

R. L. GARG

S. K. KHANNA

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## QUEUE DEPENDENT ADDITIONAL SERVER QUEUEING PROBLEM WITH BATCH ARRIVALS (\*)

by R. L. GARG and S. K. KHANNA <sup>(1)</sup>

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*Abstract.* — We consider in this paper the steady state behaviour of a Queueing System with queue length dependent additional server facility wherein arrivals occur in batches of variable size. Whenever the queue length in front of the first server reaches a certain length, the system adds another server. Steady state probabilities and expected queue lengths in Single Server System and Additional Server System are calculated explicitly. Associating the costs with the opening of new server and the waiting of the customers, a criterion to obtain the Decision Point at which the application of additional server will be profitable is discussed.

**Keywords:** Queues with several channels; Bulk queues; Design and control of queues.

*Résumé.* — Nous considérons dans cet article le comportement asymptotique d'un système disposant d'un serveur supplémentaire si la longueur de la file d'attente le nécessite. On suppose que les arrivées se font en grappes. Chaque fois que la file d'attente au premier guichet atteint une certaine longueur, le système ouvre un autre guichet. On calcule explicitement les probabilités asymptotiques et les moyennes de ces longueurs. Considérons les coûts de l'ouverture d'un nouveau guichet et de l'attente des clients, on obtient un critère pour obtenir le point et l'introduction d'un nouveau guichet est profitable.

### INTRODUCTION

A large number of queueing problems where the service is through parallel channels have been solved by various authors. In particular Kendall [1] considers the steady state behaviour of many server queueing system. Saaty [2] studies the transient behaviour of the system M/M/C. Romani [3], Phillips [4] and Murari [5] obtain the steady state probabilities of queueing problems with variable number of service channels. More recently Singh [6] discusses a Markovian queue wherein the system starts another server at some cost whenever the queue length in front of the first server reaches a certain length.

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<sup>(1)</sup> Department of Mathematics, Kurukshetra University, Kurukshetra 132 119, India.

In this paper an attempt is made to modify the results due to Singh so as to bring them more close to many practical situations, where arrivals occur in batches of variable size, which may exist in banks, election booths, etc.

Thus the queueing system studied in this paper can be described as follows:

1. The arrivals follow Poisson distribution and occur in batches of variable size with maximum size  $T$ ;  $\lambda C_i dt$  is the first order probability that a batch of arrivals consists of exactly  $i$  ( $1 \leq i \leq T$ ) units; where  $\sum_{i=1}^T C_i = 1$ .

2. The queue discipline is first come first served.

3. The service facility consists of one regular and one additional service channel. The regular service channel is always at the disposal of customers, irrespective of the queue length (the number of customers waiting including those being served). The additional service channel operates instantaneously on an arrival when there are  $N (> 2T)$  customers in the queue and stops operating when the queue length again reduces to  $N$ .

4. The service time distribution for each server is negative exponential with parameter  $\mu$ .

A relationship is developed among costs, the traffic intensity  $\rho$ , the maximum size of the arriving batch  $T$ , probabilities  $C_i$  ( $1 \leq i \leq T$ ) and the maximum allowable size  $N$  in front of the first server. For particular values of  $T$  and  $C_i$ 's, the ratios of the costs is given in a table for different values of  $N$  and  $\rho$ . The graphs for ratios of the costs against  $\rho$  for these values of  $N$  are plotted. It is demonstrated with an example that how from the graphs the decision point for  $N$  at which the application of second server is profitable can be calculated. Results of Singh are obtained as particular cases.

#### STEADY STATE SOLUTION

Let:

$P_n \equiv$  Steady state probability that there are  $n$  customers

in the system at any time.

