Revue d'histoire des mathématiques,
3 (1997), p. 1–48.

ALGEBRAIC GEOMETRY BETWEEN NOETHER AND NOETHER — A FORGOTTEN CHAPTER IN THE HISTORY OF ALGEBRAIC GEOMETRY

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ABSTRACT. — Mathematicians and historians generally regard the modern period in algebraic geometry as starting with the work of Kronecker and Hilbert. But the relevant papers by Hilbert are often regarded as reformulating invariant theory, a much more algebraic topic, while Kronecker has been presented as the doctrinaire exponent of finite, arithmetical mathematics. Attention is then focused on the Italian tradition, leaving the path to Emmy Noether obscure and forgotten.

There was, however, a steady flow of papers responding to the work of both Hilbert and Kronecker. The Hungarian mathematicians Gyula (Julius) König and József Kürschák, the French mathematicians Jules Molk and Jacques Hadamard, Emanuel Lasker and the English school teacher F.S. Macaulay all wrote extensively on the subject. This work is closely connected to a growing sophistication in the definitions of rings, fields and related concepts. The shifting emphases of their work shed light on how algebraic geometry owes much to both its distinguished founders, and how the balance was struck between algebra and geometry in the period immediately before Emmy Noether began her work.

RÉSUMÉ. — LA GÉOMÉTRIE ALGÉBRIQUE DE NOETHER À NOETHER — UN CHAPITRE OUBLIÉ DE L'HISTOIRE DE LA THÉORIE. — Mathématiciens et historiens considèrent en général que les travaux de Kronecker et de Hilbert inaugurent la période moderne de la géométrie algébrique. Mais on a souvent envisagé les articles correspondants de Hilbert comme une reformulation de la théorie des invariants, sujet de caractère nettement plus algébrique, alors que Kronecker était présenté comme promoteur doctrinaire d'une mathématique arithmétisée, finie. À partir de là, l'attention s'est portée sur la tradition italienne, laissant dans l'oubli la voie menant à Emmy Noether.

Et pourtant, il y eut un flux continu de publications, répondant aux travaux de Hilbert aussi bien que de Kronecker. Les mathématiciens hongrois Gyula (Julius) König et József Kürschák, les Français Jules Molk et Jacques Hadamard, Emmanuel

(*) Texte reçu le 17 mars 1997, révisé le 2 mai 1997.
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Lasker et enfin le professeur de lycée anglais F.S. Macaulay, ont tous publié abondamment sur le sujet. Ces travaux sont étroitement liés à une élaboration progressive des notions d'anneau, de corps, et autres concepts connexes. L'évolution des préoccupations que manifestent ces publications fait ressortir de combien la géométrie algébrique est redevable à ses deux éminents fondateurs, et la façon dont se présentaient les rapports entre algèbre et géométrie dans la période immédiatement antérieure aux débuts de l'œuvre d'Emmy Noether.

INTRODUCTION

While there has been a considerable amount of historical work done on many topics in the history of mathematics around 1900, algebraic geometry continues to evade discussion, perhaps as befits the difficulty of the subject. It is difficult if not impossible to obtain an adequate treatment, of reasonable length and sophistication, of many of the key figures in the period and, as I hope to show here, many of the interesting and important minor figures have been completely forgotten.

The best literature (Dieudonné [1974], Shafarevich [1974]) rightly tells a story with Riemann as a vital influence and the theories of Riemann surfaces and Abelian functions as central topics. This soon divided into a transcendental enquiry and two algebraic-geometrical ones, one more algebraic, the other more geometrical. From the transcendental and the geometrical perspectives, Picard in France, Castelnuovo and Enriques in Italy are the respective dominant figures at the turn of the 20th century.[1] The algebraic-geometrical aspect was presented most notably by A. Brill and M. Noether, with extensions by such as Bertini. There was then an arithmetical theory developed by Hensel and Landsberg. What is strangely hard to find is accounts of a strand that flourished at the same time, and which is more visible today in many versions of what may be called classical algebraic geometry. In this area two major theorems are associated with David Hilbert: the basis theorem and the *Nullstellensatz* (or theorem of the zeros). For a history of these results one must turn to two classic papers: Hermann Weyl's obituary of Hilbert, and van der Waerden's notes on Hilbert's geometrical work, published in the 2nd volume of Hilbert's *Gesammelte Abhandlungen*. Dieudonné suggested, and the simplest scratching around confirms, that one of the major figures in the creation of an algebraic geometry of n dimensions was

[1] See Gray [1989] and Houzel [1991].

Leopold Kronecker, and he compared Kronecker's work with the different but overlapping theories of Dedekind and Weber.

It is hardly surprising that mathematicians had their way with the history of such a difficult subject for so long, although there is now a much more comprehensive account by Corry [1996]. Van der Waerden's three pages offer a classic account: Hilbert's many papers are reduced to two that really matter, the turning point in mathematicians' interests is neatly characterised (away from explicit formulae and towards conceptual clarification). One tradition ends, another gets off to a fine start with papers by Lasker, Macaulay and, in due course Emmy Noether and her school. Since these are indeed the origins of the ideas that dominated the subject for so long, the effect is that of a master telling you all you need to know. One realises that the past was surely messier, but is lulled into thinking that the details would make no significant difference. Weyl's account confirms this impression. It gives more details of the work in invariant theory, but ends with the same brief claim that on the foundations of Hilbert's work was erected the modern theory of polynomial ideals (for which we read commutative algebra).

Historians of algebraic geometry have taken their cue from the mathematicians. The subject of invariant theory is notoriously difficult, and one is understandably reluctant to contest a story that says that Hilbert put an end to it. The incentive is to treat the topic as background, part of the pre-history of algebraic geometry and the history of something else (group representation theory in the manner of Weyl, perhaps). It might seem odd that Hilbert's famous theorems arise in such an algebraic setting, but the whole relationship between commutative algebra and algebraic geometry is shrouded in just such ambiguities. Zariski and Samuel called their famous book *Commutative algebra* the child of an unborn parent. The parent, never to be written, was a book on algebraic geometry, which they called "*the main field of applications of, and the principal incentive for new research in, commutative algebra*" [1958, p. v]. The importance of commutative algebra is only underlined by the more avowedly geometrical treatise of Hodge and Pedoe, who introduced their third and final volume by invoking "*the needs of those geometers who are anxious to acquire the new and powerful tools provided by modern algebra, and who also want to see what they mean in terms of those ideas familiar to them*" [1954, p. vii].

