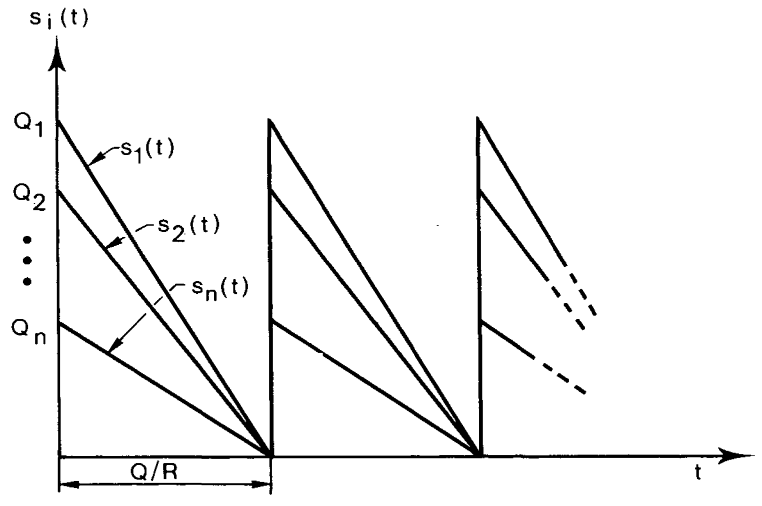

Figure 4.1. A graph of stock levels versus time.

Once again we have a n-person non-zero-sum game. Note that if $C_{0,i}=0$ for all i, then this game model reduces essentially to the base stock policy model described in section 3 (the only difference being that the annual holding cost for each vendor in this model is half that of the annual holding cost in the base stock policy model). Hence, we can consider this model as a generalization of the base stock policy model. Our analysis of this model will closely parallel the analysis in the previous section.

First we will compute an upper bound $Q_{i,m}$ on the inventory level such that vendor i is certain to incur a loss if he carries more than the upper bound. As before we have $P_i R = C_{h,i} Q_{i,m}/2$ (ignoring ordering costs), i.e.,

$$Q_{i,m} = 2 P_i R / C_{h,i} = 2 R/\alpha_i \quad \text{for all } i \tag{4.7}$$

where $\alpha_i = C_{h,i}/P_i$.

α_i represents the frequency of demand necessary to recover the cost of keeping one unit of the product in inventory for one year. Define $\beta_i = C_{0,i}/P_i$. β_i represents the number of units vendor i has to sell to recover his ordering cost per order. As in the last section, we examine the question of the optimal quantity that vendor i should order assuming that the order quantities of all other vendors are known and that these will not change as a result of vendor i's decision. The question can be posed as an optimization problem:

$$\text{(P3)} \qquad \text{maximize}\, \Pi_i (Q_i \mid Q_1, \ldots, Q_{i-1}, Q_{i+1}, \ldots, Q_n)$$

subject to $Q_i \geqq 0$.

The optimal solution to this problem is described in theorem 4.2 below.

THEOREM 4.2: *The optimal solution Q_i^* to problem P3 described above is given by*

$$Q_i^* = \begin{cases} [Q_{i,m}(\bar{Q}_i + \beta_i)]^{1/2} - \bar{Q}_i & \text{if } \bar{Q}_i \leqq [Q_{i,m} + (Q_{i,m}^2 + 4 Q_{i,m} \beta_i)^{1/2}]/2 \\ 0 & \text{if } \bar{Q}_i > [Q_{i,m} + (\bar{Q}_{i,m}^2 + 4 Q_{i,m} \beta_i)^{1/2}]/2 \end{cases}$$

where $\bar{Q}_i = Q_1 + \ldots + Q_{i-1} + Q_{i+1} + \ldots + Q_n$.

Note that this result is analogous to that of theorem 3.1 for the base stock policy model. Next, we compute the Nash Equilibrium inventory levels (which are defined as in section 3) for the continuous review model. The result is stated in theorem 4.3.

