REVUE FRANÇAISE D'AUTOMATIQUE, INFORMATIQUE, RECHERCHE OPÉRATIONNELLE. RECHERCHE OPÉRATIONNELLE

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Revue française d'automatique, informatique, recherche opérationnelle. Recherche opérationnelle, tome 6, n° V3 (1972), p. 45-59
<http://www.numdam.org/item?id=RO_1972__6_3_45_0>

R.A.I.R.O.
(6e année, novembre 1972, V-3, p. 45 à 60)

TRANSPORTATION NETWORKS WITH RANDOM ARC CAPACITIES

by P. DOULLIEZ (1) and E. JAMOULLE (2)

Summary. — *The arc capacities of a transportation network are assumed to be independent discrete random variables and the flow requirements at the sink nodes are known values. This paper presents an efficient method for finding the probability that all flow requirements be satisfied and the probability that a given arc be found in a minimal cut. The method is based upon a decomposition principle which allows the transformation of the network capacity state space into non-overlapping subsets. The expected amount of unsupplied flow is also found. The computational efficiency and domain of application of the method are discussed.*

1. INTRODUCTION

The nodes of a transportation network are joined by arcs and the capacity of an arc is an upper bound to the flow that may pass over it. The flow must be sent through the network from « supply » nodes to « demand » nodes. The largest amount of flow that can be sent from a supply node is considered as the capacity of a fictitious arc joining the supply node and a common fictitious supply node, denoted as S.

In this paper, the amounts of flow required at the different demand nodes i are known values d_i.

It is assumed that the arc capacities are independent discrete random variables.

Hence, the probability that all the flow requirements be satisfied is well defined and will be computed by an efficient method. The probability that a given arc be found in a minimal cut will also be computed (3).

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(3) Since the network has several sink nodes the definition of a minimal cut is here slightly different from the definition that can be found in [3]. We define a minimal cut as a set of arcs separating the common supply node S from one or several demand nodes i such that no supplementary flow can go from S to one of these demand nodes i.

The method is based upon a decomposition principle which allows the transformation of the network capacity state space into non-overlapping subsets.

In section 2, the decomposition principle is presented and is applied to our specific problem. It is shown in section 3 how the expected amount of unsupplied flow and the probability that a given arc be found in a minimal cut can be computed. Computational efficiency and domain of application are discussed in section 4 and 5. In section 5, a numerical example is given.

2. THE DECOMPOSITION PRINCIPLE

2.1. Notations and Definitions

Let H_j be a discrete random variable ($j = 1 \dots m$). The m random variables H_j are assumed to be independent. A random variable H_j assumes positive values $h_{j1}, h_{j2}, \dots h_{jk_j}$ with probabilites $p_{j1}, p_{j2}, \dots p_{jk_j}$ respectively. A point x of the state space can be defined as an m-tuple of values $x = (h_{1v_1}, h_{2v_2} \dots h_{mv_m})$ where v_j is a numerical index for j going from 1 to k_j. For notational convenience, the index v_j will also be used to designate the value h_{jv_j} itself so that a state point x will be denoted as $x = (v_1, v_2 \dots v_m)$. The entire state space is denoted by X. The state points $x = (1,1, \dots 1)$ and $x = (k_1, k_2, \dots k_m)$ are called « limiting state points » for X.

In our specific problem, H_j is a capacity variable for arc j and $h_{j1}, h_{j2} \dots h_{jk_j}$ are capacity values for arc j.

For simplicity, we assume that the values h_{jv_j} are such that

$$h_{j1} < h_{j2} < \dots < h_{jk_j} \quad (j = 1 \dots m).$$

A state point $x = (v_1, v_2 \dots v_m)$ is called a network state.

Let $\Pr[X = x]$ be the probability associated with a point $x \in X$.

We have :

$$\Pr[X = x] = p_{1v_1} \cdot p_{2v_2} \cdots p_{mv_m}$$

$$\sum_{v_j=1}^{kj} p_{jv_j} = 1 \quad (j = 1 \dots m)$$

$$\Sigma_{\text{over } x} \Pr[X = x] = \Pr[X] = 1$$

2.2. Statement of the problem

Suppose that any state point $x = (v_1 \ldots v_m)$ of the state space X can be recognized as either acceptable or not acceptable. Let A^0 be the set of acceptable points and B^0 the set of non-acceptable points. The problem is to compute in an efficient way the probabilities Pr $[A^0]$ and Pr $[B^0]$. We have :

$$X = A^0 \cup B^0$$

$$\Pr[X] = \Pr[A^0] + \Pr[B^0] = 1$$

In our specific problem, a network state $x = (v_1 \ldots v_m)$ is acceptable if the capacities v_j are such that all demands d_i can be satisfied. The probability Pr $[A^0]$ is the probability that all demands be satisfied and Pr $[B^0]$ is the probability that at least one demand be not entirely satisfied.

