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TWO METHODS FOR MULTICRITERIA HIERARCHIZATION OF DISCRETE ALTERNATIVES (*)

by Gheorghe PĂUN ([1])

Abstract. — *The former method starts by numerically estimating the adequacy and the inadequacy (with respect to the initial data) of each statement "the alternative i appears in an aggregate hierarchy on the j-th place" and constructs a preorder which maximizes the ratio of total sum of adequacy values by the total sum of inadequacy values for the associated hierarchy. The latter method involves usual concordances and discordances (as those in ELECTRE method) and tries to construct a preorder relation on the alternatives set by choosing a set of pairs of alternatives which again maximizes the ratio of total sum of concordances by the total sum of discordances. For both methods there are optimal algorithms.*

Keywords: Multicriteria decision making; Pseudoboolean programming; ELECTRE method.

Résumé. — *La première méthode utilise des indicateurs d'adéquation et d'inadéquation (cohérents avec les données initiales du problème) à chaque proposition du type « l'alternative i apparaît, dans la hiérarchie agrégée, en j-ième position » et construit, ensuite, la hiérarchie qui maximise le ratio de la somme totale des valeurs d'adéquation sur la somme totale des valeurs d'inadéquation. La deuxième méthode fait intervenir des concordances et des discordances similaires à celles dans les méthodes ELECTRE et vise à construire une relation de préordre (sur les alternatives) en choisissant un ensemble de paires d'alternatives qui maximise de nouveau le ratio de la somme totale des concordances sur la somme totale des discordances. Pour les deux méthodes, des algorithmes optimisants sont proposés.*

1. INTRODUCTION

Although there are so many methods for multiple criteria decision making (MCDM) (*see* [3, 5, 7, 8], etc.), there is room for further methods. In fact, the well-known Arrow's impossibility theorem [1] simply "forbides" the existence of a completely accepted method, one that can make useless any different one. More practically speaking, there are various gaps in the "periodic table" (like the Mendeleev's one) of MCDM methods. The two procedures we present here deal (of course, without escaping Arrow's theorem conclusion)

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with the problem of constructing a hierarchy on a (discrete) set of alternative actions evaluated (not necessarily numerically) from more points of view; from now, by MCDM we shall understand this particular problem. Some features of these methods deserves to be emphasized:

– they work in "concordance-discordance" terms, hence seem to use most of the information contained in initial data;

– they have an optimal character from two points of view: have explicit quality criteria which should be maximized and have algorithms which maximize these criteria;

– when there are more optimum solutions, both methods provide us all these solutions;

– the first method may produce aggregated hierarchies containing gaps, a frequent phenomenon in actual decision making ("the first rank is assigned to alternative i, the second is not granted, the third...");

– both methods are pollynomially convergent, hence they are *tractable*.

2. THE FIRST METHOD

As usual in MCDM problems, let us consider a set of action alternatives:

$$A=\{a_1, a_2, \ldots, a_n\}, \qquad n \geqq 2,$$

evaluated from m points of view (criteria):

$$p_1, p_2, \ldots, p_m, \qquad m \geqq 2,$$

whose relative importance in the concrete framework of decision is specified by means of given weights (positive real numbers):

$$w_1, w_2, \ldots, w_m.$$

The MCDM question is to construct a hierarchy of the alternatives (in most cases, a preorder relation on A is looked for) which fits as well as possible with the initial data. The passing from the words "fits as well as possible..." to a concrete unambiguous quality criterion is the real task of MCDM (as well as the practical task to construct an optimal hierarchy according to this criterion).

Generally speaking, let us assume that for each alternative a_i and for each integer j, $1 \leqq j \leqq n$, we can evaluate two parameters:

$c_{i,j}$ = the adequacy (with respect to the input data) of assigning action a_i to the place j (1 is the best place, n the worst one) in one arbitrary hierarchy on the set A;

$d_{i,j}$ = the inadequacy of assigning a_i to the rank j.

We assume $c_{i,j}$ to be as greater and $d_{i,j}$ to be as smaller as the proposition "a_i occurs on place j" is more concordant with the initial data. Moreover, we assume $c_{i,j} \in \mathbf{R}_+$, $d_{i,j} \in \mathbf{R}_+$, $d_{i,j} > 0$ for all i, j ($\mathbf{R}_+$ is the set of nonnegative real numbers).

