

LAJOS TAKÁCS

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A QUEUING MODEL WITH FEEDBACK (*)

by Lajos TAKÁCS ⁽¹⁾

Abstract. — In the time interval $(0, \infty)$ customers arrive at a service system in accordance with a Poisson process and form a queue in a waiting room. The customers are served by a single server in a service room in order of arrival. The service times are mutually independent random variables having a common exponential distribution. After each service a customer may return to the waiting room with a constant probability. Every time the service room becomes empty all the customers in the waiting room and additional r customers ($r \geq 1$) enter the service room. In this paper the distributions of the queue size, the waiting time and the total time spent in the system by a customer are determined for a stationary process.

1. INTRODUCTION

The object of this paper is to find the limit distributions (stationary distributions) of the queue size, the waiting time and the total time spent in the system by a customer for a single-server queue with feedback. It is assumed that in the time interval $(0, \infty)$ customers arrive at a service system in accordance with a Poisson process of density λ and form a queue in a waiting room. The customers are served by a single server in a service room in order of arrival. The service times are mutually independent random variables having the same distribution function

$$H(x) = \begin{cases} 1 - e^{-\mu x} & \text{for } x \geq 0, \\ 0 & \text{for } x < 0, \end{cases} \quad (1)$$

and are independent of the arrival times. After each service a customer may return to the waiting room with probability p where $0 \leq p < 1$ or may depart permanently with probability $q = 1 - p$. Every time the service room becomes empty, all the customers in the waiting room and additional r customers ($r \geq 1$) enter the service room.

Denote by $\xi_1(t)$ the number of customers in the waiting room and by $\xi_2(t)$ the number of customers in the service room at time t . The possible values of $\xi_1(t)$ are $0, 1, 2, \dots$ and the possible values of $\xi_2(t)$ are $1, 2, \dots$. We assume that the arrival times, the service times and the events of returns are independent of $\xi_1(0)$ and $\xi_2(0)$.

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(1) Case Western Reserve University, Cleveland, Department of Mathematics and Statistics.

In the particular case when $p = 0$, that is, when there is no feedback the aforementioned model is equivalent to a counter model introduced by D. G. Lampard [1]. See also R. M. Phatarfod [2], [3], and the author [5].

2. THE LIMIT DISTRIBUTION OF THE QUEUE SIZE

We have the following result.

THEOREM 1 : *If $\lambda < \mu q$, then the limit distribution*

$$\lim_{t \rightarrow \infty} \mathbf{P} \{ \xi_1(t) = i, \xi_2(t) = j \} = P_{ij} \tag{2}$$

exists for $i \geq 0$ and $j \geq 1$ and is independent of the joint distribution of $\xi_1(0)$ and $\xi_2(0)$. The generating function

$$P(w, z) = \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} P_{ij} w^i z^j \tag{3}$$

is given by

$$P(w, z) = \frac{(\mu q - \lambda)z \{ [Q(w)]^r - [zQ(z)]^r \}}{r[\mu q + \mu p w - (\lambda + \mu)z + \lambda w z]} \tag{4}$$

for $|w| \leq 1$ and $|z| \leq 1$ where

$$Q(z) = \sum_{n=0}^{\infty} A_n (\lambda z - \lambda)^n (\lambda z - \mu q)^{-n}, \tag{5}$$

$A_0 = 1$ and

$$A_n = A_{n-1} \left[p + \frac{\lambda}{\mu} - \frac{\mu q}{\lambda} \left(p + \frac{\lambda}{\mu} \right)^n \right] \left[1 - \left(p + \frac{\lambda}{\mu} \right)^n \right]^{-1} \tag{6}$$

for $n \geq 1$. If $\lambda \geq \mu q$, then $\lim_{t \rightarrow \infty} \mathbf{P} \{ \xi_1(t) = i, \xi_2(t) = j \} = 0$ regardless of the joint distribution of $\xi_1(0)$ and $\xi_2(0)$.

