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THE HOTEL OVERBOOKING PROBLEM (*)

by Gary GOTTLIEB (1) and Uri YECHIALI (2)

Abstract. — *M hotel rooms are available T days from now. Typically, a policy of overbooking is exercised. Customers may cancel their confirmed reservations and management may choose not to accept new requests for reservations. In addition, management may reserve (sell) rooms through an agent at some cost (discount) or cancel previously accepted reservations at some cost to the hotel. Each occupied room T days hence brings some revenue and the management objective is to maximize that revenue minus incurred cost (discounting and management cancelling cost).*

We study a continuous control model of the problem and show that a 4-region policy is optimal. For each point in time there exists a lower, an intermediate and an upper bound so that if the "inventory" level of confirmed reservations is below the lower bound, management should sell rooms to get the level of reservations up to the lower bound, and if the inventory level is above the upper bound, they should cancel enough reservations to reach the upper bound. Finally, they should accept new reservations if and only if the inventory level is below the intermediate bound.

Keywords: Inventory Control; Continuous Review; Control Limits.

Résumé. — *M chambres d'hotel sont disponibles T jours à partir de maintenant. Typiquement, on exerce une politique de « overbooking ». Les clients peuvent annuler leurs réservations et la direction peut choisir de ne pas accepter de nouvelles réservations. La direction peut en outre réserver (vendre) des chambres à un prix donné (escompte) à travers une agence, ou bien annuler à propre dépense des réservations préalablement acceptées. Chaque chambre occupée à partir de ce moment pendant T jours représente un revenu spécifique et l'objectif de la direction est de maximiser ce revenu moins la dépense rencontrée (dépense d'escompte et dépense d'annulation).*

Nous étudions un modèle de contrôle continu du problème et nous démontrons qu'une politique de 4-régions est optimale. Pour chaque point dans le temps il existe une limite inférieure, intermédiaire et supérieure, de façon que si le niveau « inventaire » de réservations confirmées est en dessous de la limite inférieure, la direction devrait vendre les chambres de façon à obtenir un niveau de réservations jusqu'à la limite inférieure; et si le niveau inventaire est en dessus de la limite supérieure, elle devrait annuler suffisement de réservations jusqu'à obtenir la limite supérieure. Elle devrait enfin accepter de nouvelles réservations si et seulement si le niveau inventaire est en dessous de la limite intermédiaire.

INTRODUCTION

In this paper we consider the problem of a hotel reservation manager who wishes to achieve as nearly as possible full occupancy at a given specified date T days in the future (e. g., New Year's Eve) with a minimum of hotel cancellations of reservations and with as few as possible rooms "rented" through an agent at discounted prices. We consider a continuous-time model where customer arrival

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and self-cancellation are continuous-time processes and where management has the option at any time to accept or not new reservations, to cancel confirmed reservations or to "buy" new reservations at some cost.

The model we study is related to that of Liberman and Yechiali [1978] where the overbooking problem is studied in a discrete framework. Other earlier papers include Ladany [1976] and Rothstein [1974]. A description of these papers can be found in Liberman and Yechiali [1978].

In our model we allow both continuous and impulse controls where the continuous control relates to accepting or not new requests for reservations and the impulse control relates to acquiring or cancelling reservations. To the best of our knowledge, the simultaneous consideration of both types of control is new.

We show that for any time t, $T-t$ days before the target day T, where there are $X(t)$ confirmed reservations in hand, there exist three numbers $0 \leqq n_1(t) \leqq n_2(t) \leqq n_3(t) \leqq \infty$ so that if $X(t) > n_3(t)$, no new reservations should be accepted and $-n_3(t)+X(t)$ reservations should be cancelled. If $n_2(t) \leqq X(t) \leqq n_3(t)$, no new reservations should be accepted. If $n_1(t) \leqq X(t) < n_2(t)$, any new request for a reservation should be accepted and if $X(t) < n_1(t)$, $n_1(t)-X(t)$ reservations should be bought.

In section 1, the model is presented. In section 2, it is shown that the "value" of having l reservations on hand at time t is a concave sequence in l, from which the optimality of the four-region control policy is shown to follow. We also show that for some directly determined intervals of time no buying or selling should be done, regardless of the inventory level.

