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AVAILABILITY AND RELIABILITY OF A 2-UNIT 2-SERVER SYSTEM SUBJECT TO PREVENTIVE MAINTENANCE AND REPAIR (*)

M. N. GOPALAN and P. VENKATACHALAM [1]

Abstract. — *This paper deals with the availability and reliability analysis of a 2-unit system with two service facilities-one for preventive maintenance and the other for repair. The times to failure and preventive maintenance of a unit are assumed to be arbitrarily distributed while the times for repair and preventive maintenance are exponentially distributed. The Laplace transforms of the mean down-time of the system during (0, t) and the mean time to system failure are obtained explicitly. An explicit expression for the steady-state availability of system is also obtained. The case when the distribution of the time to failure is erlangian and that of the time to preventive maintenance is exponential is discussed.*

INTRODUCTION

Gopalan and D'Souza [2] have recently developed the availability and reliability analysis of a 1-server 2-unit system subject to preventive maintenance and repair. The present paper deals with the availability and reliability analysis of a 2-unit system with two service facilities, one for preventive maintenance and the other for repair.

Initially, one unit is switched on and the other is kept as cold standby. When the operating unit is taken up either for preventive maintenance or for repair, the standby is switched on instantaneously. The times to failure and preventive maintenance of a unit are assumed to be arbitraily distributed while the times for repair and preventive maintenance are exponentially distributed. System fails when there is no unit available for operation and starts functioning as soon as a unit is available for operation. The system thus undergoes series of operating and nonoperating periods known as the up and the down periods. Characterising the system by the probability of its being in the up or the down state, integral equations are set up for these probabilities corresponding to

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different initial conditions. The Laplace transform technique has been applied to solve these equations. The Laplace transforms of the mean down-time of the system during $(0, t)$ and the mean time to system failure are obtained explicitly. Explicit expression for the steady-state availability of the system is obtained. The case when the distribution of the time to failure is erlangian and that of the time to preventive maintenance is exponential has been studied.

The following notation has been used throughout the paper:

E_0	Initial state of the system.
$f(.)$	pdf of the time to failure of a unit.
$h(.)$	pdf of the time to preventive maintenance of a unit.
$F(.)/H(.)$	Survivor function corresponding to $f(.)/h(.)$.
$a_1(.)$	$h(.)F(.)$.
$a_2(.)$	$H(.)f(.)$.
$a(.)$	$a_1(.) + a_2(.)$.
μ_1	rate of preventive maintenance of a unit.
μ_2	rate of repair of a unit.
$Z(t)$	dichotomous random variable assuming the value 1 if the system is found in the down-state at the instant t and the value 0 otherwise.
$D(t)$	total down-time of the system in $(0, t)$.
$U(t)$	total up-time of the system in $(0, t)$.
$\pi(i, j, t)$	pr $\{ Z(t) = 1 \mid Z(0) = 0, i$ units are under preventive maintenance and j units are under repair at $t = 0 \}$; $i, j = 0, 1$ and $i \neq j$.
$w(i, j, t)$	pr $\{ Z(t) = 1 \mid Z(0) = 1, i$ units are under preventive maintenance and j units are under repair at $t = 0 \}$; $i, j = 0, 2$; $i \neq j$; $i = j = 1$.
$p(i, j, t)$	pr $\{ Z(t) = 1, Z(u) \neq 1 \ \forall 0 \leqslant u < t \mid Z(0) = 0, i$ units are under preventive maintenance and j units are under repair at $t = 0 \}$; $i, j = 0, 1$ and $i \neq j$.
$a^{*\prime}(s)$	$\frac{d}{ds} a^*(s)$.
$a^{*\prime}(0)$	$\frac{d}{ds} a^*(s)\big\|_{s=0}$

AVAILABILITY ANALYSIS OF THE SYSTEM

We define the function $\rho(.)$ by $\rho(t) = \text{pr}\ \{ Z(t) = 1 \mid E_0 \}$ and obtain an expression for $\rho(.)$. In order that the system is down at the instant t, it is necessary that the unit that was switched on at $t = 0$ must either fail or go for preventive maintenance between 0 and t. Suppose the unit fails or goes for preventive maintenance between $(u, u + du; u < t)$. The epoch u is a regeneration point [1, 3].