Proof: The random variables $\{ \xi_1(t), \xi_2(t) \}$ form a homogeneous and irreducible Markov process with state space

$$\{ (i, j) : i = 0, 1, 2, \dots, j = 1, 2, \dots \}.$$

If the process $\{ \xi_1(t), \xi_2(t) \}$ has a stationary distribution, then the limit distribution (2) exists and is identical with the stationary distribution. If $\{ \xi_1(t), \xi_2(t) \}$ has no stationary distribution, then

$$\lim_{t \rightarrow \infty} \mathbf{P} \{ \xi_1(t) = i, \xi_2(t) = j \} = 0$$

for all $i \geq 0$ and $j \geq 1$.

If we suppose that $\{ P_{ij} \}$ is a stationary distribution, then it satisfies the following system of equations

$$(\lambda + \mu)P_{ij} = \lambda P_{i-1,j} + \mu p P_{i-1,j+1} + \mu q P_{i,j+1} \tag{7}$$

for $i \geq 1, j \geq 1,$

$$(\lambda + \mu)P_{0j} = \mu q P_{0,j+1} \tag{8}$$

for $1 \leq j < r,$

$$(\lambda + \mu)P_{0r} = \mu q P_{01} + \mu q P_{0,r+1} \tag{9}$$

$$(\lambda + \mu)P_{0,j+r} = \mu p P_{j-1,1} + \mu q P_{j,1} + \mu q P_{0,j+r+1} \tag{10}$$

for $j \geq 1$ and

$$\sum_{i=0}^{\infty} \sum_{j=1}^{\infty} P_{ij} = 1. \tag{11}$$

Define $P(w, z)$ by (3). By (7), (8), (9), and (10) we obtain that

$$P(w, z) = \frac{\mu z [(q + pw)G(w) - (q + pz)z'G(z)]}{\mu q - (\lambda + \mu)z + (\lambda z + \mu p)w} \tag{12}$$

for $|w| \leq 1$ and $|z| \leq 1$ where

$$G(w) = \sum_{i=0}^{\infty} P_{i1} w^i \tag{13}$$

for $|w| \leq 1.$

If a stationary distribution $\{ P_{ij} \}$ exists, then by (11) we have $P(1, 1) = 1,$ and (12) implies that

$$\lim_{z \rightarrow 1} P(1, z) = (p + r)G(1) + G'(1) = 1 \tag{14}$$

and

$$\lim_{w \rightarrow 1} P(w, 1) = \frac{\mu p G(1) + \mu G'(1)}{\lambda + \mu p} = 1. \tag{15}$$

Hence we get

$$G(1) = \frac{\mu q - \lambda}{\mu r}. \tag{16}$$

This shows at once that a stationary distribution cannot exist if $\lambda \geq \mu q.$ If $\lambda < \mu q,$ then a stationary distribution exists and its generating function is given by (12) where $G(w)$ is still to be determined. In (12) we can determine $G(w)$ for $|w| \leq 1$ by the requirement that $|P(w, z)| \leq 1$ for $|w| \leq 1$ and $|z| \leq 1.$ If $|w| \leq 1,$ then the denominator of (12) has a zero

$$z = \frac{\mu(q + pw)}{\mu + \lambda(1 - w)} \tag{17}$$

in the unit circle $|z| \leq 1$. If z is equal to (17), then the numerator of (12) should vanish too, that is,

$$(q + pw)G(w) = (q + pz)z^r G(z) \quad (18)$$

whenever z is defined by (17) and $|w| \leq 1$.

Let

$$R(w) = (q + pw)G(w)/G(1) \quad (19)$$

for $|w| \leq 1$. Then $R(1) = 1$ and by (18) we have

$$R(w) = z^r R(z) \quad (20)$$

whenever z is defined by (17) and $|w| \leq 1$. Hence it follows that

$$R(w) = [Q(w)]^r \quad (21)$$

where $Q(1) = 1$,

$$Q(w) = U(w)Q(U(w)) \quad (22)$$

and

$$U(w) = \frac{\mu(q + pw)}{\mu + \lambda(1 - w)} \quad (23)$$

for $|w| \leq 1$.