Then, a natural criterion to evaluate the quality of a hierarchy:

$$h = (a_{i_1}, a_{i_2}, \ldots, a_{i_n}), \qquad A = \{ a_{i_1}, \ldots, a_{i_n} \},$$

is

$$C(h) = \frac{\sum_{j=1}^{n} c_{i_j, j}}{\sum_{j=1}^{n} d_{i_j, j}}$$

and this value is as greater as the hierarchy h is better.

We are lead to the following (more general) pseudoboolean [4] programming problem: Let C, D be two $n \times m$ matrices, $C = (c_{i,j})$, $D = (d_{i,j})$, $c_{i,j} \in \mathbf{R}_+$, $d_{i,j} \in \mathbf{R}_+ - \{0\}$. For an n-tuple $T = (j_1, j_2, \ldots, j_n)$, $1 \leqq j_i \leqq m$, $1 \leqq i \leqq n$, consider:

$$V_{C,D}(T) = \frac{\sum_{i=1}^{n} c_{i, j_i}}{\sum_{i=1}^{n} d_{i, j_i}},$$

and:

$$M(C, D) = \max \{ V_{C,D}(T) \mid T = (j_1, j_2, \ldots, j_n), \ 1 \leqq j_i \leqq m, \ 1 \leqq i \leqq n \}.$$

Define the set:

$$O(C, D) = \{ T \mid V_{C,D}(T) = M(C, D) \}.$$

PROBLEM: *Find (all) elements in the set $O(C, D)$.*

The problem was completely solved in [2] by means of the following algorithm: For C, D as above, we construct the string of n-tuples T_k^i as follows:

(i) $T_0^0 = (1, 1, \ldots, 1)$;

(ii) if $T_k^{i-1}=(j_1, j_2, \ldots, j_n)$, then $T_k^i=(j_1, \ldots, j_{i-1}, j'_i, j_{i+1}, \ldots, j_n)$, where j'_i is choosed in such a way that:

$$V_{C,D}(T_k^i)=\max_{1 \leq r \leq m} V_{C,D}((j_1, \ldots, j_{i-1}, r, j_{i+1}, \ldots, j_n))$$

and $j'_i \neq j_i$ only when:

$$V_{C,D}(T_k^{i-1})<\max_{1 \leq r \leq m} V_{C,D}((j_1, \ldots, j_{i-1}, r, j_{i+1}, \ldots, j_n)),$$

(iii) $T_{k+1}^0=T_k^n$.

Clearly, there is k_0 such that $T_{k+1}^0=T_k^0$ for all $k \geqq k_0$. Let k^* be the smallest k_0 as above and denote $\mathscr{A}(C, D, T_0^0)=T_{k^*}^0$.

THEOREM 1 [2]: $\mathscr{A}(C, D, T_0^0) \subseteq O(C, D)$.

Consider now an n-tuple $T=(j_1, j_2, \ldots, j_n) \in O(C, D)$ and define the sets:

$$J_i(T)=\{k \mid 1 \leqq k \leqq m,\ V_{C,D}((j_1, \ldots, j_{i-1}, k, j_{i+1}, \ldots, j_n))=M(C, D)\},\quad 1 \leqq i \leqq n.$$

We denote $\mathscr{A}'(C, D, T)=\{(s_1, s_2, \ldots, s_n) \mid s_i \in J_i(T),\ 1 \leqq i \leqq n\}$.

THEOREM 2 [2]: *For any* $T \in O(C, D)$ *we have* $\mathscr{A}'(C, D, T)=O(C, D)$.

Consequently,

$$O(C, D)=\mathscr{A}'(C, D, \mathscr{A}(C, D, T_0^0)).$$

In words, the algorithm proceeds as follows:

Step 1. Construct an arbitrary initial solution T_0^0.

Step 2. Improve componentwise the current solution.

Step 3. If the above improvement is effective, then go to Step 2, otherwise go to Step 4.

Step 4. Construct the sets $J_i(T_{k^*}^0)$, $1 \leqq i \leqq n$, as described above.

Step 5. Construct all the n-tuples $T=(s_1, \ldots, s_n)$, $s_i \in J_i(T_{k^*}^0)$, $1 \leqq i \leqq n$, Stop.