It is not part of this paper to take the story up to the present day. But it should be noted that, if one examines later books on algebraic geometry, the most important novelties are surely the introduction of cohomology theories and, after Grothendieck, the language of schemes. In many ways Grothendieck's ideas produce the unification of commutative algebra and algebraic geometry that the mathematicians discussed in this paper seem to have regarded from afar.

The purpose of this paper is, rather, to explore the various historical problems that lie hidden behind the tidy histories and mathematical complexities. First, I look in more detail at the historical literature. Then we examine what Hilbert wrote, and then we consider Kronecker's contribution, notably his *Grundzüge* [1882], and try to see what it contained and what its influence was. It might seem that anyone who has radical opinions about the meaning of terms like $\sqrt{2}$, let alone π, would be hard to reconcile with a founding father of higher dimensional geometry. Indeed, most of Kronecker's contemporary geometers surely read the *Grundzüge*, if they read it at all, as if it referred to polynomials defined over the complex numbers. On the other hand, a modern mathematician feels that Kronecker's theory lacks the tools for dealing in depth with the problems of algebraic varieties. This raises questions about the response Kronecker's work could have elicited, and in pursuing them we shall find ourselves on a route that does indeed lead from Max to Emmy Noether.

1. SURVEY OF THE EXISTING HISTORICAL LITERATURE

There may not be a large historical literature, but it is still desirable not to regurgitate large amounts of it. I shall start therefore with Dieudonné's account of the two papers that Dedekind and Weber jointly and Kronecker published in 1882, and with the ideas about divisors that they contain. Dieudonné characterised these papers as opening up the whole analogy between algebraic geometry and algebraic number theory, and with introducing many ideas of abstract algebra that have become central but which in their day delayed reception of these works. As for Kronecker, Dieudonné argued [1974, pp. 60–61] that in his *Grundzüge* he gave precise definitions of the ideas of an irreducible variety and its dimension. (Dieudonné gave no precise reference, but the idea of dimension — *Stufe* — is defined in the *Grundzüge*, §10.) In order to give an intrinsic formulation of his ideas, Kronecker worked with ideals (which he called *Modulsysteme*) in polynomial rings; irreducible subvarieties give rise to prime ideals. In refining these ideas, Lasker [1905] obtained the primary decomposition theorem which became central in any discussion of the subject.

In Dieudonné's summary, the paper of Dedekind and Weber [1882] was directed to the algebraic theory of Riemann surfaces. They started from the field of functions associated to a Riemann surface, or, rather, from an algebraic extension of the field $\mathbf{C}(z)$ of rational functions in one variable. They introduced the concept of a discrete valuation (abstracting from the concrete notion of the zeros and poles of a function on a Riemann surface) and thus could associate a point set to the original field. Had they been able to topologise this set they would have been able to complete the circle and obtain a Riemann surface from a function field. But although they could not do that, they were able to show that finite sets of points, which they called polygons or divisors, and suitable equivalence classes of these, enabled one to recapture the Riemann-Roch theorem in this abstract setting. They did this by capturing at this abstract level the relevant properties of meromorphic differentials and of the canonical divisor, whence they could give a definition of the genus of the function field.

Dieudonné's account deals briskly with the first half of the paper where Dedekind and Weber drew out the analogy between number fields and function fields. Drawing on the work of their predecessors stretching back over fifty years, they defined an integer in a function field as an element ω which satisfies an equation of the form

$$\omega^e + b_1\omega^{e-1} + \cdots + b_{e-1}\omega + b_e = 0$$

where the coefficients $b_1, \ldots, b_{e-1}, b_e$ are polynomials in z. The integral elements of a function field form a ring, which they denoted $\boldsymbol{o}$. Ideals in this ring are defined, and the standard operations on them introduced, including divisibility: the ideal $\boldsymbol{b}$ divides the ideal $\boldsymbol{a}$ if and only if $\boldsymbol{a}$ is a subset of $\boldsymbol{b}$. A prime ideal is one that is only divisible by itself and $\boldsymbol{o}$. Dedekind and Weber showed that every ideal is a product of prime ideals in a unique way, and that prime ideals correspond to points on the Riemann surface. At this point they commented, not for the first time, that the theory of divisibility was much simpler for number fields than function fields, and that in this matter the analogy broke down.

The reader of Dedekind and Weber's paper in Dedekind's *Mathematische Werke* will find at the end a one-page note by Emmy Noether, one of the editors. There she wrote that the missing third part, as she regarded it, which should have gone from the point set to the topologised surface, was later supplied by Weyl. The first part, on ideals, she oddly assimilated to the theory of hypercomplex numbers. The second part, on divisors, she regarded as taken up by Hensel and Landsberg, followed by Jung and van der Waerden. Mention of Hensel takes us naturally back to Kronecker, and so to the alternative formulation. This leads into uncharted waters, and I shall resume the matter below. Bourbaki [1965] sees the contribution of Dedekind and Weber as a significant step towards giving the theory of plane algebraic curves a solid basis, and that of Kronecker, in his *Grundzüge*, as being more ambitious but also much more vague and obscure. In contrast to Dieudonné, Bourbaki says that no definition of an irreducible variety or of dimension can be found in this memoir, although it was a source of the idea that every variety is the union of irreducible varieties of various dimensions. This disparity in views may perhaps be put down to the changing composition of Bourbaki; it would be interesting to know more about that.

The other major influence in this story is Brill and Noether's theory of algebraic curves. This was first proposed in their paper of 1874 as a way of doing algebraically and rigorously what Clebsch and Gordan [1866] had earlier tried to do in a way that mixed algebra and analysis: derive the Riemann-Roch theorem and related results. The rigour of their achievement, in 1874 and later, was later and rightly questioned, as insight into the nature of singular points grew, but the theory rests on Noether's theorem [Noether 1873]. This asserts, roughly speaking (but incorrectly): Given two curves with equations $f = 0$ and $g = 0$, and a curve $h = 0$ that passes $r + s - 1$ times through every point where f has multiplicity r and g has multiplicity s, then there are polynomials A and B such that $h = Af + Bg$, and at those points the curve with equation $A = 0$ has multiplicity $(s - 1)$ and the curve with equation $B = 0$ has multiplicity $(r - 1)$. Noether's original expression of this theorem was cast in terms of power series: If at each singular point of $f = 0$ and $g = 0$ a polynomial h is such that there are power series A' and B' such that $h = A'f + B'g$, then there are polynomials A and B such that $h = Af + Bg$.