$$\rho(t) = \int_0^t \{ a_1(u)\pi(1, 0, t - u) + a_2(u)\pi(0, 1, t - u) \}\, du. \tag{1}$$

We now proceed to write an equation for $\pi(1, 0, t)$. The four possibilities are that the operating unit may either fail or go for preventive maintenance before the service facility completes the maintenance work of the unit or, the service facility may complete the maintenance work of the unit before the operating unit goes for preventive maintenance or fails. We obtain

$$\pi(1, 0, t) = \int_0^t [a_1(u) \{ \exp(-\mu_1 u) w(2, 0, t-u) + (1 - \exp(-\mu_1 u)) \pi(1, 0, t-u) \} + a_2(u) \{ \exp(-\mu_1 u) w(1, 1, t-u) + (1 - \exp(-\mu_1 u)) \pi(0, 1, t-u) \}] \, du. \tag{2}$$

Similarly we obtain

$$\pi(0, 1, t) = \int_0^t [a_1(u) \{ \exp(-\mu_2 u) w(1, 1, t-u) + (1 - \exp(-\mu_2 u)) \pi(1, 0, t-u) \} + a_2(u) \{ \exp(-\mu_2 u) w(0, 2, t-u) + (1 - \exp(-\mu_2 u)) \pi(0, 1, t-u) \}] \, du, \tag{3}$$

$$w(1, 1, t) = \int_0^t \exp(-(\mu_1 + \mu_2) u) \{ \mu_1 \pi(0, 1, t-u) + \mu_2 \pi(1, 0, t-u) \} \, du + \exp(-(\mu_1 + \mu_2) t), \tag{4}$$

$$w(2, 0, t) = \int_0^t \mu_1 \exp(-\mu_1 u) \pi(1, 0, t-u) \, du + \exp(-\mu_1 t), \tag{5}$$

$$w(0, 2, t) = \int_0^t \mu_2 \exp(-\mu_2 u) \pi(0, 1, t-u) \, du + \exp(-\mu_2 t). \tag{6}$$

Taking the Laplace transforms of the equations (1) to (6), we get

$$\rho^*(s) = a_1^*(s) \pi^*(1, 0, s) + a_2^*(s) \pi^*(0, 1, s),$$

$$\pi^*(1, 0, s) \{ 1 - a_1^*(s) + a_1^*(\mu_1 + s) \} = a_1^*(\mu_1 + s) w^*(2, 0, s) + a_2^*(\mu_1 + s) w^*(1, 1, s) + \{ a_2^*(s) - a_2^*(\mu_1 + s) \} \pi^*(0, 1, s),$$

$$\pi^*(0, 1, s) \{ 1 - a_2^*(s) + a_2^*(\mu_2 + s) \} = a_1^*(\mu_2 + s) w^*(1, 1, s) + a_2^*(\mu_2 + s) w^*(0, 2, s) + \{ a_1^*(s) - a_1^*(\mu_2 + s) \} \pi^*(1, 0, s),$$

$$w^*(1, 1, s) = \{ 1 + \mu_1 \pi^*(0, 1, s) + \mu_2 \pi^*(1, 0, s) \} / (\mu_1 + \mu_2 + s),$$

$$w^*(2, 0, s) = \{ 1 + \mu_1 \pi^*(1, 0, s) \} / (\mu_1 + s),$$

$$w^*(0, 2, s) = \{ 1 + \mu_2 \pi^*(0, 1, s) \} / (\mu_2 + s).$$

Simplifying we obtain

$$\rho^*(s) = \{ a_1^*(s)(n_1 m_2 - n_2 m_1) + a_2^*(s)(n_2 l_1 - n_1 l_2) \} / (l_1 m_2 - l_2 m_1)$$