The state space X has $\prod_{j=1}^{m} k_j$ elements, so it is hopeless to solve the problem by considering each point of the state space one after the other. Dealing with subsets in which all state points are acceptable or subsets in which all state points are not acceptable, it can be seen in [1] p. 44 that the problem is still quite complex when some subsets may overlap each other. Therefore, we now present a method for constructing sequences of non-overlapping subsets in which all state points are acceptable and non-overlapping subsets in which all state points are not acceptable.

2.3. Classification rule

Suppose that a state point $x = (v_1, \ldots v_m)$ is acceptable whenever each v_j is within the range $[v_j^0, k_j]$ where v_j^0 is a known critical value for the random event H_j. Thus, $x = (v_1 \ldots v_m)$ is an acceptable state point if

$$v_j^0 \leqslant v_j \leqslant k_j \quad (j = 1 \ldots m).$$

Suppose also that a state point $x = (v_1 \ldots v_j \ldots v_m)$ is not acceptable whenever at least one v_j is within the range $[1, v_j^*[$ where v_j^* is a known critical value. Therefore, $x = (v_1 \ldots v_j \ldots v_m)$ is a non-acceptable state point if $1 \leqslant v_j < v_j^*$ for at least one j.

If the classification rule holds, we can say that a state point $x = (v_1 \ldots v_j \ldots v_m)$ is « unspecified » whenever $v_j \geqslant v_j^*$ for all j and at least one v_j is within the range $[v_j^*, v_j^0[$, i.e, $v_j^* \leqslant v_j < v_j^0$.

It is explained in section 3 how the critical capacity values v_j^0 and v_j^* are obtained in our specific problem.

2.4. Non overlapping subsets of X

A state point is obtained by assigning numerical values to the indices v_j.

In figure 1, a state point $x = (v_1 \dots v_m)$ can be represented by a broken line through the v_j table ($j = 1 \dots m$; $m = 5$) and regions V_A, V_B, V_C are defined by the critical values v_j^* and v_j^0 as follows :

$$v_j \in V_A \quad \text{if} \quad v_j^0 \leqslant v_j \leqslant k_j$$

$$v_j \in V_B \quad \text{if} \quad 1 \leqslant v_j < v_j^*$$

$$v_j \in V_C \quad \text{if} \quad v_j^* \leqslant v_j < v_j^0$$

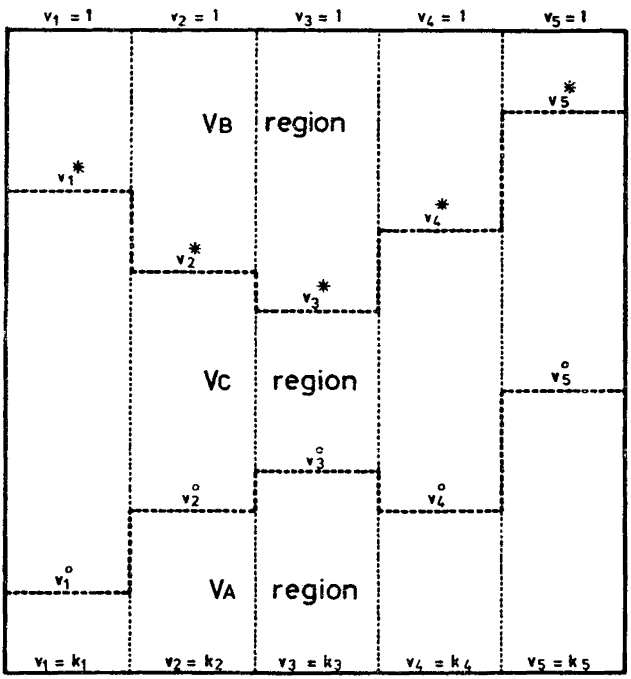

Figure 1

The v_j index table ($m = 5$)

Classifying the state points $x = (v_1, v_2 \dots v_j \dots v_m)$ with respect to the critical values v_j^0 and v_j^*, it can be seen easily that any state point necessarily belongs to one of the following classes :

— x has its values v_j entirely in V_A. Then we say that $x \in A$ where A is a set of acceptable state points ($A \subseteq A^0$);

— x has at least one v_j in V_B. Then we say that $x \in B$ where B is a set of non-acceptable state points ($B \subseteq B^0$);

— x has at least one v_j in V_C and no value in V_B. Then we say that $x \in C$ where C is a set of unspecified state points.