The novelty of Noether's insight was that singular points matter; it was to be a long time before accurate statements and rigorous proofs were supplied.[2] The version just stated is deliberately over-simple, as the reader may see by letting $f(x, y) = y$ and $g(x, y) = y - x^2$. The curve with equation $x + y = 0$ is not of the form $Af + Bg = 0$. What is missing is a statement about the curve $h = 0$ passing through the origin and having a common tangent there with the curves f and g. For example, that $h = 0$ passes k times through each k-fold intersection point of $f = 0$ and $g = 0$, or, in more traditional language, that infinitely near points are taken into consideration. This is not the occasion to enter into details. Notice, instead, that the condition on the singular points can be dropped, but then some power of h may be required. This is the geometrical reason for the appearance of powers in the *Nullstellensatz.*

2. HILBERT

It is time to confront the work of Hilbert. The relevant papers occupy about 300 pages of the second volume of his *Gesammelte Abhandlungen*; for English readers the easiest place to start may be an edition of his lecture notes of 1897 [Hilbert 1897/1993] and their useful modern introduction by B. Sturmfels. Until recently, one had to lament the lack of an adequate historical literature; Reid's biography [1970] helps, but it is hardly detailed enough. The situation has changed for the better with the appearance of Corry [1996] , which discusses Hilbert's work in the context of the rise of structural algebra. The account here is necessarily briefer, and emphasises the geometrical side of Hilbert's work.

The Hilbert basis theorem for forms in any number of variables was stated for the first time in "Zur Theorie der algebraischen Gebilde, I" [Hilbert 1888]) and it was then used in the next two papers in that series before being proved in the first of the papers singled out by van der Waerden "Über die Theorie der algebraischen Formen" [Hilbert 1890]. The reason for placing the proof last may have been the wider circulation of the later paper, published as it was in the *Mathematische Annalen* rather than the *Göttinger Nachrichten*, coupled with a natural desire to

[2] I hope to tell this story elsewhere soon; for now the reader may consult the extensive discussion in Brill and Noether [1894, pp. 367–402] and Bliss [1923].

use such a powerful result. The choice of the terms *Gebilde* (varieties) and *Formen* (forms) may also have been motivated by the more geometrical spirit of the former, shorter, papers and the more algebraic, technical nature of the proof, but it is clear from everything Hilbert wrote that he thought it at most a small step from geometry to algebra and back. The first statement of the theorem runs as follows:

"*Let $\varphi_1, \varphi_2, \varphi_3, \ldots$ be an infinite sequence of forms in n variables $x_1, x_2, \ldots, x_n$, then there is always a number m such that every form in that series can be written in the form $\varphi = \alpha_1\varphi_1 + \alpha_2\varphi_2 + \cdots + \alpha_m\varphi_m$, where $\alpha_1, \alpha_2, \ldots, \alpha_m$ are appropriate forms in the variables $x_1, x_2, \ldots, x_n$.*"[3]
The use of the term "sequence" is unfamiliar to our eyes, but the language of sets was not widely used in 1888, and the idea of a sequence permits Hilbert to saying that the first so many terms form the finite basis.

The first of these four papers is a rich one, broaching the theory of syzygies, and asking for the generalisation of Noether's theorem to arbitrary dimensions, which seems to have been an early ambition of Hilbert's. The second paper shows how to use the basis theorem to illuminate the ideas of dimension, genus, order, and rank of an algebraic variety, and so makes explicit contact with Kronecker's work. Setting the third paper on *Gebilde* aside, we come to the famous paper in which the basis theorem is proved. I do not wish to discuss the proof, but to point out the geometrical applications and illustrations to surfaces passing through a twisted cubic curve, and to multiple points on a variety. Only the final section of the paper is specifically addressed to the theory of algebraic invariants.

In 1893 Hilbert returned to the subject with a series of new ideas that occupy the second paper selected by van der Waerden, "Über die vollen Invariantsysteme" [Hilbert 1893]. The aim, as Weyl pointed out, is to indicate a way in which all the invariants associated to a given form can be obtained. Although this remained the aim of some authors (see, for example, Study [1923]), others were happy to give up. Study himself quoted one, unnamed, source saying of Hilbert's work: "*Good,*

[3] "*Ist $\varphi_1, \varphi_2, \varphi_3, \ldots$ eine unendlichen Reihe von Formen der n Veränderlichen $x_1, x_2, \ldots, x_n$, so gibt es stets eine Zahl m von der Art, daß eine jede Form jener Reihe sich in die Gestalt $\varphi = \alpha_1\varphi_1 + \alpha_2\varphi_2 + \cdots + \alpha_m\varphi_m$ bringen läßt, wo $\alpha_1, \alpha_2, \ldots, \alpha_m$ geeignete Formen der n Veränderlichen $x_1, x_2, \ldots, x_n$ sind*" [Hilbert 1888, p. 176].

now we won't have to bother with invariant theory any more."[4] However, it was not easy to convert Hilbert's ideas into an algorithm; this was only done recently (see Sturmfels in [Hilbert 1897/1993]). The conceptual clarity, which is striking, contrasted with the unresolved computational difficulties, lie at the basis of the claims of van der Waerden and Weyl that Hilbert's ideas were ultimately responsible for the growth of modern abstract algebra in the 1930s. While this is partly true, the 40 year gap between cause and effect makes one want to examine the intervening period. Some indications of the vigour of the algorithmic tradition are clearer among those influenced by Kronecker, and are described below. There is surely an element of tradition-making going on, which some have not been so ready to adopt. Indeed, Jacobson (cited in [Corry 1996, p. 146]) suggested that classical invariant theory only disappeared from the textbooks after 1930 with the publication of van der Waerden's *Moderne Algebra*.

Hilbert drew particular attention to forms all of whose invariants vanish; such forms are called nullforms. For a binary quadratic form $ax^2 + bxy + cy^2$, the only invariant is $b^2 - 4ac$, the vanishing of which is the condition for repeated points (the equation $ax^2 + bxy + cy^2 = 0$ here defines points on the projective line). The same is true of binary cubic forms. By the basis theorem, all invariants will vanish if some finite set of forms vanish. Hilbert proved the important converse to this result: if a form is such that there is a set $I_1, \ldots, I_\mu$ of invariants with the property that the vanishing of the $I_1, \ldots, I_\mu$, implies that every invariant of the form vanishes , then all the invariants of the base form are polynomials in the $I_1, \ldots, I_\mu$.