where

$$l_1 = (\mu_1 + s)(\mu_1 + \mu_2 + s)(1 - a_1^*(s)) + s(\mu_1 + \mu_2 + s)a_1^*(\mu_1 + s) - \mu_2(\mu_1 + s)a_2^*(\mu_1 + s),$$

$$l_2 = (\mu_2 + s)\{(\mu_1 + s)a_1^*(\mu_2 + s) - (\mu_1 + \mu_2 + s)a_1^*(s)\},$$

$$m_1 = (\mu_1 + s)\{(\mu_2 + s)a_2^*(\mu_1 + s) - (\mu_1 + \mu_2 + s)a_2^*(s)\},$$

$$m_2 = (\mu_2 + s)(\mu_1 + \mu_2 + s)(1 - a_2^*(s)) + s(\mu_1 + \mu_2 + s)a_2^*(\mu_2 + s) - \mu_1(\mu_2 + s)a_1^*(\mu_2 + s),$$

$$n_1 = (\mu_1 + s)a_2^*(\mu_1 + s) + (\mu_1 + \mu_2 + s)a_1^*(\mu_1 + s),$$

$$n_2 = (\mu_2 + s)a_1^*(\mu_2 + s) + (\mu_1 + \mu_2 + s)a_2^*(\mu_2 + s).$$

When $h(x) = \lambda \exp(-\lambda x)$ and $f(x) = \lambda_1^2 x \exp(-\lambda_1 x)$, we get

$$l_1 = (\mu_1 + s)(\mu_1 + \mu_2 + s)\{1 - \lambda(2\lambda_1 + \lambda + s)/(\lambda_1 + \lambda + s)^2\} + \{s\lambda(\mu_1 + \mu_2 + s)(2\lambda_1 + \lambda + \mu_1 + s) - \mu_2\lambda_1^2(\mu_1 + s)\}/(\lambda_1 + \lambda + \mu_1 + s)^2,$$

$$l_2 = \lambda(\mu_2 + s)\{(\mu_1 + s)(2\lambda_1 + \lambda + \mu_2 + s)/(\lambda_1 + \lambda + \mu_2 + s)^2 - (\mu_1 + \mu_2 + s)(2\lambda_1 + \lambda + s)/(\lambda_1 + \lambda + s)^2\},$$

$$m_1 = \lambda_1^2(\mu_1 + s)\{(\mu_2 + s)/(\lambda_1 + \lambda + \mu_1 + s)^2 - (\mu_1 + \mu_2 + s)/(\lambda_1 + \lambda + s)^2\},$$

$$m_2 = (\mu_2 + s)(\mu_1 + \mu_2 + s)\{1 - \lambda_1^2/(\lambda_1 + \lambda + s)^2\} + \{s\lambda_1^2(\mu_1 + \mu_2 + s) - \mu_1\lambda(2\lambda_1 + \lambda + \mu_2)(\mu_2 + s)\}/(\lambda_1 + \lambda + \mu_2 + s)^2,$$

$$n_1 = \{\lambda_1^2(\mu_1 + s) + \lambda(\mu_1 + \mu_2 + s)(2\lambda_1 + \lambda + \mu_1 + s)\}/(\lambda_1 + \lambda + \mu_1 + s)^2,$$

$$n_2 = \{\lambda(\mu_2 + s)(2\lambda_1 + \lambda + \mu_2 + s) + \lambda_1^2(\mu_1 + \mu_2 + s)\}/(\lambda_1 + \lambda + \mu_2 + s)^2.$$

Since $\rho^*(s)$ is known explicitly, $\rho(t)$ can be obtained after inversion. The mean down-time of the system $\mu_D(t)$ during $(0, t)$ is

$$\mu_D(t) = E[D(t)] = E\left[\int_0^t z(u)\,du\right] = \int_0^t E[z(u)]\,du = \int_0^t \rho(u)\,du$$

so that

$$\mu_D^*(s) = \rho^*(s)/s.$$

Also

$$\mu_U(t) = t - \mu_D(t).$$

STEADY–STATE AVAILABILITY

The availability function of the system $Av(t)$ is given by $Av(t) = \{1-\rho(t)\}$. The steady-state availability of the system is given by