By (12), (19) and (21) we get (4) where only $Q(w)$ remains to be determined. If we define $U_1(w) = U(w)$ and $U_{n+1}(w) = U(U_n(w))$ for $n = 1, 2, \dots$, then by (22) it follows that

$$Q(w) = \prod_{n=1}^{\infty} U_n(w) \quad (24)$$

for $|w| \leq 1$. We can easily see that

$$U_n(w) = \frac{\alpha_n w + \beta_n}{\gamma_n w + \delta_n} \quad (25)$$

for $n = 1, 2, \dots$ where

$$\left\| \begin{array}{cc} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{array} \right\| = C_n \left\| \begin{array}{cc} \mu p & \mu q \\ -\lambda & \mu + \lambda \end{array} \right\|^n \quad (26)$$

and $C_n \neq 0$. If we choose $C_n = \mu q - \lambda$, then we have

$$\begin{aligned} \alpha_n &= \mu q(\lambda + \mu p)^n - \lambda \mu^n, \beta_n = \mu q[\mu^n - (\lambda + \mu p)^n] \\ \gamma_n &= \lambda[(\lambda + \mu p)^n - \mu^n], \delta_n = \mu q \mu^n - \lambda(\lambda + \mu p)^n. \end{aligned} \quad (27)$$

However, we can also determine $Q(w)$ in a simpler way. Let us write

$$Q(w) = A \left(\frac{\lambda w - \lambda}{\lambda w - \mu q} \right) \quad (28)$$

for $|w| \leq 1$. Then by (22) we get

$$A(z) \left[1 - \left(p + \frac{\lambda}{\mu} \right) z \right] = A \left(pz + \frac{\lambda z}{\mu} \right) \left[1 - \frac{\mu q}{\lambda} \left(p + \frac{\lambda}{\mu} \right) z \right] \tag{29}$$

and $A(0) = Q(1) = 1$. By expanding $A(z)$ into a Taylor series at $z = 0$,

$$A(z) = \sum_{n=0}^{\infty} A_n z^n, \tag{30}$$

the coefficients $A_n (n = 0, 1, 2, \dots)$ can be determined by (29). If we form the coefficient of z^n in (29), then we get

$$A_n - \left(p + \frac{\lambda}{\mu} \right) A_{n-1} = \left(A_n - \frac{\mu q}{\lambda} A_{n-1} \right) \left(p + \frac{\lambda}{\mu} \right)^n \tag{31}$$

for $n = 1, 2, \dots$ and $A_0 = 1$. This proves formulas (5) and (6).

3. THE LIMIT DISTRIBUTION OF THE WAITING TIME

Denote by ξ_n the queue size, that is, the number of customers in the system, immediately before the n -th customer arrives. We can easily see that if $\lambda < \mu q$, then the limit distribution

$$\lim_{n \rightarrow \infty} \mathbf{P} \{ \xi_n = j \} = P_j \tag{32}$$

exists for $j = 1, 2, \dots$ and is independent of the joint distribution of $\xi_1(0)$ and $\xi_2(0)$. If $\lambda \geq \mu q$, then $\lim_{n \rightarrow \infty} \mathbf{P} \{ \xi_n = j \} = 0$ for $j = 1, 2, \dots$

If $\lambda < \mu q$, then the generating function

$$\Phi(z) = \sum_{j=1}^{\infty} P_j z^j \tag{33}$$

is given by

$$\Phi(z) = P(z, z) = [Q(z)]^r \left(\frac{1 - z^r}{r - rz} \right) \left(\frac{\mu q z - \lambda z}{\mu q - \lambda z} \right) \tag{34}$$

for $|z| \leq 1$ where $Q(z)$ is defined by (5) and (6).

Denote by η_n the waiting time of the n -th customer.

THEOREM 2 : *If $\lambda < \mu q$, then the limiting distribution*

$$\lim_{n \rightarrow \infty} \mathbf{P} \{ \eta_n \leq x \} = W(x) \tag{35}$$

exists and is independent of the joint distribution of $\xi_1(0)$ and $\xi_2(0)$. The Laplace-Stieltjes transform

$$\Omega(s) = \int_0^{\infty} e^{-sx} dW(x) \tag{36}$$

is given by

$$\Omega(s) = \Phi\left(\frac{\mu}{\mu + s}\right) \quad (37)$$

for $\operatorname{Re}(s) \geq 0$ where $\Phi(z)$ is defined by (34). If $\lambda \geq \mu q$, then $\lim_{n \rightarrow \infty} \mathbf{P}\{\eta_n \leq x\} = 0$ for all x .

Proof: Since η_n can be represented as a sum of ξ_n independent and identically distributed random variables with distribution function (1), (35) and (37) follow immediately from (32) and (33).