To prove this result, Hilbert introduced his *Nullstellensatz*, which he stated in substantially this form in his paper [1893, p. 294], where it was also proved, and in his lectures [Hilbert 1897/1993, p. 142]):

Given m homogeneous polynomials $f_1, f_2, \ldots, f_m$ in n variables $x_1, x_2, \ldots, x_n$, and a sequence $F_1, F_2, \ldots$ of homogeneous polynomials in those variables which vanish for all the values of the variables for which the m given polynomials $f_1, f_2, \ldots, f_m$ vanish, then one can find an r such that any product of r terms from the sequence $F_1, F_2, \ldots$ can be expressed in

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[4] "*Wie schön, daß man sich nun nicht mehr mit Invarianten zu befassen braucht*" [Study 1923, p. 5].

the form $a_1f_1+a_2f_2+\cdots+a_mf_m$*, where the* $a_1, a_2, \ldots, a_m$ *are appropriate polynomials in the* n *variables* $x_1, x_2, \ldots, x_n$.

Hilbert's treatment was confined to the case where the ground field is the complex numbers. In modern terms, over an arbitrary algebraically closed ground field, this runs: Let $\boldsymbol{a}$ be an ideal in a polynomial ring, and f a polynomial that vanishes at all the points where the elements of $\boldsymbol{a}$ vanish, then $f^r \in \boldsymbol{a}$ for some positive integer r [Atiyah and MacDonald 1969, p. 85]. As Hilbert himself said in his lectures:

"*The proof of this theorem is very cumbersome, and it would lead us too far afield to discuss a proof here. But we would like to briefly demonstrate its significance through two examples*" [Hilbert 1897/1993, p. 142].

The first example concerns m binary forms, the second two algebraic curves defined by ternary forms f_1 and f_2 of orders α and β respectively and meeting in $\alpha\beta$ distinct points. The theorem states that the equation of any curve that passes through these $\alpha\beta$ points can be written in the form $a_1f_1 + a_2f_2 = 0$. This is, as Hilbert noted, a simple case of Noether's $AF + BG$ theorem, albeit one that Noether regarded as unproblematic.

Hilbert then went on to prove the converse mentioned above and apply it to give short, elegant accounts of the full system of syzygies for invariants and covariants in many low-dimensional cases. He then considered canonical forms for the nullforms (those for which as many coefficients as possible were zero) and showed that these could be written down explicitly. In his Lectures [Hilbert 1897/1993] he described what happens in complete detail in geometrical terms for forms of degree less than 6. For example, a curve defined by a quadratic equation all of whose invariants vanish is a product of two lines, and for cubic curves (defined by cubic ternary forms) all invariants vanish if and only if the curve has a cusp. As a further demonstration of the power of this theory, Hilbert showed how to use it to obtain all the classical results about the configuration of nine inflection points on a generic cubic curve, as well as how to find the coordinates of the points and of the lines through triples of them.

Hilbert's basis theorem and *Nullstellensatz* are both of immediate geometric import. Indeed, Klein said of the work of Hilbert that it "*began a new epoch in the history of algebraic geometry*".[5] We shall now see how

[5] "*Eben darum leitet diese Arbeit von Hilbert eine neue Epoche in der algebraischen*

they give geometrical meaning to results of Kronecker that were offered in quite another spirit.

3. KRONECKER

Kronecker's *Grundzüge* has a justified reputation for difficulty. Even a work like [Edwards 1990], closely as it sticks to the spirit of Kronecker's endeavour, does not set out to do justice to the size and scope of that enterprise. The problem is compounded by Kronecker's notoriously difficult style; it is easy to get lost in the march of detail. Edwards comments that "*Kronecker's theory ... did not win wide acceptance. The presentation is difficult to follow, and the development leaves gaps that even a reader as knowledgeable as Dedekind found hard to fill*" [Edwards 1980, p. 355]. Even Hermann Weyl, who in this matter is a friend of Kronecker's, had to admit that "*Kronecker's approach ... has recently been completely neglected*" [Weyl 1940, p. iii]. Finally, there is the conscious philosophy that comes with the school and cannot fairly be cut from it. The paradigm for all who worked in this tradition is not algebra but arithmetic, and it will be worth attending carefully to what they meant by that. There is real excitement for the historian here, and we can get a sense of it by attending to the sheer ambition of his project, and why it held arithmetic in such high regard.

It might be best to try and set aside what one thinks one knows about Kronecker's philosophy of mathematics. This is usually expressed in negative terms: his factorisation theory eschewed Dedekind-style naive set theory and could therefore happily announce that some number was divisible without having an object that represented its divisors (Dedekind was appalled by this); Kronecker was a strict finitist with no place for transcendental numbers, even, on some views, algebraic numbers. Thus Felix Klein said of Kronecker that "*he worked principally with arithmetic and algebra, which he raised in later years to a definite intellectual norm for all mathematical work*"[6] and: "*With Kronecker, who for philosophical reasons recognised the existence of only the integers or at most the*

Geometrie ein" [Klein 1926, p. 330].

[6] "*Indem er sich vorwiegend mit Arithmetik und Algebra beschäftigte, in späteren Jahren aber bestimmte intellektuelle Normen für alles mathematische Arbeiten aufstellte ...*" [Klein 1926, p. 281].

rational numbers, and wished to banish the irrational numbers entirely, a new direction in mathematics arose that found the foundations of Weierstrassian function theory unsatisfactory."[7] He then alluded briefly to what has become one of the best-known feuds in mathematics, the last years of Kronecker and Weierstrass at Berlin, offered his own wisdom as an old man on these matters, and observed that although Kronecker's philosophy has always attracted adherents it never did displace the Weierstrassian point of view. Finally he quoted with approval Poincaré's judgement that Kronecker's greatest influence lies in number theory and algebra but his philosophical teaching have temporarily been forgotten.

The matter is, as so often, better put positively. The first thing is the enormous range of the project. This was emphasised by his former student Eugen Netto when he surveyed Kronecker's work for an American audience, on the occasion of the World's Columbian Exposition in Chicago in 1893, two years after Kronecker's death [Netto 1896]. Netto quoted Kronecker as having said that he had thought more in his life about philosophy than mathematics, and that the expression of his philosophical views was to be found in his ideas about arithmetic. So far as possible, Kronecker wanted a common method for dealing with all the problems of mathematics that come down to properties of polynomials in any finite number of variables over some field, usually the rational numbers. In the strict sense in which Kronecker intended to be understood, the ground field is at most an algebraic extension of a pure transcendental extension of finite transcendence degree of the rationals.