1. THE MODEL

Consider a target day T days hence with $X(t)$ the number of confirmed reservations at time $t \in [0, T]$. If there are l reservations on hand at time t, at which time we buy (cancel) q reservations, we say that $X(t)=l$ and $X(t+)=l+(-)q$. If the final state of the process is $X(T+)=j$, the reward (income) received is $f(j)$, where f attains its maximum at $j=M$, M being the number of rooms available. We only assume that f is concave. Note that it would be natural to assume that f is of the following form:

$$(1) \qquad \begin{cases} f(j)=p_1 j & \text{for } j<M, \qquad p_1>0, \\ f(j)=p_1 M-(j-M)\,p_2 & \text{for } j \geqq M, \qquad p_2>0. \end{cases}$$

p_1 = revenue per room;

p_2 = cost of last-minute management cancellation.

While we don't make such a specific assumption about f, the assumption of concavity is motivated by the above example.

Requests for new reservations arrive as a non-homogeneous Poisson Process with rate $\lambda(t)$. Each customer holding a confirmed reservation at time t, acting independently of the others, will cancel his reservation on $[t, t+\Delta t]$ with probability $\int_{t}^{t+\Delta t} \mu(s)\,ds + o(\Delta t)$.

At each time $t \varepsilon [0, T]$, the manager may buy $m \geqq 0$ reservations at a cost $g(t).m$, may cancel $r \geqq 0$ reservations at a cost $h(t).r$, and may accept or not any new requests in a deterministic or probabilistic manner. We assume that $h(t)$ and $g(t)$ are continuous and strictly positive on $[0, T]$. To avoid ambiguity, we henceforth refer to management cancelling as selling and to customer cancellations as cancellations.

The objective is to characterize the policy which maximizes the expectation of the reward received at time $T+$ minus the cost incurred by the buying and selling during the period $[0, T]$.

We now introduce the required notation. Any policy φ can be represented by the triple $\varphi(t, \omega) = (a(t, \omega), b(t, \omega), \beta(t, \omega))$ where ω is an element of the sample space Ω, and

(i) $a(t, \omega) = m$ means: buy m reservations at time t,

(ii) $b(t, \omega) = r$ means: sell r reservations at time t,

(iii) $\beta(t, \omega) = p$ means: if there is an arrival at time t, accept it with probability p.

We wish to characterize the optimal policy over the class of Markov policies. However, for technical reasons, we will occasionally consider non-Markovian policies.

Let $\tilde{\Pi}$ be the set of all policies satisfying the characterizations (i), (ii) and (iii) given above, with the additional conditions that each policy φ has an associated upper bound L, so that if $X(t) \geqq L$, no new reservations will be accepted or purchased at time t and that $\varphi(t)$ is measurable with respect to the history of the process $X(s)$ up to time t. Let $\Pi \subset \tilde{\Pi}$ be the set of all Markovian policies in $\tilde{\Pi}$.

For a given policy $\varphi \in \tilde{\Pi}$, let $N_1(t)\,(N_2(t))$ be the number of purchases (sales) made on $[0, t)$ and let $N_1(t+)(N_2(t+))$ be the number of purchases (sales) made on $[0, t]$. Let A_i be the time of the i-th purchase, $i \leqq N_1(T+)$ and B_i be the time of the i-th sale, $i \leqq N_2(T+)$ (a purchase (sale) of q reservations is handled as q simultaneous but different purchases (sales)]. Let $D(t)$ be the total number of cancellations on $[0, t]$ and let $Z(t)$ be the total number of reservations accepted on $[0, t]$.