We have :

$$A \cup B \cup C = X$$

$$A \cap B = \emptyset \qquad B \cap C = \emptyset \qquad A \cap C = \emptyset$$

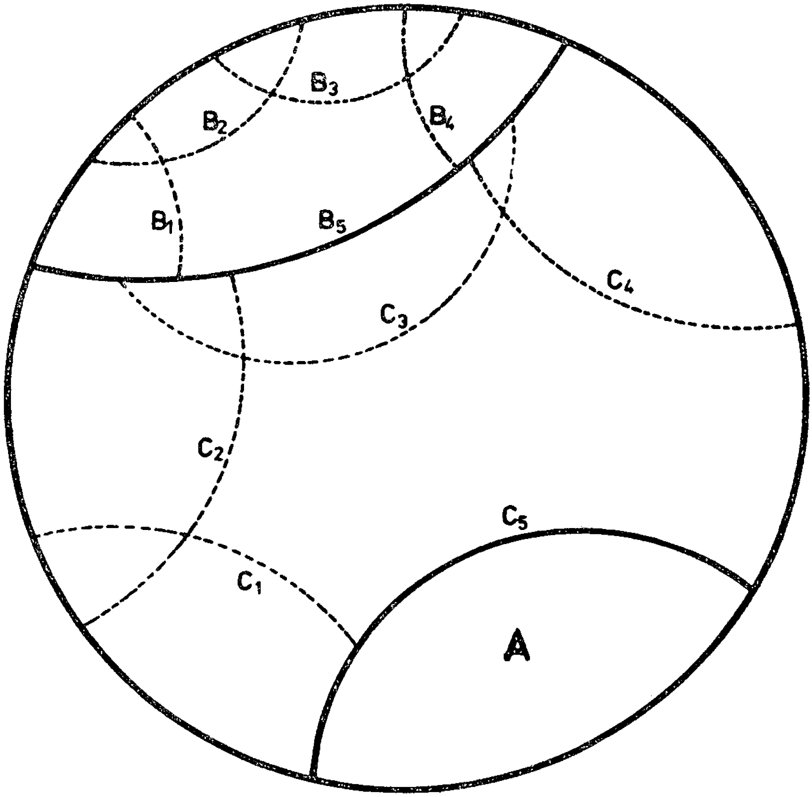

Figure 2
The state space (overlapping subsets)

The sets B and C can be defined as follows :

$$B = \bigcup_{j=1}^{m} B_j$$

$$C = \bigcup_{j=1}^{m} C_j$$

where $x = (v_1 \ldots v_m) \in B_l$ if $v_l < v_l^*$

and $x = (v_1 \ldots v_m) \in C_l$ if $v_l < v_l^0$ and if $v_k \geqslant v_k^*$ $(k = 1 \ldots m)$.

The subsets B_j and C_j are represented in figure 2. Any two sets B_j (or C_j) may overlap each other. There is a one-to-one correspondance between a state x (represented as a line through the table in figure 1) and a point of the set in figure 2 $(m = 5)$.

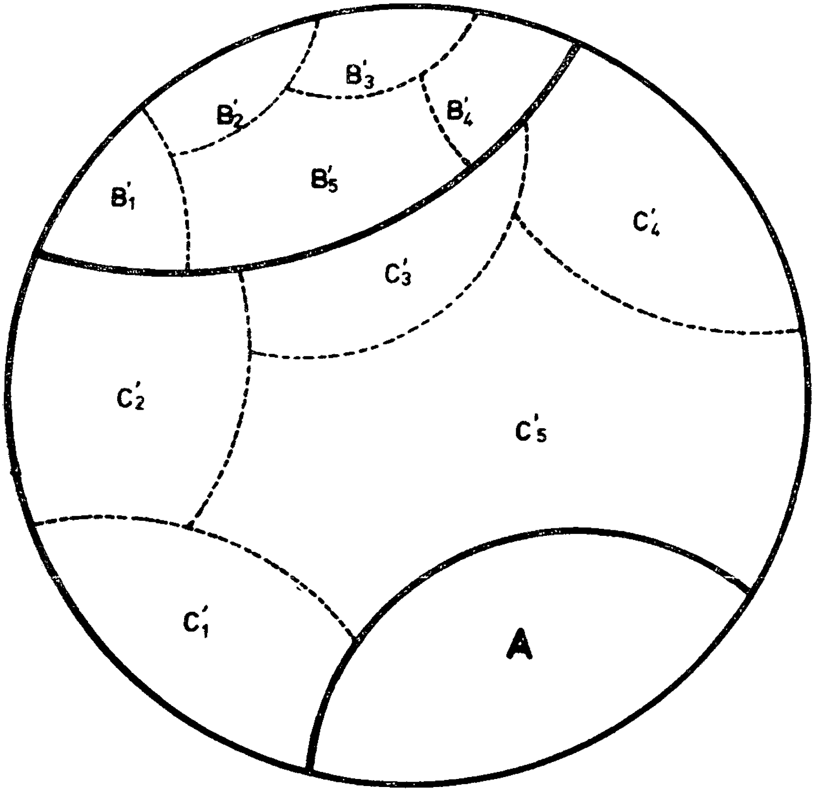

Figure 3

The state space (non-overlapping subsets)