We shall see that there were those, like Molk, who accepted this starting point, and others, like König, who preferred to start with the complex numbers. So Kronecker's subject matter included all of algebraic number theory, and, geometrically interpreted, the theory of algebraic curves and, insofar as it existed, the theory of algebraic varieties of any dimension. This is why he occupies what might otherwise seem an unexpected place in the history of early modern algebraic geometry. The fact that the ground field is not the complex numbers, nor even algebraically closed,

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[7] "*Mit Kronecker, der auf Grund philosophischer Betrachtungen nur den ganzen, höchstens noch den rationalen Zahlen wirkliche Existenz zuerkannte, die irrationalen Zahlen aber ganz und gar verbannt wissen wollte, entstand eine neue Richtung in der Mathematik, welche die Grundlagen der Weierstraßschen Funktionentheorie nicht als befriedigend ansehen wollte*" [Klein 1926, p. 284].

need not be an insuperable problem: a great deal of algebraic geometry can still be done by passing, if need be, to successive algebraic extensions. What Kronecker could not do, according to his lights, is pass to the full algebraic closure of the rational field. While this would not necessarily be a significant mathematical problem for any one drawn to this approach, the historian cannot escape so easily, however, as we shall see when we discuss the limited references Kronecker actually made to geometry.

The analogy between these algebra and arithmetic, which will be discussed a little below, is a real one, and by refining the question they share of finding common factors Kronecker sought to exploit to the benefit of all the various aspects. It is the analogy with algebraic number theory that drove him to call his theory arithmetic, rather than merely algebraic. The basic building blocks were two things: the usual integers and the rational numbers, on the one hand, and variables on the other. These were combined according to the usual four laws of arithmetic; root extraction was to be avoided in favour of equations (for example, the variable x and the equation $x^2 - 2 = 0$, rather than $\sqrt{2}$).

Kronecker on discriminants

Kronecker himself set out the thinking that led him to his general programme in a fascinating preface to a paper "Über die Discriminante algebraischer Functionen einer Variabeln" [Kronecker 1881]. The preface is a lengthy historical account indicating how much he had already proposed in lectures at the University of Berlin (and who his audience had included) and at a session of the Berlin Academy in 1862. The guiding aim, which he traced back to 1857 (the date, one notices of Riemann's paper on Abelian functions) was to treat integral algebraic numbers (for which the modern term is algebraic integers). These he defined as roots of polynomial equations with leading term 1 and integer coefficients. He encountered certain difficulties, which is where the discriminants come in. The resolution of these problems came with the insight that it was a useless, even harmful restriction to consider the rational functions of a quantity x that satisfies an algebraic equation of degree n only in the form of polynomials in x (i.e. as linear homogeneous functions of $1, x, \ldots, x^{n-1}$). It would be better, he realised, to treat them as linear homogeneous forms in any n linearly independent functions of x. This made it possible to represent complex numbers by forms, in which every algebraic integer appeared as an integer while circumventing the difficulties. The insight may be put another way: an irreducible polynomial of degree n with distinct roots defined n quantities at once, and it can be shown that it cannot share a subset of these roots with any other irreducible polynomial. By picking on one root, problems arise that can be avoided by treating all the roots simultaneously.[8] (If you like, study $\pm\sqrt{2}$, but not just $\sqrt{2}$.)

He discussed these results with Weierstrass, who was his friend at the time, and Weierstrass urged him to apply the same principles to algebraic functions of a single variable and if possible to the study of integrals of algebraic functions, taking account of all possible singularities. This set him on the road to a purely algebraic treatment, shunning geometric or analytic methods. He sent the first fruits to Weierstrass in October 1858, but Weierstrass's own results rendered his superfluous in his own eyes and so he refrained from further publication. Kronecker was brought back to the topic by discovering how much his thoughts coincided with those of Dedekind and Weber (an agreement which did not, he noted, extend to the basic definition and explanation of the concept of a divisor). Therefore he presented his old ideas, abandoned in 1862, for publication in 1881.

Kronecker's Grundzüge

Kronecker's *Grundzüge* is a lengthy work, and an unrelenting one. Happily, I may invite readers consult Edwards [1990] for a thorough mathematical commentary, where they will learn amongst other things of the unproven claims Kronecker made. This mixture of great claims for the rigour and immediacy of the theory and the absence of proofs at crucial points that surely contributed to the work's poor reception. Dedekind's rival version, attacked by Kronecker for its abstraction, was not so embarrassed. Kronecker does not seem to have had the knack of conveying in print what was known, what can be done, and what might be discovered in a way that would drive future research.

Kronecker's *Grundzüge* is about objects which are so common in mathematics that it is hard to know what to call them. They are polynomials in several variables and their quotients by other such polynomials. A collection of rational functions in some (finite set of) indeterminates R_i

[8] Netto [1896, p. 247] observed that Kronecker's factorisation theory depended on being able to work with all of a set of conjugate algebraic numbers at once, in order to use Galois theory.

closed under the four operations of arithmetic Kronecker called a *Rationalitätsbereich*, which I translate as domain of rationality. A *ganze* or integral function in a domain is a polynomial in the R's. The old name of "rational functions" (which Kronecker used) survives, although they are not thought of as functions but expressions. The coefficients may be integers, rational numbers, or elements of some field (another term we shall have to return to). The crucial thing about these objects is that they can be added, subtracted, multiplied, and divided (of course, one cannot divide by zero). Kronecker also insisted that there was no question of order here, such expressions are not greater or less than others (which rules out use of anything like the Euclidean algorithm). He also explicitly wished to avoid geometrical language.

