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A note on a two-unit standby-redundant system with imperfect switchover

A NOTE ON A TWO-UNIT STANDBY-REDUNDANT SYSTEM WITH IMPERFECT SWITCHOVER

par Shunji OSAKI (¹)

Abstract. — *This paper discusses a two-unit standby redundant model with imperfect switchover. Considering the failure of the switchover device, we derive the Laplace-Stieltjes transform of the time distribution to the first system down and the mean time to the first system down under the most generalized assumptions. We further discuss the behavior after the system down.*

1. INTRODUCTION

Reliability analysis of a two-unit standby redundant model was discussed by Gnedenko et al. [1] and Srinivasan [5]. They derived the Laplace-Stieltjes (LS) transform of the time distribution to the first system down and the mean time to the first system down under the most generalized assumptions. Further investigations were made by considering the noninstantaneous switchover, the preventive maintenance, etc. (see, e.g., Osaki [2, 3]).

In this note, we shall discuss a two-unit standby redundant model with imperfect switchover. That is, we should consider the failure of the switchover device. Considering the failure of the switchover device, we shall derive the LS transform of the time distribution to the first system down and the mean time to the first system down under most generalized assumptions. We further discuss the behavior after the system down.

The analysis of our model has recourse to Markov renewal processes. For Markov renewal processes, see Pyke [4] and papers cited there. We also apply the relationship between Markov renewal processes and signal flow graphs. The detailed discussion can be found in Osaki [2, 3].

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2. MODEL

First consider a model of two identical units. We assume that the failure time distribution of each unit is an arbitrary $F(t)$ and the repair time distribution of each unit is an arbitrary $G(t)$. A unit recovers its function perfectly upon repair. It is assumed that all switchover times from the failure to the repair, from the repair completion to the standby state, and from the standby state to the operation are instantaneous. We further consider the behavior of the switchover device. We assume that the failure time of the switchover device is distributed exponentially with parameter λ_s and the repair time of the switchover device is also distributed exponentially with parameter μ_s. The behavior of the switchover device is assumed to obey the cycle of failure and repair. When the operative unit fails and at that time the switchover device is under repair, we cannot utilize a standby unit even if we have a standby unit, which means the system down. We finally assume that the behavior of the switchover device is independent of that of the two units.

Under the above assumptions and the initial condition that one unit begins to be operative, the other in standby, and the switchover device is operative at $t = 0$, we shall derive the time distribution to the first system down.

3. ANALYSIS

TABLE I. — *The states of the model (the upper bar denotes the repair)*

ABS	state s_0
$\overline{A}$BS	state s_1
A$\overline{B}$S	state s_1
$\overline{AB}$S	state s_2
AB$\overline{S}$	....
$\overline{A}$B$\overline{S}$	state s_2
A$\overline{BS}$	state s_2
$\overline{AB}$$\overline{S}$	state s_2

Label the two units A, B, and the switchover device S. The possible states of the model is shown in table I. Noting that the two units are identical, and the failure and the repair time distributions are arbitrary, we have the following three observable states :

State s_0 : One unit begins to be operative, the other is in standby, and the switchover device is operative.

State s_1 : One unit begins to be operative, the other begins to get repaired, and the switchover device is operative.

State s_2 : The system down occurs. That is, two units are under repair or failure simultaneously, or one unit fails, the other in standby, but the switchover device is under repair.

The state transition diagram (i.e. the signal flow graph) of the model is shown in figure 1. We shall obtain each branch gain of the graph.
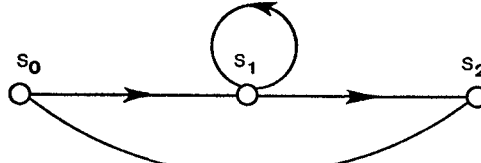

Figure 1

The state transition diagram (i. e., the signal flow graph) of the model.

First consider the behavior of the switchover device. As the switchover device repeats the cycle of failure and repair, the probability $P(t)$ that the switchover device is operative at time t, given that the switchover device was operative at $t = 0$, is given by

$$(1) \qquad P(t) = \frac{\mu_s}{\lambda_s + \mu_s} + \frac{\lambda_s}{\lambda_s + \mu_s} \, e^{-(\lambda_s + \mu_s)t},$$

which can be easily derived by the theory of queues. Further the probability $\bar{P}(t)$ that the switchover device is under repair at time t, given that the switchover device was operative at $t = 0$, is given by

$$(2) \qquad \bar{P}(t) \equiv 1 - P(t) = \frac{\lambda_s}{\lambda_s + \mu_s} [1 - e^{-(\lambda_s + \mu_s)t}].$$

The probabilities (1) and (2) will be used afterwards.