Define

$$M_\nu = \sum_{j=1}^{\infty} j^\nu P_j \quad (38)$$

for $\nu = 1, 2, \dots$ and write $M = M_1$ and $D = M_2 - M_1^2$.

THEOREM 3: If $\lambda < \mu q$, then

$$M = \frac{r(\lambda + \mu p)}{\mu q - \lambda} + \frac{r + 1}{2} + \frac{\lambda}{\mu q - \lambda} \quad (39)$$

and

$$D = \frac{r\mu(\lambda + \mu q)(\lambda + \mu p)}{(\mu q - \lambda)^2(\lambda + \mu + \mu p)} + \frac{r^2 - 1}{12} + \frac{\lambda\mu q}{(\mu q - \lambda)^2}. \quad (40)$$

Proof: By (34) we obtain that

$$M = \Phi'(1) = rQ'(1) + \frac{r + 1}{2} + \frac{\lambda}{\mu q - \lambda} \quad (41)$$

and

$$\begin{aligned} D &= \Phi''(1) + \Phi'(1) - [\Phi'(1)]^2 \\ &= r[Q''(1) + Q'(1) - (Q'(1))^2] + \frac{r^2 - 1}{12} + \frac{\lambda\mu q}{(\mu q - \lambda)^2}. \end{aligned} \quad (42)$$

If we form the derivatives of (22) at $w = 1$, then we get

$$Q'(1) = \frac{U'(1)}{1 - U'(1)} \quad (43)$$

and

$$Q''(1) + Q'(1) - [Q'(1)]^2 = \frac{U''(1) + U'(1) - [U'(1)]^2}{[1 - U'(1)]^2[1 + U'(1)]}. \quad (44)$$

Since $U'(1) = (\lambda + \mu p)/\mu$ and $U''(1) = 2\lambda(\lambda + \mu p)/\mu^2$ by the above formulas we get (39) and (40).

From (37) it follows that if $\lambda < \mu q$, then

$$\int_0^\infty x dW(x) = M/\mu \tag{45}$$

and

$$\int_0^\infty \left(x - \frac{M}{\mu}\right)^2 dW(x) = (D + M)/\mu^2. \tag{46}$$

4. THE LIMIT DISTRIBUTION OF THE SOJOURN TIME

Denote by θ_n the total time spent in the system by the n -th customer. Obviously, the distribution of θ_n depends solely on the distribution of ξ_n , the number of customers in the system immediately before the arrival of the n -th customer. If $\lambda < \mu q$, then ξ_n has a limit distribution which does not depend on the initial distribution of the process. Consequently, if $\lambda < \mu q$, then θ_n also has a limit distribution which does not depend on the initial distribution of the process. The limit distribution of θ_n is evidently the same as the distribution of θ_n in the case where $P\{\xi_n = j\} = P_j$ ($j = 1, 2, \dots$) defined by (32). Thus we assume that $\lambda < \mu q$ and that $P\{\xi_n = j\} = P_j$ ($j = 1, 2, \dots$) is given by (33) and (34), and give a method of finding the distribution of θ_n .

Denote by $\theta_n^{(k)}$ the total time spent in the system by the n -th customer until his k -th service ends if he joins the queue at least k times. Denote by $\zeta_n^{(k)}$ the number of customers in the system immediately after the k -th service of the n -th customer ends. The n -th customer is not included in $\zeta_n^{(k)}$ even if he returns at least $k + 1$ times. If ξ_n has a stationary distribution, then the expectation

$$\Phi_k(s, z) = E\{e^{-s\theta_n^{(k)}} z^{\zeta_n^{(k)}}\} \tag{47}$$

exists for $\text{Re}(s) \geq 0$ and $|z| \leq 1$ and is independent of n . We can easily see that

$$\Phi_{k+1}(s, z) = \Phi_k\left(s, \frac{\mu(q + pz)}{\mu + s + \lambda(1 - z)}\right) \frac{\mu z^r}{\mu + s + \lambda(1 - z)} \tag{48}$$

for $k = 0, 1, 2, \dots$ where $\Phi_0(s, z) = \Phi(z)$ is defined by (34). The proof of (48) follows the same lines as the proof of (28) in reference [4]. Define

$$\Phi(s, z) = q \sum_{k=1}^\infty p^{k-1} \Phi_k(s, z) \tag{49}$$

for $\text{Re}(s) \geq 0$ and $|z| \leq 1$. The function $\Phi(s, z)$ is completely determined by the recurrence formula (48) and the initial condition $\Phi_0(s, z) = \Phi(z)$.