Then:

$$X(t)=X(0)+Z(t)-D(t)+N_1(t)-N_2(t). \tag{2a}$$

$$X(t+)=X(t)+\sum_i 1_{A_i=t}-\sum_i 1_{B_i=t}=X(t)+\sum_i 1_{A_i=t}-\sum_i 1_{B_i=t}. \tag{3b}$$

Note that, as defined, $X(t)$ is neither right nor left continuous, but has both left and right limits. Having adopted a policy φ, if a reservation arrives at time t, it will e accepted with probability $\beta(t)$, and if accepted, we have:

$$X(t)=1+X(t-) \qquad \text{where} \quad X(t-)=\lim_{s\,t} X(s).$$

It $a(t)>0(b(t)>0)$, then we buy $a(t)$ (sell $b(t)$) reservations at time t and:

$$X(t+)=X(t)+a(t)-b(t).$$

Defining $C(t)$ to be the reward (income) received at time $T+$ minus the cost incurred on $[t, T]$, we have:

$$C(t)=-\sum_{A_i\geqq t} g(A_i)-\sum_{B_i\geqq t} h(B_i)+f(X(T+)). \tag{3}$$

Associated with each policy φ is an expectation operator E_φ. We set

$$v_\varphi(t, l)=E_\varphi[C(t)\,|\,X(t)=l],$$

and:

$$v(t, l)=\sup_{\varphi\in\Pi} v_\varphi(t, l).$$

We point out that $v(t, l)$ satisfies the following dynamic programming characterization:

$$v(t, l)\geqq \max_{\substack{i\geqq 0,\, k\geqq 0\\ i+k>0}} \{v(t, l+i-k)-ig(t)-kh(t)\}\equiv A(t, l) \tag{4a}$$

$$\frac{\partial v(t, l)}{\partial t}\geqq -\sup_{0\leqq\beta\leqq 1}\{l\mu(t)[v(t, l-1)-v(t, l)] + \beta\lambda(t)[v(t, l+1)-v(t, l)]\}\equiv B(t, l) \tag{4b}$$

$$(v(t, l)-A(t, l))\left(\frac{\partial v(t, l)}{\partial t}-B(t, l)\right)=0, \tag{4c}$$

$$v(T, l)=\max_{i\geqq 0,\, k\geqq 0} \{f(T+i-k)-ig(t)-kh(t)\}. \tag{4d}$$

Letting $W=\{(t, l) \mid v(t, l)>A(t, l)\}$, we note that for $(t, l) \in W$, no buying or selling should be done and that for $(t, l) \notin W$, we should buy $i^*(t, l)$ and sell $k^*(t, l)$ reservations where:

$$\max_{\substack{i \geqq 0, k \geqq 0, \\ i+k>0}} \{v(t, l+i-k)-ig(t)-kh(t)\},$$

is obtained at $i^*=i^*(t, l)$, $k=k^*(t, l)$.

Note that the optimality of the four-region control policy implies that:

$$i^*(t, l)=\begin{cases} 0 & \text{if } l \geqq n_1(t), \\ n_1(t)-l & \text{if } l<n_1(t), \end{cases}$$

$$k^*(t, l)=\begin{cases} l-n_2(t) & \text{if } l>n_2(t), \\ 0 & \text{if } l \leqq n_2(t). \end{cases}$$

Letting $\beta^*(t, l)$ be the optimal β control at t if $X(t)=l$, the four region control policy implies that:

$$\beta^*(t, l)=\begin{cases} 1 & \text{if } l \geqq n_2(t), \\ 0 & \text{if } l<n_2(t). \end{cases}$$

so, we will show that the optimal policy for each control i^*, k^* and β^* is a control limit policy.

In order to show the optimality of the four-region control limit policy we must first show that $v(t, l)$ is concave in $l \in N^+$ for all $t \in [0, T]$. This is done in the next section initially under the assumption that $\lambda(t)=\lambda>0$, $\mu(t)=\mu>0$, all $t \in [0, T]$. The methodology is to first consider those policies which only change at points of time on a lattice. The concavity of $v(t, l)$ will be shown to follow by a limit argument as the lattice is made finer. Then, loosening the conditions on $\lambda(t)$ and $\mu(t)$, one gets the same result on $v(t, l)$ by a minor modification of the arguments.

2. DERIVATION OF THE OPTIMAL POLICY

The major work of this section is proving the preliminary result that $v(t, l)$ is concave in l. Assume until otherwise stated that $\lambda(t)=\lambda>0$, $\mu(t)=\mu>0$, all $t \in [0, T]$.