REMARK 1: The best positions on each row do not provide us a global optimal solution. For, let us consider the following example:

$$C=\begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix}, \quad D=\begin{pmatrix} 2 & 1 \\ 4 & 7 \end{pmatrix}.$$

Although:

$$\frac{c_{1,2}}{d_{1,2}} > \frac{c_{1,1}}{d_{1,1}}, \qquad \frac{c_{2,2}}{d_{2,2}} > \frac{c_{2,1}}{d_{2,1}},$$

however:

$$\frac{c_{1,2}+c_{2,2}}{d_{1,2}+d_{2,2}} = \frac{6}{8} < \frac{5}{6} = \frac{c_{1,1}+c_{2,1}}{d_{1,1}+d_{2,1}}.$$

A computer program was described in [2] which solves this problem.

Returning to our MCDM problem, the only question which we have to answer is to suitably define the parameters $c_{i,j}$ and $d_{i,j}$, associated to each pair (alternative i, rank j), $1 \leq i, j \leq n$, for given initial data. We present here only some suggestions for the ordinal case (when the initial data consists of m hierarchies on A, induced by the m criteria). Denote by $pl(a_i, p_j)$ the place of the action a_i in the hierarchy induced by the criterion p_j. Then, the adequacy of assigning a_i to the place j in an aggregated hierarchy can be evaluated by:

$$c_{i,j} = \sum_{k=1}^{m} w_k (1 + |j - pl(a_i, p_k)|)^{-1},$$

or by:

$$c_{i,j} = \sum_{k=1}^{m} w_k \, 2^{-|j - pl(a_i, p_k)|}$$

whereas the inadequacy can be evaluated by:

$$d_{i,j} = \sum_{k=1}^{m} w_k (1 + |j - pl(a_i, p_k)|),$$

or by:

$$d_{i,j} = \sum_{k=1}^{m} w_k \, 2^{|j - pl(a_i, p_k)|}.$$

Other such evaluations remain to be considered and checked.

Let us remark that this approach to MCDM can lead to partially filled hierarchies (the algorithm in [2] can choose more elements on the same column, hence more alternatives on the same place and thus some unoccupied ranks appear). Moreover, the method provides us all the hierarchies of maximum quality and we can choose that solution which satisfies some further restrictions.

3. ANOTHER METHOD OF HIERARCHIZATION

This method starts by considering adequacy and inadequacy values associated to each pair (a_i, a_j) of alternatives. As in ELECTRE method, for example [6], let $c_{i,j}$ be a measure of the "concordance" of considering the alternative a_i better than the alternative a_j in an aggregated hierarchy and $d_{i,j}$ be the "discordance" of placing a_i before a_j. We assume again $c_{i,j} \in \mathbf{R}_+$, $d_{i,j} \in \mathbf{R}_+ - \{0\}$.

Our goal is to introduce a total preorder relation on the set A. Such a relation can be defined starting from a given set of pairs in $A \times A$ and transitively closing the mutual relationships induced by these pairs. Clearly, at least $n-1$ pairs must be considered in order to obtain a total relation on A. Moreover, when all $n(n-1)$ pairs (a_i, a_j), $i \neq j$, in $A \times A$ are considered, then we obtain a total preorder relation on A, but no structure is introduced in this way: all the alternatives are considered on the same level of quality. Consequently, we have to find the smallest set of pairs (a_i, a_j) which induces a total preorder relation on A and which fits in the best way with the initial data by means of our indicators of adequacy and inadequacy.

Thus we are lead again to a pseudoboolean programming problem, namely,

$$\text{maximize} \frac{\sum_{i=1}^{n} c_i x_i}{\sum_{i=1}^{n} d_i x_i},$$

where $c_i, d_i \in \mathbf{R}_+$, $d_i \neq 0$, for all i, $1 \leqq i \leqq n$, $x_i \in \{0, 1\}$, subject to:

$$\sum_{i=1}^{n} x_i = k,$$

for given $k \leqq n$.

First, we shall solve this more general problem and then we shall return to MCDM questions. We again prefer an *ad hoc* framework, instead of pseudoboolean programming theory.