THEOREM 4 : *If $\lambda < \mu q$, then the limit distribution*

$$\lim_{n \rightarrow \infty} P\{\theta_n \leq x\} = K(x) \tag{50}$$

exists and is independent of the initial distribution of the process. If $\operatorname{Re}(s) \geq 0$, then

$$\int_0^{\infty} e^{-sx} dK(x) = \Phi(s, 1) \quad (51)$$

where $\Phi(s, z)$ is defined by (49).

Proof: The probability that the n -th customer joins the queue exactly k times (including his original arrival) is qp^{k-1} for $k = 1, 2, \dots$. Thus we have

$$\mathbf{E} \{ e^{-s\theta_n} \} = q \sum_{k=1}^{\infty} p^{k-1} \mathbf{E} \{ e^{-s\theta_n^{(k)}} \} \quad (52)$$

for $\operatorname{Re}(s) \geq 0$. If ξ_n has a stationary distribution defined by (33), then (52) reduces to $\Phi(s, 1)$ and this proves (51).

We note that in (49) we can express $\Phi_k(s, z)$ in an explicit form. Let us write

$$U(s, z) = \frac{\mu(q + pz)}{\mu + s + \lambda(1 - z)} \quad (53)$$

and define $U_n(s, z)$ ($n = 0, 1, 2, \dots$) by the recurrence formula

$$U_{n+1}(s, z) = U(s, U_n(s, z)) = U_n(s, U(s, z)) \quad (54)$$

for $n = 0, 1, 2, \dots$ where $U_0(s, z) = z$. It is easy to see that if we define

$$\Phi_k(s, z) = \Phi(U_k(s, z)) \prod_{j=0}^{k-1} \left(\frac{\mu[U_j(s, z)]^r}{\mu + s + \lambda[1 - U_j(s, z)]} \right) \quad (55)$$

for $k = 1, 2, \dots$, and if $\Phi_0(s, z) = \Phi(z)$, then (48) is satisfied for $k = 0, 1, 2, \dots$. In (55) we have

$$U_k(s, z) = \frac{\alpha_k(s)z + \beta_k(s)}{\gamma_k(s)z + \delta_k(s)} \quad (56)$$

for $k = 0, 1, 2, \dots$ where

$$\begin{vmatrix} \alpha_k(s) & \beta_k(s) \\ \gamma_k(s) & \delta_k(s) \end{vmatrix} = \begin{vmatrix} \mu p & \mu q \\ -\lambda & \mu + \lambda + s \end{vmatrix}^k \quad (57)$$

From (48) it follows immediately that

$$[\mu + s + \lambda(1 - z)]\Phi(s, z) = \mu q z^r \Phi(z) + \mu p z^r \Phi\left(s, \frac{\mu(q + pz)}{\mu + s + \lambda(1 - z)}\right) \quad (58)$$

for $\operatorname{Re}(s) \geq 0$ and $|z| \leq 1$. By (58) we can easily determine the moments

$$K_m = \int_0^{\infty} x^m dK(x) \quad (59)$$

for $m = 1, 2, \dots$. Define

$$\Phi_{ij} = \frac{1}{i! j!} \left(\frac{\partial^{i+j} \Phi(s, z)}{\partial s^i \partial z^j} \right)_{s=0, z=1} \tag{60}$$

for $i = 0, 1, 2, \dots$ and $j = 0, 1, 2, \dots$. If we form the mixed partial derivatives of order m of (58), then we obtain $m + 1$ linear equations for the determination of $\Phi_{0m}, \Phi_{1,m-1}, \dots, \Phi_{m0}$. These derivatives can be determined step by step for $m = 1, 2, \dots$ and

$$K_m = (-1)^m m! \Phi_{m0} \tag{61}$$

yields the m -th moment of $K(x)$.