Fix an integer $j>0$ and define $\Delta=T/2^j$ and $\Delta_k=T-k\Delta$, for $0\leqq k\leqq 2^j$ (note that the dependence of Δ upon j is not explicitly expressed by the notation). Let $L_j=\{\Delta_k\}_{k=0}^{2^j}$. Let $\Pi_j\subset\tilde{\Pi}$ with $\varphi\in\Pi_j$ if:

(i) $\varphi\in\tilde{\Pi}$

(ii) $a(t, \omega)=0$, all $t\notin L_j$, all ω.

$b(t, \omega)=0$, all $t\notin L_j$, all ω.

(iii) $\beta(t, \omega)=\beta(\Delta_k, \omega)$, all $t\in[\Delta_k, \Delta_{k-1})$, all ω.

In other words, $\varphi\in\Pi_j$ corresponds to a policy which is only reviewed on a lattice with all purchases and sales being done on that lattice. Let $\Pi_{j,t}=\Pi_j\cap\{a(t)=0, b(t)=0\}$.

Define $v_j(t, l)=\sup_{\varphi\in\Pi_j} v_\varphi(t, l)$, and $v_j(t+, l)=\sup_{\varphi\in\Pi_{j,t}} v_\varphi(t, l)$.

THEOREM 1: *$\{v_j(\Delta_k, l)\}_{l=0}^{\infty}$ is a concave sequence in $l\in N^+$ for each $j\geqq 1$ and k satisfying $0\leqq k\leqq 2^j$.*

Proof: We will first show that $v_j(\Delta_1, l)$ is concave in $l\in N^+$. First note that $v_j(\Delta_0, l)$ is concave as $f(l)$ is concave. Given a choice of β, and given that $X(\Delta_1^+)=n$, it follows from Kleinrock [1975], p. 82, that $X(T)$ is equal in distribution to the sum of two independent random variables, U_1 and U_2, where U_1 has a Poisson distribution with parameter ρ:

$$\rho=\beta(\lambda/\mu)(1-e^{-\Delta\mu}) \tag{5}$$

and U_2 has a Binomial distribution with parameters n and $p=e^{-\Delta\mu}$.

For convenience, let $y(l)=v_j(\Delta_0, l)$. Hence:

$$v_j(\Delta_1^+, l)=\sup_{0\leq\beta\leq 1} E_{\rho,l}[y(U_1+U_2)], \tag{6}$$

where ρ satisfies (5) and $E_{\rho,l}$ is the expectation operator with respect to U_1 and U_2 having the corresponding parameters ρ, l and p.

Extend y to a concave, right-differentiable function $\bar{y}$ on R^+ with $\bar{y}(l)=y(l)$ for $l\in N^+$.

Extend l on the right-hand side of equation (6) to R^+ with the interpretation that U_2 is now the sum of two independent random variables, one binomial with parameters $[l]$ and p, where $[l]$ is the integer part of l, and the other taking values $l-[l]$ and 0, with respective probabilities p and $1-p$.

Define $\psi(\rho, l)=E_{\rho,l}[\bar{y}(U_1+U_2)]$ and let $\rho(l)$ be the smallest value of ρ $(0\leqq\rho\leqq(\lambda/\mu)(1-e^{-\Delta\mu}))$ for which $\psi(\rho, l)$ attains its maximum for a given l. Let $\varphi(l)=\psi(\rho(l), l)=\sup_{0\leq\beta\leq 1} E_{\rho,l}[\bar{y}(U_1+U_2)]$. We will show that $\varphi(l)$ is concave in

$l \in R^+$ which implies that $v_j(\Delta_1^+, l)$ is concave for $l \in N^+$, implying the concavity of $v_j(\Delta_1, l)$ for $l \in N^+$.

Assume that $\rho(l)$ is right differentiable. If not, the below arguments still hold, though they are more involved. Take all derivatives in the following to be right-derivatives.

A probabilistic argument or direct differentiation shows that:

$$\frac{\partial \psi}{\partial \rho} = E_{\rho, l}[\bar{y}(U_1+U_2+1)-\bar{y}(U_1+U_2)],$$

$$\frac{\partial^2 \psi}{\partial \rho^2} = E_{\rho, l}[\bar{y}(U_1+U_2+2)-2\bar{y}(U_1+U_2+1)+\bar{y}(U_1+U_2)].$$

As $\bar{y}(x+2)-2\bar{y}(x+1)+\bar{y}(x) \leqq 0$, all x, $\dfrac{\partial^2 \psi}{\partial \rho^2} \leqq 0$.