Since adding probabilities associated with overlapping subsets B_j and overlapping subsets C_j is meaningless, let us transform the subsets B_j and C_j into B_j' and C_j' as represented in figure 3 such that for any j and k $(j \neq k)$ we have :

$$B_j' \cap B_k' = \emptyset \text{ and } C_j' \, . \cap C_k' = \emptyset$$

The transformation of subsets B_j (C_j) into subsets $B_j'(C_j')$ can be performed sequentially as follows :

$$\begin{array}{ll} B_1' = B_1 & C_1' = C_1 \\ B_2' = B_2 - B_1' \cap B_2 & C_2' = C_2 - C_1' \cap C_2 \\ \vdots & \vdots \\ B_j' = B_j - \sum_{r<j} (B_r' \cap B_j) & C_j' = C_j - \sum_{r<j} (C_r' \cap C_j) \\ \vdots & \vdots \\ B_m' = B_m - \sum_{r<m} (B_r' \cap B_m) & C_m' = C_m - \sum_{r<m} (C_r' \cap C_m) \end{array}$$

This can be easily translated in terms of v_r, v_l and cardinal value of index l for subsets B_l' and C_l' ($l = 1 \dots m$) :

$x = (v_1 \dots v_m) \in B_l'$ if $v_l < v_l^*$ (or $v_e \leqslant v_e^* - 1$) and $v_r \geqslant v_r^*$ for any $r < l$

$x = (v_1 \dots v_m) \in C_l'$ if $v_l^* \leqslant v_l < v_l^0$, $v_r \geqslant v_r^0$ for any $r < l$ and

$$v_r \geqslant v_r^* \text{ for any } r > l$$

Thus, overlapping subsets B_j and C_j are transformed into non-overlapping subsets B_j' and C_j' by taking the increasing order of index j into account. It is easily seen that $B_j' \cap B_k' = \emptyset$ and $C_j' \cap C_k' = \emptyset$ for $j \neq k$. Also,

$$B_j' = \emptyset \text{ if } v_j^* = 1 \text{ and } C_j' = \emptyset \text{ if } v_j^0 = v_j^*.$$

2.5. Probability Computations

If all sets A, B_j' and C_j' are exhaustive and non-overlapping subsets of X, we have :

$$\Pr[X] = \Pr[A] + \sum_{j=1}^{m} \Pr[B_j'] + \sum_{j=1}^{m} \Pr[C_j'] \qquad (1)$$

It is easy to compute each term of the right-hand side in (1) since limiting state points are known for each subset, as indicated in figure 4.

If we define for each j ($j = 1 \dots m$)

$$p_j = \sum_{v_j=v_j^0}^{k_j} p_{jv_j} \quad ; \quad q_j = \sum_{v_j=v^*}^{v_j^0-1} p_{jv_j} \ ;$$