Elementary probability reasoning leads to the following difference equations governing the system:

$$P_1 - \rho P_0 = 0,$$

$$P_n - (1 + \rho) P_{n-1} + \rho \sum_{i=1}^T C_i P_{n-1-i} = 0; \quad 2 \leq n \leq N,$$

$$P_{N+1} - \frac{1}{2}(1 + \rho) P_N + \frac{\rho}{2} \sum_{i=1}^T C_i P_{N-i} = 0,$$

$$P_n - \left(1 + \frac{\rho}{2}\right) P_{n-1} + \frac{\rho}{2} \sum_{i=1}^T C_i P_{n-1-i} = 0; \quad N+2 \leq n,$$

where:

$$\rho = \frac{\lambda}{\mu} \quad \text{and} \quad P_j = 0 \quad \text{for } j < 0.$$

Solutions of these difference equations are:

$$\begin{aligned} P_n &= \sum_{i=1}^T a_i \alpha_i^n P_0 = \sum_{i=1}^T b_i \alpha_i^n P_0; \quad 1 \leq n \leq T+1 \\ &= \sum_{i=1}^T b_i \alpha_i^n P_0; \quad T+2 \leq n \leq N-T \\ &= \sum_{i=1}^T b_i \alpha_i^n P_0 = \sum_{i=1}^T d_i \beta_i^n P_0; \quad N-T+1 \leq n \leq N \\ &= \sum_{i=1}^T d_i \beta_i^n P_0; \quad N+2 \leq n. \end{aligned}$$

and:

$$P_{N+1} = \frac{1}{2} \sum_{i=1}^T b_i \alpha_i^{N+1} P_0 = \sum_{i=1}^T d_i \beta_i^{N+1} P_0$$

where:

$$\left. \begin{aligned} a_i &= \frac{\alpha_i^{T-1}}{T \prod_{\substack{r=1 \\ r \neq i}}^T (\alpha_i - \alpha_r)} \\ b_i &= \frac{\sum_{g=1}^T a_g \left[ \rho \sum_{j=1}^T \sum_{k=0}^{j-1} C_{T-k} \alpha_i^{k-j} \alpha_g^j - \alpha_g^{T+1} \right]}{\alpha_i (1 - \alpha_i) \prod_{\substack{r=1 \\ r \neq i}}^T (\alpha_i - \alpha_r)} \\ d_i &= \frac{\sum_{g=1}^T b_g \left[ \rho \sum_{j=1}^T \sum_{k=0}^{j-1} C_{T-k} \beta_i^{k-j} \alpha_g^{N-T+j} - \alpha_g^{N+1} \right]}{2 \beta_i^{N-T+1} (1 - \beta_i) \prod_{\substack{r=1 \\ r \neq i}}^T (\beta_i - \beta_r)} \end{aligned} \right\} 1 \leq i \leq T$$

and  $\alpha_i, \beta_i (1 \leq i \leq T)$  are respectively the non-unity roots of the algebraic equations:

$$Z^{T+1} - (1 + \rho) Z^T + \rho (C_1 Z^{T-1} + C_2 Z^{T-2} + \dots + C_T) = 0,$$

$$Z^{T+1} - \left(1 + \frac{\rho}{2}\right) Z^T + \frac{\rho}{2} (C_1 Z^{T-1} + C_2 Z^{T-2} + \dots + C_T) = 0.$$

Since  $\sum_{n=0}^{\infty} P_n = 1$ , each  $\beta_i$  must lie within the unit circle and by Rouché's theorem it can be easily shown that this condition holds provided  $\rho < 2/T$ .

Now assuming that this condition is satisfied and using normal condition that  $\sum_{n=0}^{\infty} P_n = 1$ , we find that:

$$P_0 = \left[ 1 + \sum_{i=1}^T \frac{b_i \alpha_i (1 - \alpha_i^N)}{1 - \alpha_i} + \frac{d_i \beta_i^{N+1}}{1 - \beta_i} \right]^{-1}.$$

#### MEAN QUEUE LENGTH

Let  $L_2$  denote the mean queue length:

$$L_2 = \sum_{n=0}^{\infty} n P_n = \sum_{n=1}^N n P_n + \sum_{n=N+1}^{\infty} n P_n.$$