Similarly:

$$\frac{\partial \psi}{\partial l} = e^{-\mu\Delta} E_{\rho, l}\left[\frac{\partial}{\partial l}\bar{y}(U_1+B_l+l-[l])\right] \quad \text{where } B_l \text{ is a binomial}$$

random variable with parameters $[l]$ and p:

$$\frac{\partial^2 \psi}{\partial l^2} = e^{-\mu\Delta} E_{\rho, l}\left[\frac{\partial^2}{\partial l^2}\bar{y}(U_1+B_l+l-[l])\right] \leqq 0.$$

Finally:

$$\frac{\partial^2 \varphi}{\partial l^2} = \frac{\partial^2 \psi}{\partial l^2} + \frac{\partial^2 \psi}{\partial p^2}\left(\frac{\partial \rho(l)}{\partial l}\right)^2 + \frac{\partial \psi}{\partial \rho}\,\frac{\partial^2 \rho(l)}{\partial l^2}.$$

Now, either $\rho(l)$ is on the interior of the set $[0, (\lambda/\mu)(1-e^{-\mu\Delta})]$, in which case $\partial\psi/\partial\rho = 0$, or if not, $\partial^2\rho(l)/\partial l^2 = 0$. In either case, $\partial^2\varphi/\partial l^2 \leqq 0$, showing the concavity of φ.

Hence, $v_j(\Delta_1^+, l)$ is concave. So $v_j(\Delta_1, l)$ is concave. Assume that $v_j(\Delta_i, l)$ is concave for $i \leqq k < 2^j$. Note that $v_j(\Delta_{k+1}^+, l)$ has exactly the same relation to $v_j(\Delta_k, l)$ as does $v_j(\Delta_1^+, l)$ to $v_j(\Delta_0, l)$. As $v_j(\Delta_k, l)$ is concave by the induction hypothesis, $v_j(\Delta_{k+1}^+, l)$ is concave. Hence, $v_j(\Delta_{k+1}, l)$ is concave, proving the theorem.

THEOREM 2: *For all* $t \in \bigcup_{j=1}^{\infty} L_j$, $l \in N^+$, $\lim_{j\to\infty} v_j(t, l) = v(t, l)$.

Proof: Clearly, $v_j(t, l) \leqq v(t, l)$ and, as Π_j is an increasing set of policies,

$$\lim_{j \to \infty} v_j(t, l) \leqq v(t, l). \tag{7}$$

To prove the reverse inequality, we choose a $l \in N^+$, an arbitrary policy $\varphi \in \Pi$ with an associated upper limit $L \geqq l$, and with $v_\varphi(0, l)$ finite.

For a fixed $j > 0$, we will construct a policy $\alpha \in \Pi_j$ which "resembles" φ such that:

$$v_\alpha(0, l) - v_\varphi(0, l) \geqq o(1) \quad \text{as } j \to \infty. \tag{8}$$

We define α as follows:

Assuming no cancellations or reservation requests on $[\Delta_k, \Delta_{k-1})$ and given the policy $\varphi \in \Pi$, and the value of $X(\Delta_k^+)$, X will have a deterministic sample path on $[\Delta_k, \Delta_{k-1})$. Let $X_k(t, \omega)$, $t \in [\Delta_k, \Delta_{k-1})$, be that sample path.

Let $\hat{\beta}(\Delta_k, \omega) = \frac{2^j}{T} \int_{\Delta_k}^{\Delta_{k-1}} \beta(s, X_k(s, \omega))\, ds.$

Note that $\hat{\beta}(\Delta_k, \omega)$ depends only on $X(\Delta_k^+, \omega)$ and φ.