$$s_j = \sum_{v_j=1}^{v_j^*-1} p_{jv_j}$$

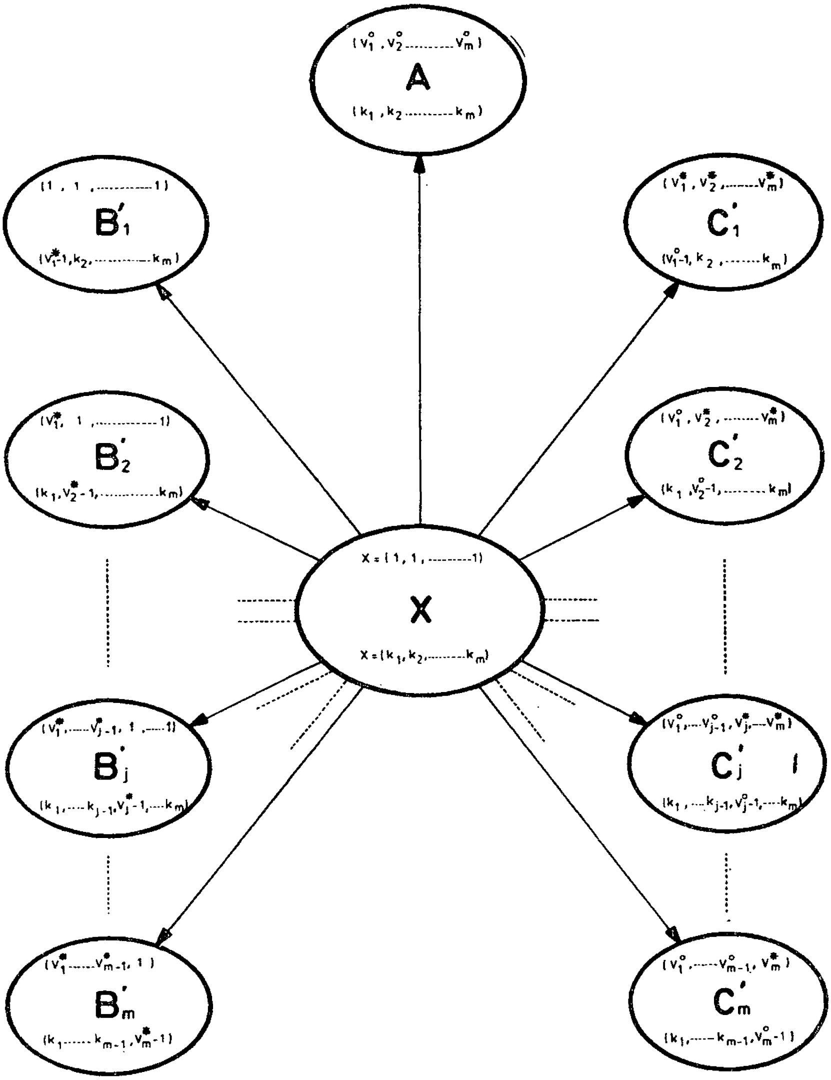

Figure 4

The subsets of X and their limiting state points

we have

$$\Pr[X] = \prod_{j=1}^{m} (p_j + q_j + s_j) \tag{2}$$

$$\Pr[A] = \prod_{j=1}^{m} p_j \tag{3}$$

$$\Pr[C_j'] = q_j \cdot \prod_{r=1}^{j-1} p_r \cdot \prod_{r=j+1}^{m} (p_r + q_r) \quad j = 1 \dots m \tag{4}$$

$$\Pr[B_j'] = s_j \cdot \prod_{r=1}^{j-1} (p_r + q_r) \cdot \prod_{r=j+1}^{m} (p_r + q_r + s_r) \quad j = 1 \dots m \tag{5}$$

Notice that $q_j = 0$ if $v_j^0 = v_j^*$ and $s_j = 0$ if $v_j^* = 1$.

In the last two equations, ignore the product $\prod_{r=a}^{b}$ whenever $b < a$. From equations (2) to (5), it can be shown algebraically that (1) holds.

2.6. The decomposition principle (1)

If values v_j^0 and v_j^* can be known for the initial state space X, non-overlapping subsets A, B_j' and C_j' can be constructed.

Any $x \in A$ is an acceptable state point and any $x \in B_j'$ $(j = 1 \dots m)$ is a non-acceptable state point. Any $x \in C_j'$ $(j = 1 \dots m)$ is an unspecified state point.

Therefore, the associated probability $\Pr[A]$ is a part of the sought value $\Pr[A^0]$ and $\sum_{j=1}^{m} \Pr[B_j']$ is a part of the sought value $\Pr[B^0]$.

Now, each non-empty subset C_j' may be considered as a new initial state space with modified but known limiting state points. In each C_j', new critical values v_j^0 and v_j^* may be determined, as they have been determined in the initial state space X. Thus, each C_j' may be again decomposed into new non-overlapping subsets B_j', that contain only non-acceptable points, a set A of acceptable points and new subsets of unspecified points. The probabilities associated with new sets A and B_j' can be computed as in (3) and (5) and are added to the former values and so on... until no subset with unspecified points can be generated and all such subsets have been considered. When this occurs, the probabilities $\Pr[A^0]$ and $\Pr[B^0]$ are obtained. The set A^0 and B^0 are respectively the union of all sets A and sets B_j' that have been constructed.

The procedure is finite. For an initial non-empty state space X, we suppose that A^0 is non-empty otherwise the problem is solved with $\Pr[A^0] = 0$. After decomposing X into A, B_j' and C_j', we have $A \subseteq X$ but if $A = X$, $\Pr[A^0] = 1$ and the problem is solved. Thus, $A \subset X$ and also each $B_j' \subset X$ and $C_j' \subset X$.

Similarly, at any stage of the procedure, each subset of unspecified state points is strictly contained within the set from which it has been generated.

(1) What is here called the « Decomposition Principle » is by no means related to lhe decomposition principle introduced by Dantzig and Wolfe for solving large-scale tinear programs.

Consequently, the procedure is finite.