THEOREM 4.3: *Let the vendors be labeled such that $\alpha_1 \leqq \ldots \leqq \alpha_n$. If there are no vendors who are too cost inefficient, i.e., $\alpha_n \leqq \alpha/(n-1)$, and the total demand R is high enough to accommodate the fixed ordering costs of all the vendors, e.g.,*

$$R \geqq (\alpha_n \beta - \alpha \beta_{\min})/2(n-1) f_n \tag{4.8}$$

where

$$\alpha = \sum_{j=1}^{n} \alpha_j, \qquad \beta = \sum_{j=1}^{n} \beta_j, \qquad \beta_{\min} = \min\{\beta_1, \ldots, \beta_n\},$$

and $f_n = [\alpha - (n-1)\alpha_n]/\alpha$, then the Nash Equilibrium inventory levels $(Q_i^*, \ldots, Q_n^*)$ are given as follows.

$$Q_i^* = Q^* f_i + e_i \quad \textit{for all } i \tag{4.9}$$

where

$$Q^* = \sum_{j=1}^{n} Q_j^* = [R(n-1)/\alpha] + [R^2(n-1)^2 + 2R\alpha\beta]^{1/2}/\alpha,$$

$$f_i = [\alpha - (n-1)\,\alpha_i]/\alpha,$$

and

$$e_i = (\alpha\beta_i - \alpha_i\,\beta)/\alpha.$$

This result is similar in many respects to the Nash Equilibrium inventory levels for the base stock policy model. The main difference is the presence of an additional condition (4.8) on the total demand that arises due to the fact that the vendors have to recover the fixed ordering costs that was not present in the base stock policy model. Condition (4.8) is by no means a very tight condition, i. e., the demand does not necessarily have to be as large as indicated. However, it is certainly sufficient. In any case for most applications, this condition is academic, i. e., we expect that this condition will be easily satisfied.

Comparing with theorem 3.2, expression (4.9) has an additional term e_i. Note that e_i may either be positive or negative. More precisely, for vendor i, if $\beta_i/\beta > \alpha_i/\alpha$ then e_i is positive and if $\beta_i/\beta < \alpha_i/\alpha$ then e_i is negative. Furthermore, $\sum_{j=1}^{n} e_j = 0$. Comparing the expression for total inventory Q^* carried by all vendors with the corresponding expression in theorem 3.2, we note that Q^* in this model is more than twice the Q^* in the base stock policy model. The factor of 2 is explained by the fact that the annual holding cost component in the objective function of each vendor in the continuous review model is half that of the annual holding cost component in the objective function in the base stock policy model. The fixed ordering cost parameters β_j's in the continuous review model explain why Q^* is more than twice greater.

V. CONCLUSIONS

In both the models we studied, we assume that the total demand for the product is fixed and independent of the total inventory. In some markets, this may not be valid. It would be interesting to examine the models assuming that total demand is an increasing function of the total inventory. Another possible extension is to examine the competitive situation as a dynamic, multi-period game. Some work done in industrial economics may be relevant here. See, for example, Kreps and Scheinkman (1983).

APPENDIX

In all the proofs stated here, we assume that $C_{h,i}>0$ for all i, $C_{0,i}\geqq 0$ for all i, $P_i>0$ for all i and that $R>0$.

Proof of theorem 3.1: The first and the second derivatives of the objective function of problem P1 with respect to Q_i are as follows:

$$\begin{aligned}\Pi_i'(Q_i\,|\,\bar{Q}_i>0)&=P_i R\bar{Q}_i/Q^2-C_{h,i}\\ \Pi_i''(Q_i\,|\,\bar{Q}_i>0)&=-2P_i R\bar{Q}_i/Q^3.\end{aligned}\tag{A.1}$$

Since the second derivative is strictly negative, the objective function of problem P1 is strictly concave. Hence, problem P1 has a unique global optimal solution Q_i^* characterized as follows:

$$\Pi_i'(Q_i^*\,|\,\bar{Q}_i>0)=0 \qquad \text{if} \quad Q_i^*>0$$

and (A.2)

$$\Pi_i'(0\,|\,\bar{Q}_i>0)\leqq 0 \qquad \text{if} \quad Q_i^*=0.$$

Solving (A.2) using (A.1), we get the result.