$$\begin{aligned} Av &= \operatorname*{limit}_{t\to\infty} Av(t) \\ &= \operatorname*{limit}_{s\to 0} sAv^*(s) \\ &= 1 - \lim_{s\to 0} s\rho^*(s) \\ Av &= N/D \end{aligned}$$

where

$$N = \{ a_1^*(0) - \mu_1 a_1^*(\mu_2)/(\mu_1 + \mu_2) \} \{ a^{*\prime}(0) + \mu_2 a_2^*(\mu_1)/(\mu_1 + \mu_2)^2 \}$$
$$+ \{ a_2^*(0) - \mu_2 a_2^*(\mu_1)/(\mu_1 + \mu_2) \} \{ a^{*\prime}(0) + \mu_1 a_1^*(\mu_2)/(\mu_1 + \mu_2)^2 \}$$

and

$$\begin{aligned} D = {} & \{ a_1^*(0) - \mu_1 a_1^*(\mu_2)/(\mu_1 + \mu_2) \} \\ & \qquad \{ a^{*\prime}(0) - a_1^*(\mu_1)/\mu_1 - \mu_1 a_2^*(\mu_1)/(\mu_1 + \mu_2)^2 \} \\ & + \{ a_2^*(0) - \mu_2 a_2(\mu_1)/(\mu_1 + \mu_2) \} \\ & \qquad \{ a^{*\prime}(0) - a_2^*(\mu_2)/\mu_2 - \mu_2 a_1^*(\mu_2)/(\mu_1 + \mu_2)^2 \}. \end{aligned}$$

When $h(x) = \lambda \exp(-\lambda x)$ and $f(x) = \lambda_1^2 x \exp(-\lambda_1 x)$, we get

$$\begin{aligned} N = {} & \{ (2\lambda_1 + \lambda)/(\lambda_1 + \lambda)^2 \} \\ & \{ \mu_1\lambda(2\lambda_1 + \lambda + \mu_2)/(\lambda_1 + \lambda + \mu_2)^2 + \mu_2\lambda_1^2/(\lambda_1 + \lambda + \mu_1)^2 - 1 \} \\ & - 2\mu_1\mu_2\lambda\lambda_1^2(2\lambda_1 + \lambda + \mu_2)/(\mu_1 + \mu_2)^3(\lambda_1 + \lambda + \mu_1)^2(\lambda_1 + \lambda + \mu_2)^2 \\ & + \{ \lambda_1^2\lambda/(\mu_1 + \mu_2)^2(\lambda_1 + \lambda)^2 \} \{ \mu_2(2\lambda_1 + \lambda)/(\lambda_1 + \lambda + \mu_1)^2 \} \\ & + \mu_1(2\lambda_1 + \lambda + \mu_2)/(\lambda_1 + \lambda + \mu_2)^2 \} \end{aligned}$$

and

$$\begin{aligned} D = {} & \{ (2\lambda_1 + \lambda)/(\lambda_1 + \lambda)^2 \} \\ & \{ \mu_1\lambda(2\lambda_1 + \lambda + \mu_2)/(\lambda_1 + \lambda + \mu_2)^2 + \mu_2\lambda_1^2/(\lambda_1 + \lambda + \mu_1^2) - 1 \} \\ & + \lambda\lambda_1^2(2\lambda_1 + \lambda + \mu_2)(\mu_1^2 + \mu_2^2)/(\mu_1 + \mu_2)^2(\lambda_1 + \lambda + \mu_1)^2(\lambda_1 + \lambda + \mu_2)^2 \\ & - \{ \lambda_1^2\lambda/(\mu_1 + \mu_2)^2(\lambda_1 + \lambda)^2 \} \{ \mu_1(2\lambda_1 + \lambda)/(\lambda_1 + \lambda + \mu_1)^2 \\ & + \mu_2(2\lambda_1 + \lambda + \mu_2)/(\lambda_1 + \lambda + \mu_2)^2 \} \\ & \qquad + \mu_2(2\lambda_1 + \lambda + \mu_2)/(\lambda_1 + \lambda + \mu_2)^2 \} \\ & - \lambda^2 \{ (2\lambda_1 + \lambda)^2 + \mu_1(2\lambda_1 + \lambda) \}/\mu_1(\lambda_1 + \lambda)^2(\lambda_1 + \lambda + \mu_1)^2 \\ & - \lambda_1^4/\mu_2(\lambda_1 + \lambda)^2(\lambda_1 + \lambda + \mu_2)^2 \\ & + \{ \lambda^2(2\lambda_1 + \lambda + \mu_1)(2\lambda_1 + \lambda + \mu_2) + \lambda_1^4 \} \\ & \qquad /(\mu_1 + \mu_2)(\lambda_1 + \lambda + \mu_1)^2(\lambda_1 + \lambda + \mu_2)^2. \end{aligned}$$