The procedure could be summarized as follows :

— Step 1 : Start with initial set X. Let Pr $[A^0]$ = Pr $[B^0] = 0$.

— Step 2 : Given the limiting state points of X, compute the critical values v_j^0 and v_j^*. The sets A, B_j', C_j' can then be defined with their own limiting state points. Compute Pr $[A]$ and Pr $[B_j']$ for each j.

$$\text{Let Pr}\,[A^0] = \text{Pr}\,[A^0] + \text{Pr}\,[A] \text{ and Pr}\,[B^0] = \text{Pr}\,[B^0] + \sum_{j=1}^{m} \text{Pr}\,[B_j']$$

— Step 3 : Choose a set C_j' that has not yet been considered. If no such set exists, go to step 4. Otherwise, let $X = C_j'$ and go to step 2.

— Step 4 : Pr $[A^0]$ and Pr $[B^0]$ are the sought values. We necessarly have Pr $[A^0]$ + Pr $[B^0] = 1$. END

3. TRANSPORTATION NETWORK WITH RANDOM ARC CAPACITIES

3.1. Critical capacity values

In this section, a method for finding critical values v_j^0 and v_j^* is presented for our specific network problem.

They will be found with respect to initial network state set X with limiting state points $x = (1, 1 \dots 1)$ and $x = (k_1 \dots k_m)$ but the procedure would be the same for any set of network states with known limiting state points. The probability Pr $[A]$ and probabilities Pr $[B_j']$ can be computed once the critical capacity values v_j^0 and v_j^* have been determined for each random capacity H_j.

Let us join each demand node i to a common fictitious demand node T by way of an arc with capacity d_i. A flow $D = \sum_{over\ i} d_i$ is sent from the common source node S to T and the flow f_j in any arc j cannot be higher than k_j.

The values v_j^0 are obtained as follows :

Let v_j^0 be the smallest capacity value that can be assumed by the discrete random variable H_j and which is not less than f_j. The network state $x = (v_1^0 \dots v_j^0 \dots v_m^0)$ is obviously acceptable. The set A is a set of network states $x = (v_1 \dots v_j \dots v_m)$ with $v_j^0 \leqslant v_j \leqslant k_j$ for any j. Therefore, any $x \in A$ is acceptable since it has been assumed in the previous section that capacity values are increasing with their indices.

The j^{th} value v_j^* is obtained as follows :

The set B_j' is a set of non-acceptable network states $x = (v_1 \dots v_j \dots v_m)$ with $1 \leqslant v_j < v_j^*$ for arc j.

Given the limiting capacity values k_r $(r \neq j)$ and the existing flow D in the network, find the maximal flow F_j that can still go from the origin to the end of arc j without using arc j. The value F_j can be obtained by Ford-Fulkerson's method [3] and corresponds to the largest feasible decrease in flow for arc j. If $F_j < f_j$, arc j is in a minimal cut separating node S from node T and v_j^* is the smallest capacity value which is not less than $(f_j - F_j)$. Obviously, any $x \in B_j'$ is not able to satisfy all demands and is non-acceptable. If $F_j \geqslant f_j$, then $v_j^* = 1$ and $B_j' = \emptyset$ since the entire range of capacity values for arc j is feasible for values d_i.

3.2. Presence of an arc in a minimal cut

We are now interested in finding the probability Pr $[B^0|r]$ that arc r be found in a minimal cut $(r = 1 \dots m)$. Since any non-acceptable network state has a minimal cut associated with it, only those with a minimal cut that contains arc r must be retained and their associated probabilities must be summed up.

For any set of non-acceptable network states B_j' that has been generated by the procedure explained in section 2, let us make the following decomposition :

The limiting state points of a set B_j' are known and let us designate them as $x = (g_1 \dots g_m)$ and $x = (k_1 \dots k_m)$ with $g_j \leqslant k_j$ for all j. Let M be a minimal cut associated with a network state x that is defined by the highest capacity values, i.e., $x = (k_1 \dots k_m)$. The value of the cut M is necessarily lower than the flow D which must go from S to T but cut M is not necessarily a minimal cut for network states with capacity values v_j with $g_j \leqslant v_j \leqslant k_j$. The decomposition principle must now be applied to the set B_j' with a different criterion, i.e. B_j' must be divided into (non-overlapping) subsets of network states with identical minimal cut.

Let $X = B_j'$. The set X can be decomposed into a subset A and new subsets B_j' $(j = 1 \dots m)$ when critical capacity values $(v_i \dots v_j^0 \dots v_m^0)$ are determined, we have :

$$x = (v_1 \dots v_m) \in A \quad \text{if } v_j \geqslant v_j^0 \quad \text{for all } j$$

$$x = (v_1 \dots v_m) \in B_j' \text{ if } v_j < v_j^0 \quad \text{for at least one } j.$$

The critical values v_j^0 are obtained as follows :

For any arc $j \notin M$, let v_j^0 be the smallest capacity value which is not less than the flow value in arc j. For any arc $j \in M$, let $v_j^0 = g_j$.