Let $C=(c_i)$, $D=(d_i)$, $1 \leqq i \leqq n$, be two vectors of real numbers, $c_i \geqq 0$, $d_i > 0$, and let $T=(j_1, j_2, \ldots, j_k)$, $1 \leqq j_i \leqq n$, $j_i \neq j_r$ for all $i \neq r$. We denote:

$$V_{C,D}(T)=\frac{\sum_{i=1}^{k} c_{j_i}}{\sum_{i=1}^{k} d_{j_i}}$$

$$M_k(C, D)=\max \{ V_{C,D}(T) \mid T=(j_1, \ldots, j_k),\ 1 \leqq j_i \leqq n,\ j_i \neq j_r \text{ for all } i \neq r \}.$$

PROBLEM: *Find (all) elements in the set:*

$$O_k(C, D)=\{ T \mid V_{C,D}(T)=M_k(C, D) \}.$$

Let us note that the first k positions in the decreasing order of the numbers c_i/d_i do not constitute a solution. Indeed, let us consider the problem characterized by:

$$C=(4, 2, 3), \qquad D=(7, 4, 2).$$

Although:

$$\frac{c_3}{d_3} > \frac{c_1}{d_1} > \frac{c_2}{d_2},$$

however:

$$\frac{c_1+c_3}{d_1+d_3}=\frac{7}{9}<\frac{c_2+c_3}{d_2+d_3}=\frac{5}{6},$$

therefore $(1, 3) \notin O_2(C, D)$.

The problem can be solved by an algorithm similar to that described in the previous section. Let us consider the following string of k-tuples (denoted by T_i^j):

(i) $T_0^0=(1, 2, \ldots, k)$,

(ii) if

$$T_s^{r-1}=(j_1, \ldots, j_{r-1}, j_r, j_{r+1}, \ldots, j_k),$$

then

$$T_s^r=(j_1, \ldots, j_{r-1}, j_r', j_{r+1}, \ldots, j_k),$$

where j'_r is choosed in such a way that:

$$V_{C,D}(T^r_s)=\max\{V_{C,D}((j_1,\ldots,j_{r-1},t,j_{r+1},\ldots,j_k))\mid 1\leqq t\leqq n,\ t\neq j_i \text{ for } i=1,\ldots,r-1,r+1,\ldots,k\},$$

$j'_r\neq j_i$, for all $1\leqq i\leqq k$, $i\neq r$, and $j'_r\neq j_r$ only if:

$$V_{C,D}(T^{r-1}_s)<V_{C,D}(T^r_s),$$

(iii) $T^0_{s+1}=T^k_s$.

Clearly, there exists s_0 such that $T^0_s=T^0_{s+1}$ for all $s\geqq s_0$. We denote $\mathscr{A}_k(C, D, T^0_0)=T^0_{s^*}$, where s^* is the smallest s_0 as above.

THEOREM 3: *In the above circumstances,* $\mathscr{A}_k(C, D, T^0_0)\subseteq O_k(C, D)$.

Proof: Let $\mathscr{A}_k(C, D, T^0_0)=(j_1, j_2, \ldots, j_k)$ and consider a k-tuple $(s_1,\ldots,s_k)\in O_k(C, D)$. Suppose that $\mathscr{A}_k(C, D, T^0_0)\notin O_k(C, D)$, hence:

$$\frac{\sum_{t=1}^{k} c_{s_t}}{\sum_{t=1}^{k} d_{s_t}}>\frac{\sum_{t=1}^{k} c_{j_t}}{\sum_{t=1}^{k} d_{j_t}}.$$

Let $f:\{1,2,\ldots,k\}\to\{1,2,\ldots,k\}$ be an one-to-one mapping such that $f(i)=r$ whenever $s_i=j_r$. For each $i=1,2,\ldots,k$ we have:

$$\frac{\sum_{t=1}^{k} c_{j_t}}{\sum_{t=1}^{k} d_{j_t}}\geqq\frac{\sum_{t=1}^{k} c_{j_t}-c_{j_{f(i)}}+c_{s_i}}{\sum_{t=1}^{k} d_{j_t}-d_{j_{f(i)}}+d_{s_i}}.$$

Indeed, when $j_{f(i)}=s_i$, then we obtain an equality and when $j_{f(i)}\neq s_i$, then we obtain a (not necessarily proper) inequality because the algorithm of constructing the vector $\mathscr{A}_k(C, D, T^0_0)$ does not change the vector $(j_1,\ldots,j_k)$ when one tries to replace $c_{j_{f(i)}}$ by c_{s_i}.