Define a sequence of random time-transformations:

ρ_k: $[\Delta_k, \Delta_{k-1}] \to [\Delta_k, \Delta_{k-1}]$ as follows:

$$\rho_k(t, \omega) = \begin{cases} \Delta_k + \dfrac{\int_{\Delta_k}^{t} \beta(s, X_k(s, \omega))\, ds}{\hat{\beta}(\Delta_k, X(\Delta_k^+, \omega))}, & \text{if } \hat{\beta}(\Delta_k, X(\Delta_k^+, \omega)) > 0. \\ t, \quad \text{if } \hat{\beta}(\Delta_k, X(\Delta_k^+, \omega)) = 0. \end{cases}$$

Let:

$$\hat{a}(\Delta_{k-1}, \omega) = \sum_{t \in (\Delta_k, \Delta_{k-1}]} a\{t, X_k(t, \omega) - [D(t, \omega) - D(\Delta_k^+, \omega)] + [Z(\rho_k(t, \omega), \omega) - Z(\Delta_k^+, \omega)]\}.$$

$$\hat{b}(\Delta_{k-1}, \omega) = \sum_{t \in (\Delta_k, \Delta_{k-1}]} b\{t, X_k(t, \omega) - [D(t, \omega) - D(\Delta_k^+, \omega)] + [Z(\rho_k(t, \omega), \omega) - Z(\Delta_k^+, \omega)]\}.$$

In a sense that we will soon specify, the policy $(\hat{a}, \hat{b}, \hat{\beta}) \in \Pi_j$ "resembles" φ except that, if there is more than one reservation request or cancellation in an interval, this "resemblance" no longer holds. So, we must first go through some further technical details. We explicitly construct the cancellation times. For

$n=1, 2, \ldots, L$ let $\{T_{n,j}\}_{j=1}^{\infty}$ be i. i. d. Poisson renewal sequences with rate μ. Let $T_{n,j}$ be the time of a real cancellation if $X(T_{n,j}^-) \geqq n$. Otherwise, refer to $T_{n,j}$ as the time of an imaginary cancellation.

Define:

$\eta = \max\{2^j \geqq k > 0$: there is more than one event on $[\Delta_k, \Delta_{k-1}]$ where an event is a reservation request or a cancellation (real or imaginary)$\}$,

where $\max \emptyset = 0$. Note that η does not depend upon the policy used. We now complete the construction of α.

We let:

$$\tilde{a}(\Delta_k, \omega) = \hat{a}(\Delta_k, \omega),\ k \geqq \eta,$$
$$\tilde{b}(\Delta_k, \omega) = \hat{b}(\Delta_k, \omega),\ k \geqq \eta,$$
$$\tilde{\beta}(\Delta_k, \omega) = \hat{\beta}(\Delta_k, \omega),\ k \geqq \eta$$

and:

$$\tilde{a}(\Delta_k, \omega) = 0,\ k < \eta,$$
$$\tilde{b}(\Delta_k, \omega) = 0,\ k < \eta,$$
$$\tilde{\beta}(\Delta_k, \omega) = 0,\ k < \eta.$$

Finally, let $\alpha \in \Pi_j$ with associated triple $(\tilde{a}, \tilde{b}, \tilde{\beta})$. The key observation is that the joint distribution of the vectors:

$$\{(X(0+), X(\Delta+), X(2\Delta+), \ldots, X(T+)), (N_1(0+), N_1(\Delta+), \ldots, N_1(T+)), (N_2(0+), N_2(\Delta+), \ldots, N_2(T+))\}$$

conditioned on $\{\eta = 0\}$, is identical under policies φ and α.

Using this observation we have:

$$(9)\quad v_\alpha(0, l) - v_\varphi(0, l) = E_\alpha[C(0) \mid X(0) = l] - E_\varphi[C(0) \mid X(0) = l]$$
$$= E_\alpha[C(0)\, 1_{\eta=0} \mid X(0) = l] - E_\varphi[C(0)\, 1_{\eta=0} \mid X(0) = l]$$
$$+ E_\alpha[(C(0) - C(\eta\Delta))\, 1_{\eta>0} \mid X(0) = l] - E_\varphi[(C(0) - C(\eta\Delta))\, 1_{\eta>0} \mid X(0) = l]$$
$$+ E_\alpha[C(\eta\Delta)\, 1_{\eta>0} \mid X(0) = l] - E_\varphi[C(\eta\Delta)\, 1_{\eta>0} \mid X(0) = l].$$