RELIABILITY ANALYSIS OF THE SYSTEM

We introduce the function

$$P(t) = pr \{ Z(t) = 1, Z(u) \neq 1 \ \forall 0 \leq u < t \mid E_0 \}$$

and obtain the following relation.

$$P(t) = \int_0^t [a_1(u)p(1, 0, t - u) + a_2(u)p(0, 1, t - u)]\, du \tag{7}$$

The argument for setting up the above relation is very similar to the one adopted while equation (1) was being set up. We obtain

$$p(1, 0, t) = \int_0^t \{ a_1(u)(1 - \exp(-\mu_1 u))p(1, 0, t - u) + a_2(u)(1 - \exp(-\mu_1 u))p(0, 1, t - u) \}\, du + a(t)\exp(-\mu_1 t) \tag{8}$$

and

$$p(0, 1, t) = \int_0^t \{ a_1(u)(1 - \exp(-\mu_2 u))p(1, 0, t - u) + a_2(u)(1 - \exp(-\mu_2 u))p(0, 1, t - u) \}\, du + a(t)\exp(-\mu_2 t). \tag{9}$$

so that

$$P^*(s) = a_1^*(s)p^*(1, 0, s) + a_2^*(s)p^*(0, 1, s),$$

$$p^*(1, 0, s) \{ 1 - a_1^*(s) + a_1^*(\mu_1 + s) \} = a^*(\mu_1 + s) + \{ a_2^*(s) - a_2^*(\mu_1 + s) \}\, p^*(0, 1, s)$$

and

$$p^*(0, 1, s) \{ 1 - a_2^*(s) + a_2^*(\mu_2 + s) \} = a^*(\mu_2 + s) + \{ a_1^*(s) - a_1^*(\mu_2 + s) \}\, p^*(1, 0, s).$$

Solving these equations, we get

$$P^*(s) = N_1/D_1$$

where

$$N_1 = a_1^*(s)[a_1^*(\mu_1 + s) \{ 1 + a_2^*(\mu_2 + s) \} + a_2^*(\mu_1 + s) \{ 1 - a_1^*(\mu_2 + s) \}] + a_2^*(s)[a_2^*(\mu_2 + s) \{ 1 + a_1^*(\mu_1 + s) \} + a_1^*(\mu_2 + s) \{ 1 - a_2^*(\mu_1 + s) \}]$$

and

$$D_1 = 1 + a_1^*(\mu_1 + s) + a_2^*(\mu_2 + s) + a_1^*(\mu_1 + s)a_2(\mu_2 + s) - a_1^*(\mu_2 + s)a_2^*(\mu_1 + s) - a_1^*(s)\{1 + a_2^*(\mu_2 + s) - a_2^*(\mu_1 + s)\} - a_2^*(s)\{1 + a_1^*(\mu_1 + s) - a_1^*(\mu_2 + s)\}.$$