Kronecker's *Grundzüge* places three obstacles in the way of comprehension. One is the number of unproven claims that are made. Another is the style, which mixes up what is proved with unproven claims about what is true. The third is the delicate, and often heavily computational nature of the material. Each of these difficulties calls for comment. The existence of unproven claims is not necessarily a barrier to the acceptance of a work; it may function as a challenge to later workers. However, the failure to meet these challenges, coupled with the opportunity of switching to a rival theory that did not have these disadvantages, was to prove crucial in the demise of Kronecker's approach. As for the style, unexpected though it may be in a mathematician with a strong axe to grind about what is the right way to do mathematics, it is typical of the period. Long papers and books were designed to be read; they were seldom presented in the style of definition, theorem, proof that came in later, with Landau. It is tempting to imagine that it was the obstacles presented by such works as Kronecker's *Grundzüge* that pushed Landau to accentuate the division between mathematics and literature. It is also noticeable that Hilbert's writings were much more lucid and carefully structured so that they could be easily understood. The final obstacle would be a virtue if the computational machine gave acceptable answers. The problem seems to have been that it did not.

Edwards' analysis of Kronecker's theory praises it for being based on the idea of divisors, and in particular the idea of greatest common divisors (when they exist). The greatest common divisor of two elements of a field is, Edwards points out [1990, p. v], independent of the field: if two objects have a third as their greatest common divisor this third object remains their greatest common divisor even if the field is extended. This is not the case of prime elements: an element may be prime in one field but factorise in an extension of that field.[9] Kronecker claimed to have a method for factorising a given divisor as a product of prime divisors. His successor Hensel gave proof that this method works in the context of algebraic number fields, and echoed Kronecker's claim that the method worked in general. But he never gave such a proof, and Edwards wrote in 1990 that he did not know of one.

Edwards makes light, however, of what seemed to every one at the time to be an immediate problem with divisor theory: there simply may not be a greatest common divisor of two elements. Rings and fields with greatest common divisors include all the so-called "natural domains"; these are the domains which are either the rational numbers or pure transcendental extensions of the rational numbers. They are to be contrasted with algebraic number fields, which typically do not have greatest common divisors. All writers (Kronecker, Molk, König) give the same example, due originally to Dedekind, because it is the simplest: algebraic integers of the form $m + n\sqrt{-5}$. It is easy to show that $2 - \sqrt{-5}$ is irreducible, but it is not prime. Indeed, $(2-\sqrt{-5}\,)(2+\sqrt{-5}\,) = 9 = 3\times 3$, but $2-\sqrt{-5}$ does not divide 3 (the solutions, x and y, of the equation $(2-\sqrt{-5})(x+y\sqrt{-5}) = 3$ are not integers). So the algebraic integers 9 and $3 \times (2 - \sqrt{-5}\,)$ have no greatest common divisor: their common divisors are 1, 3 and $2-\sqrt{-5}$, but neither of 3 and $2 - \sqrt{-5}$ divides the other. It was exactly this problem that caused Dedekind to formulate his theory of ideals, which invokes ideals that are not principal (generated by a single element) precisely

[9] The theory of factorisation takes place in rings, usually rings of integers in some field. When this field is extended, the ring of integers may be enlarged. This process has no effect on greatest common divisors, but it may produce factors of previously prime elements of the initial ring. Edwards' use of the term "field" is typical of Kronecker's day.

to the get round the problem. In König's terminology, rings and fields having greatest common divisors are called complete, those like $\mathbf{Q}(\sqrt{-5})$ are incomplete. In the Kroneckerian approach the whole analysis grinds to a halt with incomplete domains. Netto [1896, p. 252] observed that finding irreducible factors was an open question once the existence of greatest common divisors failed.

In this paper, which is devoted to the history of some ideas in algebraic geometry, it might be possible to set problems with algebraic number fields on one side. It was generally agreed throughout this period that in geometry the subject matter was polynomials in a certain number of indeterminates and whose coefficients were complex numbers. The field of complex numbers is a natural one, and so the difficulty with greatest common divisors does not arise. But before this problem can be set aside, we must at least notice that doing so violates Kronecker's philosophy. Whatever his motivation might be, his *Grundzüge* is a non-geometrical work. In reading it for its geometrical meaning and seeking to connect it to contemporary geometry, mathematicians were reading it against the grain. They had first to jump from the kind of fields that Kronecker was prepared to contemplate to the complex numbers; when writing in the spirit of his *Grundzüge*, Kronecker did not recognise that there was such an object as the field of all complex numbers. Then mathematicians had to turn their back on the unity that Kronecker was trying to forge: a coherent domain of objects studied (he hoped) by uniform methods.

Kronecker's most powerful influence was exerted on the handful of students around him at any time in Berlin. For that reason it is worthwhile looking at the posthumous volume of his Lectures [Kronecker 1901], edited by Hensel. This was largely based on his lectures in the 1880s, but they have the advantage over the *Grundzüge* of being both more elementary, clearer about what has been proved, and more geometric. It may be the case that the lectures were more influential that the *Grundzüge*, or at least that the combination of lectures and conversation was more potent (Kronecker was a sociable mathematician, see [Biermann 1973]). Best known among his followers are Hensel (who knew him personally, succeeded him at Berlin, and edited a volume of Kronecker's *Vorlesungen über Zahlentheorie* in 1901) and Landsberg. Another was Eugen Netto, whose two-volume *Algebra* [1896–1900] is his account of ideas in the

Grundzüge. Based on his two years of study in Berlin, Molk wrote his account of those ideas as his doctoral thesis and as an article in *Acta mathematica* [1885]. Slightly less readable, in English or French, are the presentations of the American mathematician Harris Hancock, who heard the last lecture course Kronecker gave, and subsequently published his version as part of his thesis (see [Hancock 1902]). His interminable book [1931–1932] is an extended comparison of the approaches of Dedekind and Weber and Kronecker.[10]

Kronecker's lectures

In his *Vorlesungen über Zahlentheorie* [1901, Lecture 13] Kronecker defined a domain of rationality determined by an indeterminate R as the totality of all products and quotients of polynomials in R (division by 0 being excluded). He denoted it (R) — we would write it as $\mathbf{Q}(R)$. He showed that it consisted precisely of all rational functions of R with integer coefficients. If division was not allowed, a subdomain of (R) was constructed which he called a domain of integrity (*Integritätsbereich*) and denoted $[R]$ — we would write it as $\mathbf{Z}[R]$. Kronecker observed that if in particular the indeterminate R is set equal to 1, then $(R) = (1)$ is the usual rational numbers, and $[R]$ is the integers. The same construction can also be carried out with finitely many indeterminates.