Substituting the value of  $P_n$  already obtained we have

$$L_2 = \sum_{i=1}^T \left[ \frac{b_i \alpha_i \{ 1 - (N+1) \alpha_i^N + N \alpha_i^{N+1} \}}{(1 - \alpha_i)^2} + \frac{d_i \beta_i^{N+1} (N+1 - N \beta_i)}{(1 - \beta_i)^2} \right] P_0.$$

#### SINGLE SERVER SYSTEM

When additional server does not operate  $\alpha_i$  must lie in a unit circle, for which  $\rho < 1/T$ . The mean queue length  $L_1$  for the system can be obtained by

making  $N \rightarrow \infty$  in the expression for  $L_2$ :

$$L_1 = \sum_{i=1}^T \frac{b_i \alpha_i}{(1-\alpha_i)^2} Q_0$$

where:

$$Q_0 = \left[ 1 + \sum_{i=1}^T \frac{b_i \alpha_i}{1-\alpha_i} \right]^{-1}$$

**PARTICULAR CASE**

When  $T=1$ ,  $C_1=1$ ,  $C_i=0 (2 \leq i \leq T)$ , i. e. when the arrival occur singly the situation corresponds to Queues Dependent Servers by V. P. Singh [6].

In this case we have:

$$\alpha_1 = \rho, \quad a_1 = 1, \quad b_1 = 1, \quad \beta_1 = \frac{\rho}{2}, \quad d_1 = 2^N$$

and:

$$P_n = \rho^n P_0; \quad 0 \leq n \leq N = \frac{\rho^n}{2^{n-N}} P_0; \quad N < n.$$

where:

$$P_0 = \frac{(1-\rho)(2-\rho)}{2-\rho-\rho^{N+1}}$$

and:

$$L_2 = \frac{\rho(2-\rho)}{(1-\rho)(2-\rho-\rho^{N+1})} - \frac{(2-\rho)(N-\rho N+1) \rho^{N+1}}{(1-\rho)(2-\rho-\rho^{N+1})} + \frac{(1-\rho)(2N-\rho N+2) \rho^{N+1}}{(2-\rho)(2-\rho-\rho^{N+1})}$$

These results are in confirmity with that of V. P. Singh [6].

**PROFITABILITY CRITERION**

Let  $R_1$  be the unit profit associated with  $L_1-L_2$  and  $R_2$  is the cost to the system of providing the additional server. It is profitable for the system to

have additional server only if:

$$R_1(L_1 - L_2) > R_2(\text{Prob. } n > N),$$

$$\text{Prob. } (n > N) = \sum_{n=N+1}^{\infty} P_n = \sum_{i=1}^T \frac{d_i \beta_i^{N+1}}{1 - \beta_i} P_0,$$

$$L_1 - L_2 = \sum_{i=1}^T \frac{b_i \alpha_i}{(1 - \alpha_i)^2} (Q_0 - P_0)$$

$$+ \sum_{i=1}^T \left[ \frac{b_i \alpha_i^{N+1} \{ (N+1) - N \alpha_i \}}{(1 - \alpha_i)^2} - \frac{d_i \beta_i^{N+1} (N+1 - N \beta_i)}{(1 - \beta_i)^2} \right] P_0.$$

$$\frac{R_2}{R_1} < \left[ \frac{\left\{ \sum_{i=1}^T \frac{b_i \alpha_i}{(1 - \alpha_i)^2} \right\} \left\{ \sum_{j=1}^T (d_j \beta_j^{N+1} / 1 - \beta_j) - (b_j \alpha_j^{N+1} / 1 - \alpha_j) \right\}}{1 + \sum_{j=1}^T b_j \alpha_j / 1 - \alpha_j} + \sum_{i=1}^T \frac{b_i \alpha_i^{N+1} \{ (N+1) - N \alpha_i \}}{(1 - \alpha_i)^2} - \frac{d_i \beta_i^{N+1} \{ (N+1) - N \beta_i \}}{(1 - \beta_i)^2} \right] \times \left[ \sum_{i=1}^T \frac{d_i \beta_i^{N+1}}{1 - \beta_i} \right]^{-1}.$$

It may be noted that the above relation depends only on  $T, N, \rho, R_2/R_1$  and  $C_i (1 \leq i \leq T)$ . Thus in terms of any four value fifth can be calculated.