The sum of the first four terms of (9) is bounded below by

$$(10)\qquad E_\varphi(N_1(T+) + N_2(T+)).\, o(1) \qquad (\text{as } j \to \infty),$$

where the $o(1)$ term is due to the uniform continuity of g and h on $[0, T]$. On $\{\eta = 0\}$, the processes evolve similarly in the sense that the number of purchases (sales) on $(\Delta_k, \Delta_{k-1}]$ under policy φ has the same distribution as the number of purchases (sales) at Δ_{k-1} under policy α. In addition, $X(T+)$ has the same distribution under either policy.

Now:

$$E_\alpha[C(\eta\Delta)\,1_{\eta>0}\,|\,X(0)=l] \geqq \min_{0\leqq i\leqq L} f(i)\; P(\eta>0), \tag{11}$$

$$E_\varphi[C(\eta\Delta)\,1_{\eta>0}\,|\,X(0)=l] \leqq f(M)\; P(\eta>0). \tag{12}$$

Further, $P(\eta>0)=o(1)$.

So, combining (10), (11) and (12) gives:

$$v_\alpha(0, l)-v_\varphi(0, l)\geqq o(1) \tag{13}$$

By identical reasoning, for any $t\in \bigcup_{j=1}^{\infty} L_j$.:

$$v_\alpha(t, l)-v_\varphi(t, l)\geqq o(1). \tag{14}$$

The theorem now follows from (7) and (14).

LEMMA 3: *$v(t, l)$ is a continuous function in $t\in[0, T]$ for each $l\in N^+$.*

Proof: We begin by noting a set of inequalities:

(i) $v(t, l)\geqq \min_{i\leqq l} f(i)$;

(ii) $|v(t, l+r)-v(t, l)|\leqq g^* r$ where $g^*=\max_{0\leqq t\leqq T}\{g(t), h(t)\}$.

(iii) $v(t-\Delta t, l)\geqq v(t, l)-l\mu\Delta t g^*+o(\Delta t)$.

To see (i), choose $\gamma\in\Pi$ with $\gamma=(0, 0, 0)$. Trivially, $v_\gamma(t, l)\geqq \min_{i\leqq l} f(i)$. To see (iii), use the above policy γ on the time interval $[t-\Delta t, t]$.

Let:

$$\xi(l)=\frac{f(M)-\min_{i\leqq l} f(i)}{g_*},$$

where $g_*=\min_{0\leqq t\leqq T}\{g(t), h(t)\}$. Recall that $f(M)=\max_{i\geqq 0} f(i)<\infty$. We can see from (i) that if $X(t)=l$, it can never be optimal to buy or sell more than $\xi(l)$ reservations at time t.

Thus:

$$v(t, l) = \max_{0 \leqq r \leqq \xi(l)} \{ v(t, l+r) - rg(t), v(t, l-r) - rh(t) \}. \tag{15}$$

Let T_1 be the time of the first reservation request or cancellation (real or imaginary) after $t-\Delta t$:

$$\begin{aligned} v(t-\Delta t, l) = \sup_{\varphi \in \Pi} \{ & E_\varphi [C(t-\Delta t); T_1 \leqq t \mid X(t) = l] \\ & + E_\varphi [C(t-\Delta t); T_1 > t \mid X(t) = l] \} \leqq f(M) P(T_1 \leqq t) \\ & + P(T_1 > t) \max_{0 \leqq r \leqq \xi(l)} \{ v(t, l+r) - rg(t-\Delta t, t), \\ & v(t, l+r) + rh(t-\Delta t, t) \}. \end{aligned} \tag{16}$$

where:

$$g(t-\Delta t, t) = \min_{s \in [t-\Delta t, t]} g(s),$$

$$h(t-\Delta t, t) = \min_{s \in [t-\Delta t, t]} h(s).$$

Now, $P(T_1 \leqq t) = O(\Delta t)$.

Combining (15) and (16) and using the continuity of g and h leads to:

$$v(t-\Delta t, l) - v(t, l) \leqq f(M) O(\Delta t) + \xi(l) o(1) = o(1) \quad (\text{as } \Delta t \to 0). \tag{17}$$

Finally, equation (17) and inequality (iii) prove the lemma.