Divisibility in any domain of integrity had the natural meaning that m divides a if and only if there is an integer c in the domain such that $a = cm$. Kronecker chose to write this in the formalism of congruences to a modulus, and therefore spoke of modular systems (*Modulsysteme*). When $a = cm$ he said that a was contained in the modular system (m), and more generally he said that a form a was contained in a modular system $(m_1, m_2, \ldots, m_r)$ or was divisible by the modular system if $a = c_1m_1 + c_2m_2 + \cdots + c_rm_r$, for some integers $c_1, c_2, \ldots, c_r$. He said that two modular systems were equivalent if each was contained in the other.

He then turned to divisors. The greatest common divisor of the

[10] In all of this activity there are some curious, even unexpected, allegiances. One might have expected Hensel and Landsberg's book to be dedicated to Kronecker, but in fact their book is dedicated to Dedekind. Likewise one might guess that Heinrich Weber's book on algebra would be in Dedekind's style, and so it is, but it was regularly and rightly cited in its day for its thorough account of Kronecker's approach.

modular systems $(m_1, m_2, \ldots, m_r)$ and $(n_1, n_2, \ldots, n_s)$ was, he showed, the modular system $(m_1, m_2, \ldots, m_r, n_1, n_2, \ldots, n_s)$, and the composition of two modular systems $(m_1, m_2, \ldots, m_r)$ and $(n_1, n_2, \ldots, n_s)$ was, he also showed, the modular system $(m_i n_j\ ;\ 1 \le i \le r,\ 1 \le j \le s)$. This was later called the product of the two modular systems.

In Lecture 14, Kronecker restricted his attention to polynomials in a single variable and integer coefficients. In this domain a generalisation of the Euclidean algorithm permitted him to find the greatest common divisor of two polynomials, as follows. Given $f_1(x)$ and $f_2(x)$, with $f_1(x)$ of higher degree than $f_2(x)$, one can write

$$f_1 = q(x)f_2 + r(x),$$

where the degree of $r(x)$ is less than that of f_2, but it is not certain that the coefficients of $q(x)$ and $r(x)$ will be integers. By following this complication through, Kronecker deduced that: If $f_1(x)$ and $f_2(x)$ are polynomials with integer coefficients, then by successive division one can find a third polynomial $f_n(x)$ with integer coefficients and two integers s_1 and s_2 such that $f_n(x) \equiv 0 \pmod{f_1(x), f_2(x)}$ and $s_1 f_1(x) \equiv s_2 f_2(x) \equiv 0 \pmod{f_n(x)}$. The polynomial is the greatest common divisor of $f_1(x)$ and $f_2(x)$ if and only if $s_1 = s_2 = 1$.

Divisor or modular systems $(f_1(x), \ldots, f_n(x))$ that are equivalent to a system with just one element $(f(x))$ he called modular systems of the first level or rank, all others of the second level. An example of the first kind was $(3x - 3, x^2 - 1, x^2 + x - 2)$; of the second kind $(m, x - n)$, where $m > 1$. A modular system was said to be pure if the defining terms have no common factor, otherwise mixed. So $(3, x-1)$ is pure, and $(3(x^2+1), (x-1)(x^2+1))$ is mixed. To use Dedekind's language harmlessly here, an element of a domain of integrity belonged to a modular system if it was in the ideal the modular system generated.

Pure divisor systems of the first level lead to the study of polynomials with integer coefficients and their irreducible factors. In Lecture 15, Kronecker proved the unique decomposition theorem in this context. Let the given polynomial be

$$F(x) = c_0 + c_1 x + \cdots + c_n x^n.$$

The Euclidean algorithm finds the greatest common divisor of the coefficients, m say. Factoring it out leads to the polynomial with integer coefficients

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n.$$

The possible degrees of any factor of $f(x)$ can be found easily, but finding the coefficients of any factor would be harder. Kronecker offered an argument based on the Lagrange interpolation formula to show how one could find out if $f(x)$ had a divisor of any given degree μ, and to determine the coefficients of the divisor if it exists, in a finite number of steps. This insistence on exhibiting a finite process is characteristic of Kronecker.

He then defined a prime function as an integer or a polynomial that is not divisible by any other in the domain, and proved that if a product $\phi(x)\psi(x)$ is divisible by a prime function $P(x)$ then at least one of $\phi(x)$ and $\psi(x)$ is so divisible. Finally he showed that a polynomial with integer coefficients can be written in essentially one way as a product of prime functions.

In Lecture 20 he broached the generalisation to modular systems in more than one variable, without going into the proofs in detail. He showed that an integral quantity $F(x, y)$ can be factored uniquely into irreducible or prime functions, by treating it as a polynomial in y with coefficients that are polynomials in x. Kronecker now admitted what he called arbitrary, not merely integral coefficients (it is not clear this means complex numbers!) so every element in the domain of integrity $\{x, y\}$ therefore corresponds to an algebraic equation $F(x, y) = 0$ and so to an algebraic curve. Similarly, the elements in the domain of integrity $\{x, y, z\}$ correspond to algebraic surfaces.[11]

Kronecker considered the equations divisible by the modular system $(f_1(x, y), \ldots, f_n(x, y))$ in $\{x, y\}$, and showed that they corresponded to curves through the common points of the curves corresponding to the equations $f_1(x, y) = 0$, $\ldots$, $f_n(x, y) = 0$. Such points he called the fundamental or base points of the system. Two modular systems are equivalent if they generate the same ideal; for this to happen in $\{x, y\}$ it is necessary but not sufficient that they have the same base points;

[11] Kronecker or his editors denoted a domain here by parentheses { }, and in his *Grundzüge* by brackets [ ].

he gave the example of (x^2, y) and (x, y^2), which are not equivalent but have the same base point (the coordinate origin $(0, 0)$). Similar if vaguer remarks followed about $\{x, y, z\}$.

The defining elements of a modular system in two variables may have a common curve and also meet in isolated points. If there are no such isolated points, Kronecker called the system is pure, otherwise mixed.

If the elements of a modular system are polynomials in several variables with integer coefficients, then whether the domain is $[x, y, z]$ or $\{x, y, z\}$ matters. Rather than recapitulate Kronecker's definitions, here is one of his examples. In the domain $\{x, y, z\}$ (with arbitrary coefficients) a quantity $f(x, y, z)$ is a prime divisor of the first level if and only if it is irreducible, and so the equation $f(x, y, z) = 0$ defines an indecomposable algebraic surface F in 3-dimensional space. If $g(x, y, z)$ is another integral quantity from the same domain and G the corresponding surface, then either g is divisible by f, or (f, g) is a modular system of the second level. In the first case, the surface F is part of the surface G. In the second case the modular system (f, g) corresponds to the complete intersection of the two surfaces F and G, and therefore to a space curve C. In this latter case, if the modular system is a prime modular system then the curve is irreducible.

