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PROGRAMMING WITH PARAMETRIC ELEMENTS
OF THE MATRIX COEFFICIENTS

by Ramon M. ARANA

Abstract — *Many studies have been carried out concerning parametric programming of the objective function coefficients and also the parametric programming of the constant terms of the constraints has been extensively studied*

However, many authors state that parametric programming of elements of the matrix coefficients is simple and useful only when the coefficients that vary correspond to nonbasic vectors, and that when, on the other hand, they correspond to basic vectors the complexity of the topic does not make this type of parametric programming very useful

In this article one obtains a formula which determines the critical values of a parameter which affects the coefficients of a basic vector While the parameter does not reach these limits, the new vector continues forming a feasible basis with the remaining vectors of the optimal basis

Five useful examples of parametric programming of elements of the matrix coefficients have been given in 1963 by Henri Maurin [1] with determination of the critical values of the parameter

INTRODUCTION

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However, many authors state that parametric programming of elements of the matrix coefficients is simple and useful only when the coefficients that vary correspond to nonbasic vectors, and that when, on the other hand, they correspond to basic vectors the complexity of the topic does not make this type of parametric programming very useful.

In this article one obtains a formula which determines the critical values of a parameter which affects the coefficients of a basic vector. While the parameter does not reach these limits, the new vector continues forming a feasible basis with the remaining vectors of the optimal basis.

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NOTATIONS

Let us consider a problem of lineal programming presented in the following manner :

Minimize $z = cx$, subject to :

$$Ax = b$$
$$x \geqslant 0$$

Let us designate by a_{ij} the element of the matrix A located in row i and in column j. The matrix A is composed of m rows and n columns.

Let B be the matrix whose columns are the m vectors of the optimal basis. Let us suppose that the m vectors of the optimal basis correspond to the m first columns of the matrix A [1].

Let B^{-1} be the inverse matrix of B.

The solution of the problem of linear programming is defined by the following formula :

$$x_B = B^{-1}b.$$

The vector a_j can be expressed, as a linear combination of the vectors that form the basis B, in the following way :

$$a_j = By_j \qquad y_j = B^{-1}a_j.$$

Let us designate by β_i the row vector composed of the elements of row i of the matrix B^{-1}.

PARAMETRIC PROGRAMMING OF A BASIC VECTOR

Let us suppose that a_h is the vector that will be affected by the parameter θ.

Since this vector belongs to the optimal basis (calculated for $\theta = 0$), it results that :

$$y_h = B^{-1}a_h = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad \text{(column vector of } m \text{ elements. The unit} \qquad (1)$$
(column vector of m elements. The unit element is located in row h)

y_h is a column vector of m elements, all of which are zero, except one element, located in row h, whose value is 1.

Let us suppose that vectors y_t corresponding to the calculated optimal basis are organized in such a way so that the unit element of the column vector y_t would be located in row i ([2]).

In order to perform the parametric analysis one substitutes a_h for a'_h. Let us suppose that between a_h and a'_h there exists the linear function defined by the following expression :

$$a'_h = a_h + \theta e \qquad e = \begin{bmatrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ \cdot \\ e_m \end{bmatrix}$$

Then :

$$y'_h = B^{-1} a'_h = \begin{bmatrix} \theta \beta_1\, e \\ \theta \beta_2\, e \\ \vdots \\ 1 + \theta \beta_h\, e \\ \vdots \\ \theta \beta_m\, e \end{bmatrix} \qquad (2)$$

GEOMETRIC INTERPRETATION

In linear programming problems reasonings must be developed in a space of m dimensions, where m can be larger than 3.

The graphic representations can only be made in a space of, as a maximum, 3 dimensions.

The principal difficulty in obtaining, by geometric means, deductions applicable to the problems of linear programming, consists in that only those deductions obtained in a space of 3 dimensions which are also valid in an m-dimensional space are useful.

However, on passing from the space of 3 dimensions to that of m dimensions, certain concepts that were perfectly clear and understandable, and that were also amenable to an easy mental representation, become abstract ideas, much more complex.

In spite of everything, geometric reasoning is at times an efficient method of investigation, or at least a means which permits one to deduce subjects which are to be the object of investigation.

([1]) and ([2]) : These suppositions simplify the notations, and do not diminish the general character of neither reasonings nor deductions.

The non-negativity restrictions of the variables requires that the solution vector x_B be contained in the polyhedral convex cone determined by vectors y_i which form the basis.

Therefore, if in a feasible basis one of the vectors varies, the modified basis will continue to be feasible basis as long as vector x_B be contained in the polyhedral convex cone determined by the vectors that form the new basis.
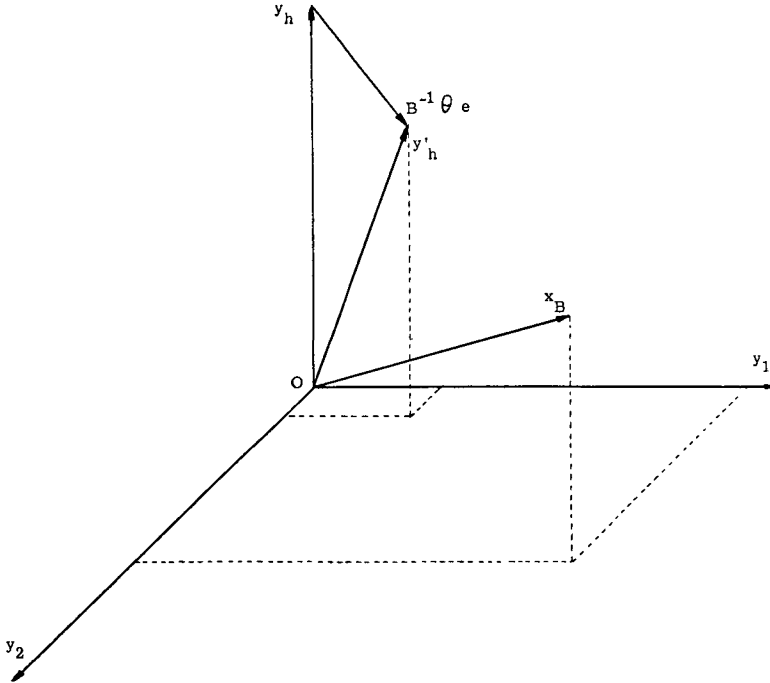
Let us consider a space of 3 dimensions, and let us take as the axis the right lines that contain the vectors $y_i = B^{-1}a_i$ that form the basis.