We eliminate the denominators and sum all the above $k+1$ inequalities (at least the first one is proper) and we obtain:

$$\sum_{i=1}^{k} c_{s_i}\sum_{i=1}^{k} d_{j_i}+\sum_{i=1}^{k}\sum_{t=1}^{k} c_{j_t}\left(\sum_{t=1}^{k} d_{j_t}-d_{j_{f(i)}}+d_{s_i}\right)$$
$$>\sum_{i=1}^{k} c_{j_i}\sum_{i=1}^{k} d_{s_i}+\sum_{i=1}^{k}\sum_{t=1}^{k} d_{j_t}\left(\sum_{t=1}^{k} c_{j_t}-c_{j_{f(i)}}+c_{s_i}\right).$$

and finally $0>0$, a contradiction.

In conclusion, the above algorithm (start by an arbitrary initial solution and improve it componentwise while this is possible) constructs an optimal solution of our problem. *All* the optimal solutions can be found in the following way. For $T \in O_k(C, D)$, $T=(j_1, \ldots, j_k)$, let us consider the sets:

$$J_i^*(T)=\{r \mid r \neq j_s, s=1, 2, \ldots, i-1, i+1, \ldots, k, \text{ and } V_{C,D}((j_1, \ldots, j_{i-1}, r, j_{i+1}, \ldots, j_k))=M_k(C, D)\},$$

for each i, $1 \leqq i \leqq k$, and define:

$$\mathcal{A}'_k(C, D, T)=\{(s_1, s_2, \ldots, s_k) \mid s_i \in J_i^*(T), 1 \leqq i \leqq k, s_i \neq s_j \text{ for all } i \neq j, 1 \leqq i, j \leqq k\}.$$

THEOREM 4: *For each $T \in O_k(C, D)$ we have $\mathcal{A}'_k(C, D, T) \subseteqq O_k(C, D)$.*

Proof: Let $j_i, j'_i \in J_i^*(T)$ and $j_t, j'_t \in J_t^*(T)$, for $i<t$, $j'_i \neq j'_t$. We denote:

$$\alpha=\sum_{\substack{r=1\\ r\neq i,t}}^{k} c_{j_r}, \qquad \beta=\sum_{\substack{r=1\\ r\neq i,t}}^{k} d_{j_r}$$

therefore:

$$M_k(C, D)=\frac{\alpha+c_{j_i}+c_{j_t}}{\beta+d_{j_i}+d_{j_t}}=\frac{\alpha+c_{j'_i}+c_{j_t}}{\beta+d_{j'_i}+d_{j_t}}=\frac{\alpha+c_{j_i}+c_{j'_t}}{\beta+d_{j_i}+d_{j'_t}},$$

that is:

$$(1) \qquad \alpha+c_{j_i}+c_{j_t}=M_k(C, D)(\beta+d_{j_i}+d_{j_t}),$$

$$(2) \qquad \alpha+c_{j'_i}+c_{j_t}=M_k(C, D)(\beta+d_{j'_i}+d_{j_t}),$$

$$(3) \qquad \alpha+c_{j_i}+c_{j'_t}=M_k(C, D)(\beta+d_{j_i}+d_{j'_t}).$$

Calculating (1) minus (2) we obtain:

$$(4) \qquad c_{j_i}-c_{j'_i}=M_k(C, D)(d_{j_i}-d_{j'_i})$$

and from (3) minus (4) we obtain:

$$\alpha+c_{j'_t}+c_{j'_i}=M_k(C, D)(\beta+d_{j'_t}+d_{j'_i}),$$

that is:

$$M_k(C, D)=\frac{\alpha+c_{j'_t}+c_{j'_i}}{\beta+d_{j'_t}+d_{j'_i}}.$$

Consequently,

$$(j_1, \ldots, j_{i-1}, j'_i, j_{i+1}, \ldots, j_{t-1}, j'_t, j_{t+1}, \ldots, j_k) \in O_k(C, D).$$

Consider now $j_i, j'_i \in J^k_i(T)$, $j_t, j'_t \in J^k_t(T)$, $j_s, j'_s \in J^k_s(T)$, for $i<t<s$, $j'_i \neq j'_t \neq j'_s \neq j'_i$. From the above argument we get that both the following k-tuples are in $O_k(C, D)$:

$$(j_1, \ldots, j_{i-1}, j'_i, j_{i+1}, \ldots, j_{t-1}, j'_t, j_{t+1}, \ldots, j_{s-1}, j_s, j_{s+1}, \ldots, j_k),$$
$$(j_1, \ldots, j_{i-1}, j_i, j_{i+1}, \ldots, j_{t-1}, j'_t, j_{t+1}, \ldots, j_{s-1}, j'_s, j_{s+1}, \ldots, j_k).$$