Introduce the notation

$$a_{ij} = \frac{1}{i! j!} \left[\frac{\partial^{i+j}}{\partial s^i \partial z^j} \left(\frac{\mu(q + pz)}{\mu + s + \lambda(1 - z)} \right) \right]_{s=0, z=1} \tag{62}$$

for $i \geq 0$ and $j \geq 0$. Then $a_{00} = 1$, $a_{10} = -1/\mu$, $a_{01} = (\lambda + \mu p)/\mu$, $a_{20} = 1/\mu^2$, $a_{11} = -(2\lambda + \mu p)/\mu^2$ and $a_{02} = \lambda(\lambda + \mu p)/\mu^2$.

By (58) we obtain

$$1 + \mu\Phi_{10} = \mu p\Phi_{10} - p\Phi_{01}, \tag{63}$$

and

$$\mu\Phi_{01} - \lambda = \mu r + \mu q\Phi'(1) + (\lambda + \mu p)p\Phi_{01}. \tag{64}$$

From (63)

$$\Phi_{10} = -(1 + p\Phi_{01})/\mu q \tag{65}$$

and from (64)

$$\Phi_{01} = [\lambda + \mu r + \mu q\Phi'(1)][\mu - (\lambda + \mu p)p]^{-1} \tag{66}$$

where $\Phi'(1) = M$ is given by (39). Since $K_1 = -\Phi_{10}$, by (65) and (66) we get

$$K_1 = \frac{1}{\mu q} + \frac{p}{\mu q} \left[\frac{2(\mu q - \lambda)(\lambda + \mu r) + \mu q\lambda(r + 1) + \mu^2 q(2r - (r - 1)q)}{2(\mu q - \lambda)[\mu - (\lambda + \mu p)p]} \right] \tag{67}$$

If we form the partial derivatives of order two of (58), then we obtain the following equations for the determination of $K_2 = 2\Phi_{20}$:

$$\mu\Phi_{20} + \Phi_{10} = \mu p a_{20}\Phi_{01} + \mu p\Phi_{20} + \mu p a_{10}\Phi_{11} + \mu p a_{10}^2\Phi_{02}, \tag{68}$$

$$\begin{aligned} \Phi_{01} - \lambda\Phi_{10} + \mu\Phi_{11} = \\ = \mu p [a_{11}\Phi_{01} + a_{01}\Phi_{11} + 2a_{10}a_{01}\Phi_{02} + r\Phi_{10} + ra_{10}\Phi_{01}] \end{aligned} \tag{69}$$

and

$$\mu\Phi_{02} - \lambda\Phi_{01} = \mu q \left[\frac{\Phi''(1)}{2} + r\Phi'(1) + \binom{r}{2} \right] + \mu p \left[a_{02}\Phi_{01} + a_{01}^2\Phi_{02} + ra_{01}\Phi_{01} + \binom{r}{2} \right]. \quad (70)$$

Now Φ_{02} can be determined by (70), Φ_{11} by (69) and finally Φ_{20} by (68).

5. AN EXAMPLE

Let us suppose that in the queuing process, $p = 1/3$ and $\lambda/\mu = 1/6$. Then $\lambda/\mu q = 1/4$ and ξ_n, η_n and θ_n have a limit distribution. The probabilities $\lim_{n \rightarrow \infty} \mathbf{P} \{ \xi_n = j \} = P_j$ ($j = 1, 2, \dots$) are determined by (34) where now

$$Q(z) = 3(z+2)(4-z)^{-2}. \quad (71)$$

This follows from (5) where now $A_0 = 1, A_1 = -3, A_2 = 2$ and $A_n = 0$ for $n \geq 3$. In this case (34) reduces to

$$\Phi(z) = \left(\frac{1-z^r}{r(1-z)} \right) \binom{3}{4-z} \left(\frac{3(z+2)}{4-z} \right)^r. \quad (72)$$

The Laplace-Stieltjes transform of the limiting distribution of η_n is given by (37), and the Laplace-Stieltjes transform of the limiting distribution of θ_n is given by (51).

By (39) and (40) we obtain that

$$M = (9r + 5)/6 \quad (73)$$

and

$$D = (3r^2 + 40r + 13)/36. \quad (74)$$

By (67) we get

$$\mu K_1 = (18r + 29)/15 \quad (75)$$

and by (68), (69) and (70) it follows that

$$\mu^2 K_2 = (16065r^2 + 37025r + 23827)/2475. \quad (76)$$

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