THEOREM 4: *$v(t, l)$ is concave in $l \in N^+$ for each $t \in [0, T]$.*

Proof: The theorem is an immediate consequence of Theorems 1 and 2, Lemma 3 and the fact that the limit of concave functions is concave.

We now drop the assumption that $\lambda(t)$ and $\mu(t)$ are constant and make instead the weaker assumption that each function is piecewise constant with points of jumps all on L_J, some $J > 0$. We further assume that $\lambda(t)$ is bounded above by $\lambda < \infty$ and that $\mu(t)$ is bounded above by $\mu < \infty$.

THEOREM 5: *Theorem 4 holds under the new assumptions on* $\lambda(t)$ and $\mu(t)$.

Proof : For $j \geqq J$, Theorem 1 holds under the new assumptions with only a trivial modification of the proof. The proofs of Theorems 2 and 4, and of Lemma 3 are identical under the new assumptions on $\lambda(t)$ and $\mu(t)$.

THEOREM 6: *There exist three functions* $\{n_1(t), n_2(t), n_3(t), 0 \leqq t \leqq T\}$, *each integer valued so that at time* t, *given that* $X(t)=l$, *the optimal policy is as follows:*

(1) *If* $l>n_3(t)$, *set* $\beta=0$ *and sell* $l-n_3(t)$ *reservations.*

(2) *If* $n_2(t)\leqq l\leqq n_3(t)$, *set* $\beta=0$.

(3) *If* $n_1(t)\leqq l<n_2(t)$, *set* $\beta=1$.

(4) If $l<n_1(t)$, *set* $\beta=1$ *and buy* $n_1(t)-l$ *reservations.*

Proof: From Lemma 5.1 of Yushkevich [1977], if $X(t)=l$, β should be choosen to maximize:

$$\lambda(t)\beta v(t, l+1)-[\lambda(t)\beta+\mu(t)]v(t, l)+\mu(t)\, v(t, l-1). \tag{18}$$

Setting $n_2(t)$ to be the smallest value of l for which $v(t, l)$ attains its maximum, the result about β follows directly from the concavity in l of $v(t, l)$.

As for the impulse control, we have from a modification of Theorem 2.2 of Robin [1976] that we should do no buying or selling if:

$$v(t, l)>\max\{\max_{i>0}[v(t, l+i)-ig(t)], \max_{k>0}[v(t, l-k)-kh(t)]\}. \tag{19}$$

If (19) does not hold, we should choose an i (or k) which maximizes the right-hand-side of (19) and then buy i (or sell k) reservations. So, set:

$$n_3(t)=\sup_{k>0}\{k: v(t, k)>v(t, k-1)-h(t)\}$$

and:

$$n_1(t)=\inf_{i\geq 0}\{i: v(t, i)>v(t, i+1)-g(t)\},$$

where $\sup \emptyset=0$ and $\inf \emptyset=\infty$.

The result about buying and selling then follows from the concavity of $v(t, l)$.

THEOREM 7: (i) *Let* $B=\left\{s:\ s\in[0, T) \text{ where } \exists\ t\in(s, T] \text{ with } g(s)>g(t).\exp\left[-\int_s^t \mu(z)\,dz\right]\right\}$. *Then, if* $s\in B$, $n_1(s)=0$, (*i.e., never buy for* $s\in B$).

(ii) *Let*

$$S=\left\{s:\ s\in[0, T) \text{ where } t\in(s, T] \text{ with } h(s)>h(t).\exp\left[-\int_s^t \mu(z)\,dz\right]\right\}.$$

Then, if $s\in S$, $n_3(s)=\infty$, (*i.e., never sell for* $s\in S$).

Proof: We only outline the proof of (ii). A particular reservation can be cancelled by management at time s at cost $h(s)$. Alternatively, management can wait until time t and then cancel that reservation unless it has already cancelled itself. The expected cost of the later course of action is $h(t)\exp\left[-\int_s^t \mu(z)\,dz\right]$. Assertion (ii) follows from this observation.

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