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K-CLASS ASSIGNMENTS (1)

by Charles S. TAPIERO *

Abstract. — *A multi-dimensional assigment problem is resolved by computing the maximal internally stable set of the dual of a K-partite graph.*

The Assignment Problem

The classical assignment problem is formulated in graph theory as one determining the internal stability of a dual of the bipartite graph (Berge [1]). This graph is found by replacing each arc in the bi-partite graph by a point, and letting these points be connected if they have common points in the bi-partite graph. Define G* as the dual of the bi-partite graph. Then :

1. The set of feasible assignments is defined by the set of internally stable sets in G*.

2. The set of feasible maximal assignments is defined by the maximal internally stable sets in G*, and equals the internal stability number of G*.

This particular property of the graph has been used by Hammer and Rudeanu [2] and Maghout [3] to solve the assignment problem by boolean methods. This problem is briefly formulated as follows : Given the adjacency matrix $((a_{ij}))$ of G* and given the cost of an assignment c_i,

$$\text{Minimize} \sum_{i=1}^{n} c_i x_i$$

subject to

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j = 0$$

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where n is the number of pairs in the bi-partite graph. The computational solution of such problems by boolean methods is difficult when the number of internally stable sets is large.

In this note, we shall consider a similar formulation for the multi-dimensional assignment problem. Pierskalla [4] provided a linear programming method for this problem which involves many variables. Here, a boolean approach is shown to be, in some cases, more efficient than linear programming. This occurs when the number of feasible assignments (or multi-dimensional groupings) are not too numerous. In general, the greater the dimensions of the assignment problem, the less the number of feasible grouping. Thus, while the linear programming method becomes more cumbersome with an increase in dimensions, the boolean approach becomes more efficient.

K-Class Assignments

K-class assignments are assignments along K-dimensions. For example, assume that n persons can be assigned to m machines and perform p different tasks. Each task, moreover, cannot be performed randomly on any machine. An exhaustive grouping of men-machines-tasks is called a 3-class assigment (1). The linear programming formulation is well known [4] and has n.m.p. variables whether or not the problem has many restrictions. A similar formulation can also be found by graph theory.

Define a K-partite graph as a graph whose vertices can be partitioned into K-disjoint sets $X_1 \dots X_K$ in such a way that no edge joins two vertices of the same set; and define $\Gamma_{ij} : X_i \to X_j$ as the mapping of sets X_i and X_j.Then :

Definition: Dual of a K-Partite Graph, G^*.

A K-partite graph of K disjoint sets of vertices X_1, $X_2 \dots X_K$ has a dual G^* whose sets of vertices V is the set of K-cycles in the K-partite graph. Two vertices v_1, $v_2 \in V$ are connected in G^* if they have at least a point in common in the K-partite graph.

Theorem

A feasible K-class assignment is an internally stable set of G^*, the dual of the K-partite graph.

Proof: The proof is immediate and identical to that of the classical assignment problem [1, 2].

This assignment problem can, as before, be resolved by boolean methods. Let c_i be the weight attached to all cycles of length K and let b_{ij} be the adjacency matrix of graph G^*. Then the problem is :

$$\text{Maximize} \sum_{i=1} c_i v_i$$

(1) Such groupings are naturally defined a-priori. Here a grouping is defined by a boolean equation for K-cycles.

subject to

$$\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}v_iv_j = 0$$

Therefore, it is clear that in graph theoretic (and boolean) terms, K-class assignments are identical to the classical assignment problem. The boolean approach would however be superior to the method of linear programming if there are only a few K-cycles in the K-partite graph.

EXAMPLE : Men-Jobs-Machines.

Consider a 3-partite graph with points $X = (x_1, x_2)$, $Y = (y_1, y_2, y_3)$, $Z = (z_1, z_2, z_3)$ where X are men, Y jobs and Z machine sets. The feasible matchings of men-jobs are given by a matrix $A = ((a_{ij}))$; the feasible matchings of jobs and machines are given a matrix $B = ((b_{jk}))$, while the feasible matchings of machines and men are given by a matrix $C = ((c_{ki}))$. The set of cycles of length 3 satisfy the equality

$$d_{ijk} = a_{ij} \cdot b_{jk} \cdot c_{ki} = 1$$

where ' $\cdot$ ' is the conjunction boolean operator. Edges are discovered by noting that whenever $i = j$ or $j = k$ or $i = k$, or any combination of these equalities hold, then the vertices defined by $d_{ijk} = 1$ have an edge in common. Thus if

$$A = X \overset{Y}{\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{matrix}} \qquad B = Y \overset{Z}{\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{matrix}} \qquad C = Z \overset{X}{\begin{matrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{matrix}}$$

then $d_{ijk} = 1$ for

$$v_1 = d_{111} \qquad v_2 = d_{131} \qquad v_3 = d_{222}$$
$$v_4 = d_{223} \qquad v_5 = d_{231} \qquad v_6 = d_{232}$$

A graph can be constructed by letting any two points v_k ($k = 1,6$) be connected if they have a common index in d_{ijk}. In our case, it is obvious that the adjacency matrix is as given below :

$((e_{ij}))$		v_1	v_2	v_3	v_4	v_5	v_6
	v_1	0	1	0	0	1	0
	v_2	1	0	0	0	1	1
	v_3	0	0	0	1	1	1
	v_4	0	0	1	0	1	1
	v_5	1	1	1	1	0	1
	v_6	0	1	1	1	1	0

If we further attach a weight p_i to each grouping v_i, the optimal assignment is found by solving :

$$\text{Maximize} \sum_{i=1}^{6} p_i v_i$$

subject to

$$\sum_{j=1}^{6} \sum_{i=1}^{6} e_{ij} v_i v_j = 0$$

which is in the form of the classical assignment.

Conclusion

In this note, a K-class assignment problem was formulated and resolved by boolean methods. The essential result of this note is that from a graph theoretic point of view, classical assignments and K-class assignments are similar. Computationally, although little can be claimed for boolean methods, in this particular case when the number of feasible assignments is not too large, this method is superior to linear programming methods.

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