For  $T=5$ :

$$C_{i+1} = \binom{4}{i} q^{4-i} p^i; \quad i = 0, 1, 2, 3, 4$$

where:

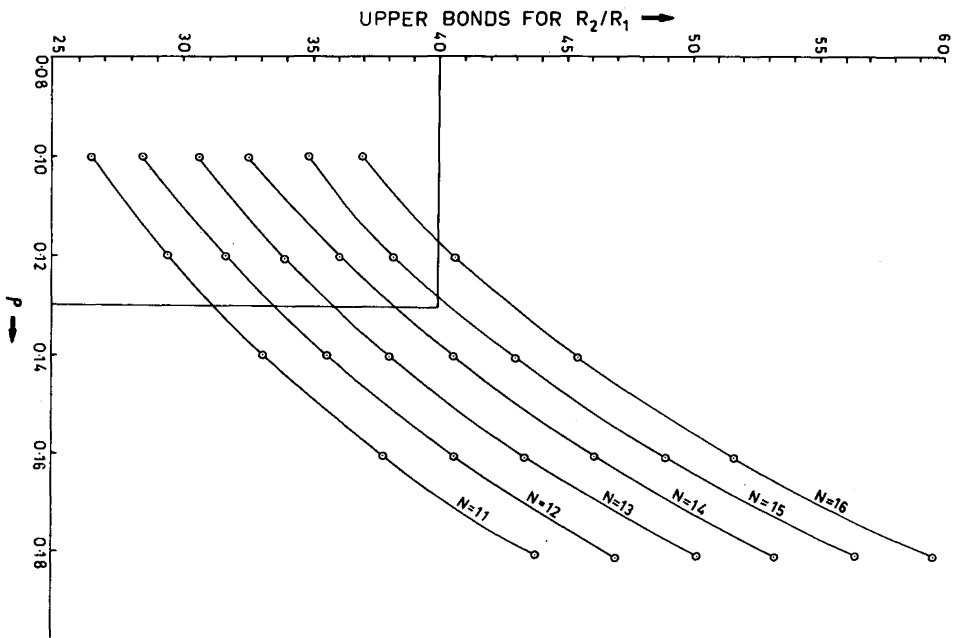
$$p + q = 1, \quad p = \frac{1}{2}, \quad q = \frac{1}{2}.$$

The upper bounds of  $R_2/R_1$  for various values of  $N$  and  $\rho$  are given in the Table.

TABLE  
 Values of the upper bounds of  $R_2/R_1$  for various  $\rho$  and  $N$

$\rho$ . . . . . $N$	.10	.12	.14	.16	.18
11. . . . .	26.45	29.47	33.17	37.86	43.94
12. . . . .	28.47	31.69	35.62	40.64	47.1
13. . . . .	30.61	33.88	38.09	43.42	50.29
14. . . . .	32.45	36.11	40.61	46.22	53.42
15. . . . .	34.84	38.22	43.00	48.99	56.59
16. . . . .	37.06	40.65	45.49	51.70	59.75

From the tabular values the graphs for the upper bounds of  $R_2/R_1$  against  $\rho$ , for various values of  $N$ , are plotted.



From the graphs the decision point for  $N$  at which the application of second server is profitable can be calculated for the given values of  $R_2/R_1$  and  $\rho$ , for example, if  $\rho = .13$ ,  $R_2/R_1 = 40$ ; the point  $(.13, 40)$  lies between the graphs for  $N = 14$  and  $N = 15$ . Therefore, in this case, it will be profitable for the system to provide the additional server when  $N \geq 15$ .



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