Q.E.D.

Proof of theorem 3.2: We need to prove that for each i, Q_i^* given by (3.4) is the optimal solution to problem P2. Using theorem 3.1, the optimal solution to problem P2 is given by:

$$Q_i^*=\begin{cases}(Q_{i,\max}\bar{Q}_i^*)^{1/2}-\bar{Q}_i^* & \text{if} \quad \bar{Q}_i^*\leqq Q_{i,\max}\\ 0 & \text{if} \quad \bar{Q}_i^*>Q_{i,\max}\end{cases}$$

where

$$\bar{Q}_i^*=\sum_{\substack{j=1\\ j\neq i}}^{n} Q_j^*=Q^*\sum_{\substack{j=1\\ j\neq i}}^{n} f_j=\frac{R(n-1)}{\alpha}\cdot\frac{(n-1)\alpha_i}{\alpha}=\frac{R(n-1)^2\alpha_i}{\alpha^2}.$$

Since

$$0<\alpha_i\leqq\alpha_n\leqq\alpha/n-1, \quad \frac{R(n-1)^2\alpha_i}{\alpha^2}\leqq\frac{R}{\alpha_i}, \qquad \text{i.e.} \quad \bar{Q}_i^*\leqq Q_{i,\max}.$$

Therefore

$$Q_i^*=(Q_{i,\max}\bar{Q}_i^*)^{1/2}-\bar{Q}_i^*$$

$$=\left(\frac{R}{\alpha_i}\frac{R(n-1)^2\alpha_i}{\alpha^2}\right)^{1/2}-\frac{R(n-1)^2\alpha_i}{\alpha^2}$$

$$=\frac{R(n-1)}{\alpha}\left[1-\frac{(n-1)\alpha_i}{\alpha}\right]=Q^* f_i.$$

Q.E.D.

Proof of theorem 3.3: By theorem 3.2, it is sufficient to prove that given the optimal order quantities for the first m vendors, vendors $m+1, \ldots, n$ should not carry any inventory at the NE. By theorem 3.1, this will be true if $\bar{Q}_j^* > Q_{j,\max}$ for $j=m+1, \ldots, n$. Now

$$\bar{Q}_j^* = \sum_{i=1}^{m} Q_i^* = \frac{R(m-1)}{\sum_{i=1}^{m} \alpha_i} \quad \text{and} \quad Q_{j,\max} = \frac{R}{\alpha_j}.$$

Since m is the largest integer such that

$$\alpha_m \leqq \frac{\alpha_1 + \ldots + \alpha_{m-1}}{m-2}, \qquad \alpha_j \geqq \alpha_{m+1} > \frac{\alpha_1 + \ldots + \alpha_m}{m-1} \quad \text{for } j=m+1, \ldots, n.$$

Therefore,

$$\frac{R(m-1)}{\sum_{i=1}^{m} \alpha_i} > \frac{R}{\alpha_j}, \quad \text{i.e.} \quad \bar{Q}_j^* > Q_{j,\max}.$$

Q.E.D.

Proof of theorem 4.1: It is easy to verify that (4.3) and (4.4) satisfy systems (4.1) and (4.2).

Q.E.D.

The proofs of theorems 4.2 and 4.3 are analogous to the proofs of theorems 3.1 and 3.2 respectively and are therefore omitted.

REFERENCES

1. J. H. Case, *Economics and the Competitive Process*, New York University Press, 1979.
2. D. Kreps and J. Scheinkman, *Quantity Precommitment and Bertrand Competition Yield Cournot Outcomes*, Bell Journal of Economics, Vol. 14, 1983, pp. 326-337.

3. R. D. LUCE and H. RAIFFA, *Games and Decisions: Introduction and Critical Survey*, John Wiley and Sons, New York, 1957.

4. H. M. WAGNER and T. M. WHITIN, *Dynamic Version of the Economic Lot Size Model*, Management Science, Vol. 5, 1958, pp. 89-96.