Let us represent vector $x_B = B^{-1}b$, vector $y'_h = B^{-1}a'_h$ and the variable vector $B^{-1}\theta e$ (and remember that $y'_h = B^{-1}a_h + B^{-1}\theta e$).

$$y_h$$

In order that the vectors y_1, y_2 and y'_h form a feasible basis it is necessary that vector x_B be contained in the polyhedral convex cone determined by these three vectors.

On considering the plane that contains vectors y_h and y_1 and the plane which contains y'_h and y_1, the end of vector x_B must be on the same side with respect to both planes.

Likewise, on considering the plane that contains vectors y_h and y_2 and the plane which contains y_h' and y_2, the end of vector x_B must be on the same side with respect to both planes.

DEDUCTIONS IN THE M-DIMENSIONAL SPACE

The previous reasoning can be extended to the space of m dimensions $(m > 3)$ if, instead of considering the planes which contain vector y_h and another vector of the basis (and the planes which contain vector y_h' and this other vector of the basis), one considers the hyperplanes which contain vector y_h and the remaining vectors of the basis except one (and the hyperplanes which contain vector y_h' and those same remaining vectors of the basis).

The equation of the hyperplane that contains vector y_h' and all of the vectors of the previous basis except vector y_h and another vector (y_i) is :

$$\frac{y_i}{y_h} = \frac{\theta \beta_i e}{1 + \theta \beta_h e}$$

$$y_i(1 + \theta \beta_h e) - y_h \theta \beta_i e = 0$$

The vector x_B (in the case of a non-degenerate solution) has a positive component x_i.

In order that the modified basis be a feasible basis, the end of x_B must be on the side of the positive values of y_i, with respect to the hyperplane which contains all of the vectors of the previous basis except y_i, as well as with respect to the hyperplane that contains vector y_h' and all the vectors of the previous basis except y_h and y_i.

Consequently, it must be satisfied that :

$$x_i(1 + \theta \beta_h e) - x_h \theta \beta_i e \geqslant 0 \tag{3}$$

where x_i and x_h are the components of vector x_B corresponding to rows i and h respectively.

The limit (or limits) of the parameter is obtained by setting the first term of the above inequality to zero, and solving θ.

$$\theta = \frac{x_i}{x_h \beta_i e - x_i \beta_h e} \tag{4}$$

CRITICAL VALUES OF THE PARAMETER

The inequality (3) must be fulfilled for all values of i corresponding to the vectors that form the calculated optimal basis (calculated for $\theta = 0$).

In the particular case $i = k$, the inequality (3) is satisfied for any finite value of θ.

Therefore, from the formula (4), one obtains $m - 1$ values of θ.

Some of these values may be positive and others negative.

In the case in which $\theta = 0$, the calculated optimal basis is, of course, feasible basis. As we increase the value of the parameter, the modified basis continues being feasible basis until θ reaches the lowest value of the positive values obtained in the formula (4). Starting from this value, for greater values of the parameter, the previous optimal basis no longer is a feasible basis.

On decreasing the value of the parameter starting from $\theta = 0$, another limiting value is reached in the maximum value of those negative obtained in the formula (4).

The critical values of the parameter are, therefore, the lowest of the positive values and the highest of the negative values obtained from the formula (4).

If all of the values of θ obtained from the formula (4) are of the same sign, the modified optimal basis continues being a feasible basis for any finite value of θ of the opposite sign.

CASE IN WHICH THE MODIFIED BASIS NO LONGER IS A BASIS

It can occur also that, on varying one of the vectors of the optimal basis, the transformed previous basis no longer is a basis.

This will occur when the vector affected by the parameter becomes a linear combination of the remaining vectors of the previous optimal basis.

In order for this to occur one must nullify y'_h. That is to say, it will occur that :

$$1 + \theta \beta_h e = 0.$$

Therefore, in the case in which $\theta = -\dfrac{1}{\beta_h e}$, the modified optimal basis ceases being a basis.

FULFILLMENT OF THE OPTIMALITY CONDITION

The formula (4) determines the limits of θ between which the modified optimal basis continues being a feasible basis.

Naturally, although one may know that a modified basis is a feasible basis, one must check to see also if it fulfills the optimality condition, by previously substituting in the basis the vector y_h for the y'_h. The components of these vectors are determined by the formulas (1) and (2) respectively.

BIBLIOGRAPHY

1. H. MAURIN, *Paramétrisation Générale d'un Programme Linéaire* (Thèse présentée le 19/6/1963 à la Fac. des Sciences de l'Université de Paris).