Obviously, any network state $x \in A$ has the minimal cut M associated with it. If arc r is in cut M, the value Pr $[A]$ is a part of the sought value Pr $[B^0 \mid r]$ and is to be added to the part already computed.

The network states x in a new set B'_j do not necessarily have an identical minimal cut. Therefore, each new set B'_j is to be decomposed in the same manner as the initial ones. The value $\Pr[B^0 \mid r]$ is obtained when no subset remains for decomposition.

3.3. Existence of a minimal cut

It is interesting to notice that probability $\Pr[B^0 \mid L]$ that a cut L be minimal can be found similarly by summing up all probabilities associated with sets in which the network states have L as minimal cut. Thus the problem that has been formulated in [4] by Frank and Hakimi can be solved by our approach.

3.4. Expected value of unsupplied flow

Let A be a set of non-acceptable network states with identical minimal cut and with known limiting state points. It has been shown above how an exhaustive sequence of sets A can be generated.

Let the limiting state points of A be $x = (g_1 \dots g_j \dots g_m)$ and $x = (k_1 \dots k_j \dots k_m)$ with $k_j \geqslant g_j$ for all j.

Let D be the total demand and $G(x)$ be the value of the minimal cut for state $x \in A$.

Each network state $x \in A$ gives rise to an unsupplied flow $(D - G(x))$ with a probability $\Pr[A = x] = p_x$.

The expected value $\mu[A]$ of the unsupplied flow associated with A is :

$$\mu[A] = \sum_{x \in A} p_x(D - G(x)) = \Pr[A] \cdot D - \sum_{x \in A} p_x G(x)$$

The term $\sum_{x \in A} p_x G(x)$ is the expected value of minimal cut M within set A.

Let

$$q_j = \sum_{v_j = g_j}^{k_j} p_{jv_j}$$

$$\bar{h}_j = \sum_{v_j = g_j}^{k_j} h_{jv_j} \cdot p_{jv_j}$$

M be the minimal cut for any $x \in A$.

From the fact that the expected value of a sum of independant random variables is the sum of the expected values of these variables, we have :

$$\sum_{x \in A} p_x G(x) = \prod_{j \notin M} q_j \left(\sum_{j \in M} \bar{h}_j \cdot \prod_{\substack{r \in M \\ r \neq j}} q_r \right)$$

$$= \sum_{j \in M} \bar{h}_j / q_j \cdot \Pr[A]$$

Therefore, we have :

$$\mu[A] = \Pr[A] \cdot \left(D - \sum_{j \in M} \bar{h}_j / q_j\right) \tag{1}$$

Computing the right-hand side of (1) is straightforward. Summing up the values in (1) for all generated sets A gives the expected value of the unsupplied flow.

Similarly, summing up the values in (1) for generated sets A for which the minimal cut contains arc r gives the expected value of the flow which is unsupplied when arc r is present in a minimal cut.

Finally, summing up the values in (1) for generated sets A for which cut M is minimal for any $x \in A$ gives the expected value of the flow which is unsupplied when cut M is minimal.

4. CONCLUDING REMARKS

The decomposition principle presented in this paper is general in the sense that it could be applied in any other system in which a point of the state space is defined when several random events occur simultaneously and when a point of the state space can be classified according to a given criterion.

The problem of finding the probability of meeting the demands at the nodes of a network has been treated up to now by making a large number of successive random trials (Monte-Carlo approach). The method presented in this paper is able to find an exact answer to the problem with a small computational effort. Anyway, Monte-Carlo approach is not appropriate for estimating properties of a set of state points which is reached very rarely even after a great number of random trials. For some networks that have been treated by our approach, the probability associated with all non acceptable state points was not higher than 0.001. A main reason for efficiency is that a sequence of non-overlapping sets of network states are considered instead of a sequence of single network states as in the Monte-Carlo approach.

When the demands at the demand nodes are increasing with time, the method can be easily adapted for finding the largest value of time up to which all demands can be satisfied with a given probability level.

Also, a best location for an investment on a network can be found since the responsibility of each arc for non satisfying the demands is evaluated with respect to the entire network.

The present paper can be considered as a probabilistic extension of reference [2]. In [2], the required probability level with which all demands must be satisfied has been taken into account only in an empirical way — i.e. — all demands must be satisfied even if any one arc assumes a given lower capacity.