From $j_t, j'_t \in J^k_t(T)$ we obtain:

$$T_t = (j_1, \ldots, j_{i-1}, j_i, j_{i+1}, \ldots, j_{t-1}, j'_t j_{t+1}, \ldots, j_{s-1}, j_s, j_{s+1}, \ldots, j_k) \in O_k(C, D),$$

therefore $j_i, j'_i \in J^k_i(T_t)$ and $j_s, j'_s \in J^k_s(T_t)$. Using again the first part of the proof for T_t we obtain:

$$(j_1, \ldots, j_{i-1}, j'_i, j_{i+1}, \ldots, j_{t-1}, j'_t, j_{t+1}, \ldots, j_{s-1}, j'_s, j_{s+1}, \ldots, j_k) \in O_k(C, D).$$

In the same way one can prove that each k-tuple in $\mathscr{A}'_k(C, D, T)$ belongs to $O_k(C, D)$.

THEOREM 5: *For each $T \in O_k(C, D)$ we have $O_k(C, D) \subseteq \mathscr{A}'_k(C, D, T)$.*

Proof: Consider $T=(i_1, i_2, \ldots, i_k) \in O_k(C, D)$ and let $J^k_i(T)$, $1 \leqq i \leqq k$, be the sets constructed as above. Let us suppose that $O_k(C, D) - \mathscr{A}'_k(C, D, T) \neq \varnothing$ and let $T' = (j_1, \ldots, j_k)$ be an element of $O_k(C, D)$ which does not belong to $\mathscr{A}'_k(C, D, T)$ and no permutation of T' belong to this set. Consequently, there exists indices t, $1 \leqq t \leqq k$, for which $j_t \notin J^k_t(T)$. We permute the sequence T' in such a way that whenever $j_s \in \{i_1, \ldots, i_k\}$, then $j_s = i_s$. Let us denote by S the set $\{1, 2, \ldots, k\}$, by S_1 the sequence of indices t for which $j_t \in J^k_t(T)$ and by S_2 the sequence of indices t for which $j_t \notin J^k_t(T)$. Clearly, $S_1 \cup S_2 = S$, $S_1 \cap S_2 = \varnothing$, $S_2 \neq \varnothing$ and S_1 also contains those positions for which $j_s = i_s$; moreover, in any sequence of the form $(i_1, \ldots, i_{r-1}, j_r, i_{r+1}, \ldots, i_k)$ we have $j_r \neq i_s$ for each $s \neq r$ (in other words, each such sequence is a—not necessarily optimal—solution of the problem).

As $j_t \in J_t^k(T)$, $t \in S_1$, we obtain:

$$\frac{\sum_{\substack{s=1\\ s\neq t}}^{k} c_{i_s}+c_{j_t}}{\sum_{\substack{s=1\\ s\neq t}}^{k} d_{i_s}+d_{j_t}}=M_k(C, D).$$

As $j_t \notin J_t^k(T)$, $t \in S_2$, we have:

$$\frac{\sum_{\substack{s=1\\ s\neq t}}^{k} c_{i_s}+c_{j_t}}{\sum_{\substack{s=1\\ s\neq t}}^{k} d_{i_s}+d_{j_t}}<M_k(C, D).$$

We eliminate the denominators of these relations (they are strictly positive), then we sum them member by member, and, by denoting:

$$\alpha=\sum_{t=1}^{k} c_{i_t}, \qquad \beta=\sum_{t=1}^{k} d_{i_t},$$

we obtain:

$$\sum_{s=1}^{k} (\alpha-c_{i_s})+\sum_{s=1}^{k} c_{j_s}<M_k(C, D) \sum_{s=1}^{k} (\beta-d_{i_s})+M_k(C, D) \sum_{s=1}^{k} d_{j_s},$$

that is:

$$(k-1)\,\alpha+\sum_{s=1}^{k} c_{j_s}<M_k(C, D)\,(k-1)\,\beta+M_k(C, D) \sum_{s=1}^{k} d_{j_s}.$$