When $h(x) = \lambda \exp(-\lambda x)$ and $f(x) = \lambda_1^2 x \exp(-\lambda_1 x)$ we get

$$N_1 = [\lambda(2\lambda_1 + \lambda + s)/(\lambda_1 + \lambda + s)^2(\lambda_1 + \lambda + \mu_1 + s)^2] [\lambda(2\lambda_1 + \lambda + \mu_1 + s)\{1 + \lambda_1^2/(\lambda_1 + \lambda + \mu_2 + s)^2\} + \lambda_1^2\{1 - \lambda(2\lambda_1 + \lambda + \mu_2 + s)/(\lambda_1 + \lambda + \mu_2 + s)^2\}] + [\lambda_1^2/(\lambda_1 + \lambda + s)^2(\lambda_1 + \lambda + \mu_2 + s)^2] [\lambda_1^2\{1 + \lambda(2\lambda_1 + \lambda + \mu_1 + s)/(\lambda_1 + \lambda + \mu_1 + s)^2\} + \lambda(2\lambda_1 + \lambda + \mu_2 + s)\{1 - \lambda_1^2/(\lambda_1 + \lambda + \mu_1 + s)^2\}]$$

and

$$D_1 = 1 + [\lambda(2\lambda_1 + \lambda + \mu_1 + s)\{1 + \lambda_1^2/(\lambda_1 + \lambda + \mu_2 + s)^2\} - \lambda\lambda_1^2(2\lambda_1 + \lambda_2 + \mu_2 + s)/(\lambda_1 + \lambda + \mu_2 + s)^2]/(\lambda_1 + \lambda + \mu_1 + s)^2 + \lambda_1^2/(\lambda_1 + \lambda + \mu_2 + s)^2 - \{\lambda(2\lambda_1 + \lambda + s)/(\lambda_1 + \lambda + s)^2\} \{1 + \lambda_1^2(\mu_1 - \mu_2)(2(\lambda_1 + \lambda + s) + \mu_1 + \mu_2)/(\lambda_1 + \lambda + \mu_2 + s)^2(\lambda_1 + \lambda + \mu_2 + s)^2\} - \{\lambda_1^2/(\lambda_1 + \lambda + s)^2\}\{1 + \lambda(2\lambda_1 + \lambda + \mu_1 + s)/(\lambda_1 + \lambda + \mu_1 + s)^2 - \lambda(2\lambda_1 + \lambda + \mu_2 + s)/(\lambda_1 + \lambda + \mu_2 + s)^2\}$$

The function $P(t)$ is obtained after inversion. The reliability function $R(.)$ is given by

$$R(t) = \int_t^\infty P(u)\, du$$

so that

$$R^*(s) = \{1 - P^*(s)\}/s.$$

The mean time T_0 to system failure is given by

$$T_0 = R^*(0) = -\frac{d}{ds} P^*(s)\Big|_{s=0}$$

$$T_0 = N_2/D_2$$

where

$$N_2 = a^{*\prime}(0)\{1 + a_1^*(\mu_1) + a_2^*(\mu_2) + a_1^*(\mu_1)\, a_2^*(\mu_2) - a_1^*(\mu_2)\, a_2^*(\mu_1)\}$$

and

$$D_2 = a^*(\mu_2)\{ a_2^*(\mu_1) - a_2^*(0) \} - a^*(\mu_1)\{ a_1^*(\mu_2) + a_1^*(0) \}$$

When $h(x) = \lambda \exp(-\lambda x)$ and $f(x) = \lambda_1^2 x \exp(-\lambda_1 x)$, we get

$$N_2 = (2\lambda_1 + \lambda)[\{ \lambda_1^2 + (\lambda_1 + \lambda + \mu_2)^2 \} \{ (\lambda_1 + \lambda + \mu_1)^2 + \lambda(2\lambda_1 + \lambda + \mu_1) \} - \lambda\lambda_1^2(2\lambda_1 + \lambda + \mu_2)]$$

and

$$D_2 = \lambda \{ (\lambda_1 + \lambda)^2 + \lambda\mu_1 \} \{ 2\lambda_1 + \lambda)(\lambda_1 + \lambda + \mu_2)^2 + (2\lambda_1 + \lambda + \mu_2)(\lambda_1 + \lambda)^2 \} + \lambda_1^2\mu_1 \{ (\lambda_1 + \lambda)^2 + \lambda\mu_2 \} \{ 2\lambda_1 + 2\lambda + \mu_1 \}.$$

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