We shall derive each branch gain of the graph in figure 1. First consider the transitions from state s_0. Two transitions can be considered from state s_0 : One is to state s_1 and the other to state s_2. A transition from state s_0 to state s_1 is an event that the operative unit fails and at that time the switchover device is operative. The other transition from state s_0 state s_2 is a similar event except that the switchover device is under repair at that time. Using the probabilities $P(t)$ and $\bar{P}(t)$, respectively, we have

$$(3) \quad q_{01}(s) = \int_0^\infty e^{-st} P(t) \, dF(t) = \frac{\mu_s}{\lambda_s + \mu_s} \, \hat{F}(s) + \frac{\lambda_s}{\lambda_s + \mu_s} \, \hat{F}(s + \lambda_s + \mu_s),$$

$$(4) \quad q_{02}(s) = \int_0^\infty e^{-st} \bar{P}(t) \, dF(t) = \frac{\lambda_s}{\lambda_s + \mu_s} \, [\hat{F}(s) - \hat{F}(s + \lambda_s + \mu_s)],$$

where the LS transform of the distribution is denoted by a circumflex ($\hat{\ }$).

Next consider the transitions from state s_1. Two transitions can be considered from state s_1 : One transition from state s_1 to state s_1 is an event that the operative unit fails after the repair completion and the switchover device is operative at that time. Thus we have

$$(5) \qquad\qquad q_{11}(s) = \int_0^\infty e^{-st} P(t) G(t) \, dF(t).$$

The other transition from state s_1 to state s_2 is an event that the operative unit fails after the repair completion and the switchover device is under repair at that time, or the operative unit fails before the repair completion (the latter event is independent of the behavior of the switchover device), whichever occurs first. Thus we have

$$(6) \qquad q_{12}(s) = \int_0^\infty e^{-st} \bar{P}(t) G(t) \, dF(t) + \int_0^\infty e^{-st} \bar{G}(t) \, dF(t).$$

Noting that $P(t) + \bar{P}(t) = 1$, we rewrite (6) as follows :

$$(7) \qquad q_{12}(s) = \int_0^\infty e^{-st} P(t) \bar{G}(t) \, dF(t) + \int_0^\infty e^{-st} \bar{P}(t) \, dF(t),$$

the right-hand side of which can be similarly interpreted.

We note that

$$(8) \qquad\qquad\qquad q_{01}(0) + q_{02}(0) = 1,$$

$$(9) \qquad\qquad\qquad q_{11}(0) + q_{12}(0) = 1,$$

because we consider all the possibilities from states s_0 and s_1, respectively.

Defining that state s_0 is a source and state s_2 is a sink, and applying Mason's gain formula, we have

$$(10) \qquad\qquad \varphi_0(s) = q_{02}(s) + \frac{q_{01}(s) q_{12}(s)}{1 - q_{11}(s)},$$

which is the LS transform of the time distribution to the first system down starting from state s_0 at $t = 0$.

Modifying some branch gains and applying Mason's gain formula again, we have the mean time to the first system down

$$(11) \qquad\qquad \hat{T}_0 = E(X) + \frac{q_{01}(0)}{1 - q_{11}(0)} E(X),$$

where

$$(12) \qquad\qquad\qquad E(X) = \int_0^\infty t \, dF(t)$$

is the mean repair time. We can similarly obtain the higher moments (see Osaki [2, 3]).

4. A MODEL WHICH IS ERGODIC

In the preceding section we restricted our attention to the first system down. In this section we consider the behavior after the system down. That is, we consider a model which is ergodic.

For tractability of analysis we assume the following assumption :

(13) $$G(t) = 1 - \exp(-\mu t).$$

In the preceding section we defined only one state which denotes the system down. However, instead of state s_2, we define the following three states which denote all the system down.

State s_3 : One unit is in the operating condition, the other is under repair, and the switchover device is under repair. This state is caused by an event that we cannot utilize the standby unit because of the repair of the switchover device, which means the system down.

State s_4 : The two units are under repair simultaneously and the switchover device is operative.