Several numerical examples have been treated on an IBM 370-155 by a program written in FORTRAN IV. An example with about 20 random arcs usually needs less than one second of computing time.

5. NUMERICAL EXAMPLE

Consider the network in figure 5. In the same figure, the flow requirements at the demand nodes are indicated. Table 1 gives the capacity values that can be assumed by the arcs of the network and their associated probabilities.

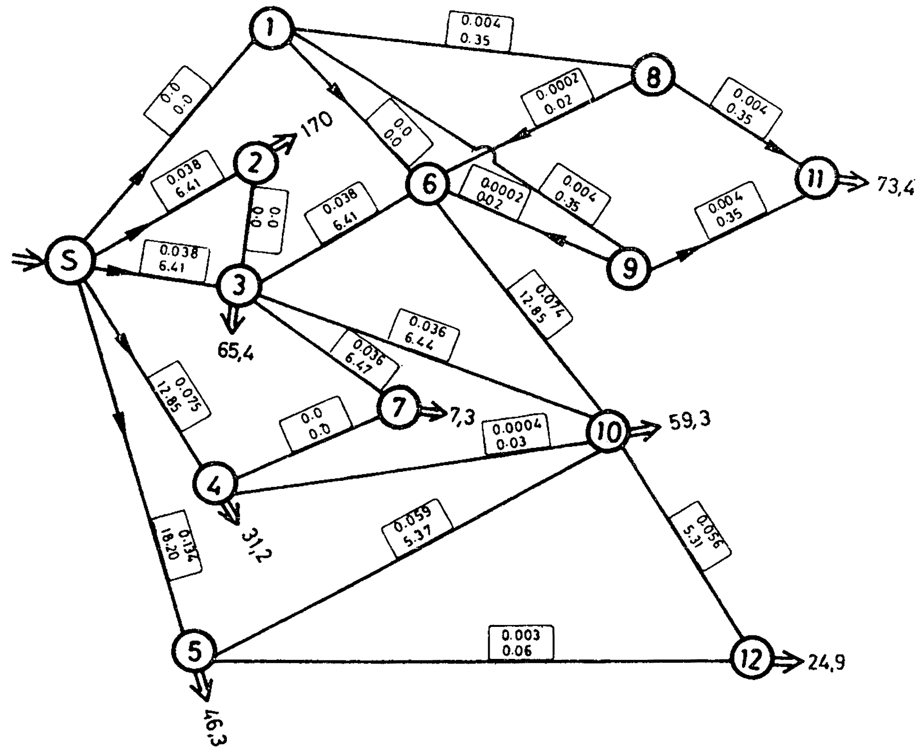

Figure 5

In figure 5, two data are associated with earch arc. The first one is the probability that the arc be found in any minimal cut and the second is the expected value of the flow which is unsupplied when the arc is present in a minimal cut.

In this example, the probability that at least one demand be not entirely satisfied is found to be 0.1418 and the expected value of the unsupplied flow is 18.93.

Looking at figure 5, it can be concluded that arc S-5 must be reinforced in priority since it is present in a minimal cut for almost each non-acceptable network state (0.134 ≃ 0.1418). Also, the expected value of the flow which is unsupplied when the arc *S*-5 is present in a minimal cut is almost equal to the unconditional expected value of unsupplied flow (18.20 ≃ 18.93).

TABLE 1

ARC	CAPACITY VALUES AND ASSOCIATED PROBABILITIES				
S-1	800 (1.0)				
S-2	0 (0.0035)	5.6 (0.0899)	11.1 (0.0709)	14.5 (0.2288)	17.0 (0.6067)
S-3	0 (0.04)	29 (0.32)	58 (0.64)		
S-4	0 (0.05)	20 (0.95)			
S-5	0 (0.05)	20 (0.95)			
1-6	0 (0.0053)	220 (0.9947)			
1-8	0 (0.0047)	145 (0.9953)			
1-9	0 (0.0047)	145 (0.9953)			
2-3	24 (1.0)				
3-6	0 (0.0225)	65 (0.9975)			
3-7	0 (0.0040)	12 (0.9960)			
3-10	0 (0.0001)	18 (0.0068)	36 (0.9931)		
4-7	0 (0.0026)	12 (0.9974)			
4-10	0 (0.0001)	18 (0.00638)	36 (0.99361)		
5-10	0 (0.0031)	14 (0.9969)			
5-12	0 (0.00001)	10 (0.00379)	28 (0.00309)	38 (0.99311)	
6-10	0 (0.0004)	65 (0.0392)	130 (0.9604)		
8-6	∞ (1.0)				
8-11	∞ (1.0)				
9-6	∞ (1.0)				
9-11	∞ (1.0)				
10-12	0 (0.00001)	24 (0.00479)	48 (0.99520)		

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