As $T \in O_k(C, D)$, we have:

$$\frac{\alpha}{\beta}=M_k(C, D),$$

therefore:

$$(k-1)\,\alpha=M_k(C, D)\,(k-1)\,\beta$$

and, from the previous inequality, we obtain:

$$\sum_{s=1}^{k} c_{j_s} < M_k(C, D) \sum_{s=1}^{k} d_{j_s}.$$

This means that $(j_1, \ldots, j_k)$ is not an optimal solution of the problem, a contradiction. In conclusion, $T' \in \mathscr{A}'_k(C, D, T)$, hence

$$O_k(C, D) \subseteq \mathscr{A}'_k(C, D, T)$$

and the proof is over.

Consequently, $O_k(C, D) = \mathscr{A}'_k(C, D, \mathscr{A}_k(C, D, T_0^0))$.

The following theorem can be useful when we actually have to solve the previous problem:

THEOREM 6: *For C, D, k as above, let us consider the set:*

$$M = \left\{ j \,\middle|\, \frac{c_j}{d_j} = \max_{1 \leq i \leq n} \frac{c_i}{d_i} \right\}, \qquad \text{card } M = p.$$

1. *If $k \leq p$, then:*

$$O_k(C, D) = \{(i_1, \ldots, i_k) \mid i_j \neq i_t \text{ and } i_j \in M \text{ for each } j\}.$$

2. *If $k > p$, then for each:*

$$T \in O_k(C, D), \qquad T = (i_j, \ldots, i_k),$$

we have $M\{i_1, \ldots, i_k\}$.

Proof: We use the following *assertion* whose proof is left to the reader: if

$$S = (j_1, \ldots, j_k) \in O_k(C, D),$$

then

$$V_{C,D}(S) \geq \frac{c_s}{d_s} \quad \text{for all } s \notin S.$$

1. Clearly,

$$\frac{\sum_{s=1}^{k} c_{i_s}}{\sum_{s=1}^{k} d_{i_s}} \leq \max_{1 \leq t \leq k} \frac{c_{i_t}}{d_{i_t}},$$

and the inequality is proper if and only if there is s, $1 \leqq s \leqq k$, such that $c_{i_s}/d_{i_s} < \max_{1 \leqq t \leqq k} (c_{i_t}/d_{i_t})$. Point 1 of the theorem follows from the above assertion.

2. If $k > p$, then T contains at least an element which does not belong to M, hence:

$$\frac{\sum_{j=1}^{k} c_{i_j}}{\sum_{j=1}^{k} d_{i_j}} < \max_{1 \leqq i \leqq n} \frac{c_i}{d_i}.$$

Using again the above mentioned assertion, we obtain point 2 too and the proof is complete.

A significant result with respect to our initial purpose, that of MCDM, is the next theorem, which enlightenes the dependence of $M_k(C, D)$ on the values of k.

LEMMA 7: *For C, D as above, we have*

$$M_1(C, D) \geqq M_2(C, D) \geqq \ldots \geqq M_n(C, D).$$

Proof: Let us consider the vectors:

$$T_k = (i_1, \ldots, i_k) \in O_k(C, D),$$
$$T_{k+1} = (j_1, \ldots, j_{k+1}) \in O_{k+1}(C, D).$$

We denote:

$$\alpha_k = \sum_{t=1}^{k} c_{i_t}, \qquad \beta_k = \sum_{t=1}^{k} d_{i_t}$$
$$\alpha_{k+1} = \sum_{t=1}^{k+1} c_{j_t}, \qquad \beta_{k+1} = \sum_{t=1}^{k+1} d_{j_t},$$

hence:

$$\frac{\alpha_k}{\beta_k} = M_k(C, D), \qquad \frac{\alpha_{k+1}}{\beta_{k+1}} = M_{k+1}(C, D),$$

and for each $T' = (i'_1, \ldots, i'_k)$, $1 \leqq i'_j \leqq n$, we have:

$$\frac{\alpha_k}{\beta_k} \geqq V_{C,D}(T').$$

Therefore, for each $t=1, 2, \ldots, k+1$, we obtain:

$$\frac{\alpha_k}{\beta_k} \geqq \frac{\alpha_{k+1}-c_{j_t}}{\beta_{k+1}-d_{j_t}}.$$