State s_5 : The two units are under repair simultaneously and the switchover device is also under repair.
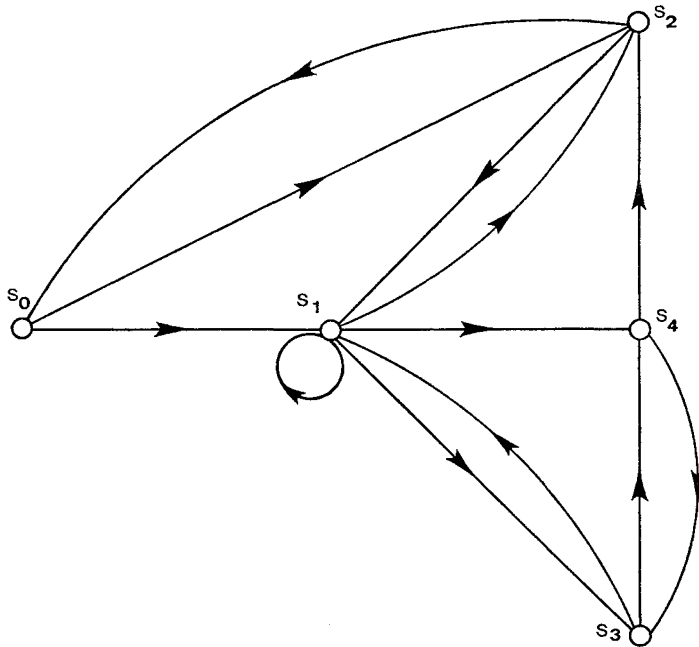

Figure 2
The state transition diagram of the model which is ergodic.

Note that the system down state is separated three states just mentioned above by its cause. Then the signal flow graph for our model is shown in figure 2. The branch gains $q_{01}(s)$ and $q_{11}(s)$ are the same in (3) and (5), respectively. The other branch gains are given by

$$(14) \qquad q_{02}(s) = \int_0^\infty e^{-st} \bar{P}(t) \, dF(t),$$

$$(15) \qquad q_{12}(s) = \int_0^\infty e^{-st} \bar{P}(t) G(t) \, dF(t),$$

$$(16) \qquad q_{13}(s) = \int_0^\infty e^{-st} P(t) \bar{G}(t) \, dF(t),$$

$$(17) \qquad q_{14}(s) = \int_0^\infty e^{-st} \bar{P}(t) \bar{G}(t) \, dF(t),$$

$$(18) \qquad q_{20}(s) = \int_0^\infty e^{-st} (1 - e^{-\mu t}) \mu_s e^{-\mu_s t} \, dt$$
$$= \mu_s/(s + \mu_s) - \mu_s/(s + \mu + \mu_s),$$

$$(19) \qquad q_{21}(s) = \int_0^\infty e^{-st} e^{-\mu t} \mu_s e^{-\mu_s t} \, dt = \mu_s/(s + \mu + \mu_s),$$

$$(20) \qquad q_{31}(s) = \int_0^\infty e^{-st} e^{-\lambda_s t} (2\mu) e^{-2\mu t} \, dt = 2\mu/(s + 2\mu + \lambda_s),$$

$$(21) \qquad q_{34}(s) = \int_0^\infty e^{-st} e^{-2\mu t} \lambda_s e^{-\lambda_s t} \, dt = \lambda_s/(s + 2\mu + \lambda_s),$$

$$(22) \qquad q_{42}(s) = \int_0^\infty e^{-st} e^{-\mu_s t} (2\mu) e^{-2\mu t} \, dt = 2\mu/(s + 2\mu + \mu_s),$$

$$(23) \qquad q_{43}(s) = \int_0^\infty e^{-st} e^{-2\mu t} \mu_s e^{-\mu_s t} \, dt = \mu_s/(s + 2\mu + \mu_s).$$

As we have obtained each $q_{ij}(s)$, we can obtain the transition probabilities, the limiting probabilities, and the renewal functions from the results of Markov renewal processes [4]. We, however, omit the results.

5. DISSIMILAR UNIT CASE

In the preceding sections we have discussed a model of two identical units. We can extend a similar model of two dissimilar units. We first consider a model which is absorbing. Then we can define four states and show the corresponding signal flow graph. Obtaining each branch gain similarly for

the graph and applying Mason's gain formula, we can obtain the system gain, which is the LS transform of the time distribution to the first system down. We can similarly obtain the mean time to the first system down.

We next consider a model which is ergodic. Then we assume that the repair time is distributed exponentially. We can define seven states and show the corresponding signal flow graph. We can similarly obtain the transition probabilities, and the renewal functions from the results of Markov renewal processes.

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