We eliminate the denominators, add the obtained $k+1$ inequalities and we obtain:

$$(k+1)\,\alpha_k\,\beta_{k+1}-\alpha_k\,\beta_{k+1} \geqq (k+1)\,\alpha_{k+1}\,\beta_k-\beta_k\,\alpha_{k+1},$$

hence:

$$k\,\alpha_k\,\beta_{k+1} \geqq k\,\alpha_{k+1}\,\beta_k,$$

therefore:

$$\frac{\alpha_k}{\beta_k} \geqq \frac{\alpha_{k+1}}{\beta_{k+1}}$$

that is $M_k(C, D) \geqq M_{k+1}(C, D)$ and the proof is terminated.

Let us now return to our initial problem. For given vectors:

$$C=(c_{1,2}, c_{1,3}, \ldots, c_{1,n}, c_{2,1}, \ldots, c_{n,n-1})$$
$$=(c'_1, c'_2, \ldots, c'_{n(n-1)}) \text{ (concordances)},$$
$$D=(d_{1,2}, d_{1,3}, \ldots, d_{1,n}, d_{2,1}, d_{2,3}, \ldots, d_{n,n-1})$$
$$=(d'_1, d'_2, \ldots, d'_{n(n-1)}) \text{ (discordances)},$$

we use the above algorithm for $k=n-1$ and construct the set of k-tuples of pairs of action alternatives corresponding to vectors $(i_1, \ldots, i_k)$, $1 \leqq i_j \leqq n(n-1)$, such that $i_r \neq i_s$ for $r \neq s$ and $\sum_{j=1}^{k} c'_{i_j} / \sum_{j=1}^{k} d'_{i_j}$ is maximum. If the reflexive transitive closure of some such k-tuple of pairs of alternatives is a total preorder relation on the set A, then the problem is solved. In the opposite case, we either increase k by one and look for $(k+1)$-tuples of pairs, or we construct all k-tuples of pairs associated to vectors $(i_1, \ldots, i_k)$ for which $V_{C,D}((i_1, \ldots, i_k)) \geqq M_{k+1}(C, D)$ (better than a $(k+1)$-solution, but not necessarily optimal for k — *see* Theorem 7). By repeating this procedure, we can eventually find an s-tuple of pairs which leads to a total preorder relation on A, is optimum with respect to the quality criterion $V_{C,D}$ and has the smallest s. This is the aggregated hierarchy which solves our MCDM problem.

4. FINAL REMARKS

Both the above methods request a large amount of calculus, hence they need a computer in order to be practically implemented. However, the algorithms are easy to be programmed and they seem to be rapidly convergent.

Indeed, let us assume that $C=(c_i)$, $D=(d_i)$ are vectors of positive rational numbers. By multiplying these numbers by a positive constant we obtain two vectors C', D' which lead to an equivalent problem, in the sense that

$$M_k(C', D')=M_k(C, D), \qquad O_k(C', D')=O_k(C, D),$$
$$\mathscr{A}_k(C, D, T_0^0)=\mathscr{A}_k(C', D', T_0^0).$$

Consequently, it is sufficient to examine the convergence for C, D constituted of integers. Clearly,

$$V_{C,D}(T_0^0)\geqq \frac{k}{k \, . \max\limits_{1\leqq i\leqq n} d_i}=\frac{1}{\max\limits_{1\leqq i\leqq n} d_i}>0,$$

$$M_k(C, D)\leqq \frac{k \, . \max\limits_{1\leqq i\leqq n} c_i}{k}=\max_{1\leqq i\leqq n} c_i.$$

At each step of the algorithm (a step means the replacing of a component of the current solution, thus strictly improving its quality), the current solution quality increases at least by:

$$\frac{1}{k^2 \max\limits_{1\leqq i\leqq n} d_i^2}.$$

Consequently, the algorithm finds an optimum solution in at most:

$$\left(\max_{1\leqq i\leqq n} c_i-\frac{1}{\max\limits_{1\leqq i\leqq n} d_i}\right) k^2 \max_{1\leqq i\leqq n} d_i^2<k^2 \max_{1\leqq i\leqq n} c_i \max_{1\leqq i\leqq n} d_i^2$$

steps. Each such step involves at most $n-k$ trials to improve the current solution, hence the algorithm converges pollynomially.

A similar argument works for the algorithm in [2] presented in Section 2.

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