If a third quantity $h = h(x, y, z)$ is taken in $\{x, y, z\}$ then either h is divisible by the prime modular system (f, g) or the modular system (f, g, h) is of the third level, and the three surfaces F, G, and H have only isolated points in common. Whence the theorem: an irreducible space curve and an algebraic surface either have a finite number of common points, or else the curve lies completely in the surface. Similar considerations allowed Kronecker to give what he called a complete overview of the geometrical interpretation of the purely arithmetical idea of a prime divisor: divisors of the first, second and third kinds in the domain $\{x, y, z\}$ correspond to algebraic surfaces, algebraic curves, and points; prime divisors of these kinds correspond to indecomposable surfaces, irreducible space curves, and isolated points.

The *Grundzüge* goes over the ground of the *Vorlesungen*, in a more visionary way. As far as geometry is concerned, he noted (§21) that there was a connection with the theory of hypercomplex numbers. When three variables are taken as coordinates of space, divisors of the first level

(*Stufe*) are either numbers or polynomials in x, y, z, the vanishing of which represents a surface. Modular systems of the second level represent either a number or a curve, of the third level, sets of points. A modular system of level n was defined by Kronecker to be of the principal class if it was defined by n elements. So, in the principal class of each divisor system of the second level are those curves which are the complete intersection of two surfaces. Kronecker commented that this, surprisingly, is a higher viewpoint from which the representation of integers as norms of complex numbers and the isolated representation of geometric figures are seen to be intimately related.[12]

4. UNCHARTED WATERS

Let us now look at the work of those who took up Kronecker's ideas. An influential, if perhaps unexpected, follower was Jules Molk, who studied in Berlin from 1882–1884, where he was drawn above all to the teaching of Kronecker. On his return to Paris he took his *Doctorat ès sciences* at the Sorbonne in 1884; we may read this thesis, lightly revised, in *Acta mathematica* [Molk 1885]. It is a summary, with a few simplifications, of Kronecker's ideas, coupled with strongly worded claims for its merits; Netto [1896, p. 247] called it a very thorough and well arranged presentation (*sehr eingehende und übersichtliche Darstellung*). Some years later, Molk arranged for an extensive reworking of Lansdberg's article on divisor theory in the *Encyklopädie der mathematischen Wissenschaften* for the French *Encyclopédie des sciences mathématiques pures et appliquées*, of which Molk was editor-in-chief. The authors of that article were Kürschák and Hadamard. In 1903 and 1904 the Hungarian mathematician Gyula König wrote the first textbook on the subject (first in Hungarian and then in German); in Molk and König we have two valuable guides to the *Grundzüge*, with the essays by Landsberg and by Hadamard and Kürschák to take us further. A paper and a book by the English mathematician F.S. Macaulay bring this journey to an end.

5. MOLK

In his long paper of 1885, Molk distinguished between natural and

[12] The point was echoed almost verbatim by Netto [1896, p. 249].

general domains of rationality. In the general domain there may be algebraic relations between the indeterminates. Having shown that there was a good theory of divisibility in the natural case, he set about decomposing integral quantities in a general domain. As he put it, he wished to show both the important role played in modern research by algebraic functions, and the great simplifications which their use allows, and also,

"*la méthode à suivre pour éviter précisément l'emploi de ces fonctions. La complication de cette méthode n'est qu'apparente et ne porte* que *sur le mécanisme de la démonstration; loin de rendre la démonstration elle-même plus difficile, elle nous fait, au contraire, apercevoir plus clairement le lien entre les hypothèses que nous faisons et le résultat qui en découle, entre notre point de départ et notre point d'arrivée; et* seule, *elle mérite le nom de méthode algébrique, car, seule, elle se meut dans le domaine particulier à l'algèbre*" [Molk 1885, p. 65].

Given the elements of a domain (which I shall denote D) formed from the indeterminates $R_1, \dots, R_n$ that are related by an irreducible algebraic equation $\Psi = 0$, the problem is to decompose a polynomial $F(z)$ with coefficients in the domain D into factors. To do this, Molk first found the resultant, which he called S, of the functions $F(z+tR_1)$ and Ψ with respect to the indeterminates $R_1, \dots, R_n$. He then considered the resultant which he called T, of S and its derivative $\partial S/\partial z$ with respect to z, and showed that T was necessarily non-zero. At the end of the lengthy proof, which I shall not attempt to summarise here, Molk commented:

"*Cette recherche est nouvelle. Elle offre un exemple frappant de l'avantage qu'il y a à se servir de méthodes naturelles, sans introduire aucun élément étranger au domaine dans lequel on se meut. C'est, en effet, l'impossibilité dans laquelle je me suis trouvé, de démontrer que le résultant* $T(t, \psi_1, \psi_2, \dots, \psi_n)$ *est différent de zéro, sans supposer l'existence des racines des équations algébriques, et à l'aide de la généralisation des idées de contenant et de contenu donnée au début de ce chapitre, qui a amené M. Kronecker à généraliser d'avantage encore les idées de contenant et de contenu en découvrant le théorème* auxiliaire *nécessaire à notre démonstration, théorème qui, en réalité, est* fondamental *en Algèbre*" [*Ibid.*, p. 75].

Molk then observed that it was now easy in a natural domain of rationality to decompose a system $[F(z), \Psi]$ into other systems each containing Ψ. The proof, he said, followed Kronecker's Lectures of 1883. And finally, on p. 79, he concluded the entire proof, observing that it had made no use of the ideas of algebraic numbers or algebraic functions.

As to the decomposition of modular systems or divisor systems, Molk began by reviewing the discussion provided by Kronecker of points, curves, and surfaces (he did not mention Kronecker by name at this point). Then he switched to algebra. He took a domain of integrity as fixed, and said that a function in the domain *contains* a system of divisors when it can be represented by a homogenous linear function of elements of the system with coefficients that also belong to the domain. The theory of the resultant in this setting rapidly becomes complicated, requiring certain linear changes of variable to avoid misleading results. Molk agreed with Kronecker that the best way forward was to work systematically with new indeterminates — a move that was not to win converts to the theory. They permit the introduction of a technical tool called the resolvent (which I shall not define here); it can be thought of as extending the use of the resultant of two polynomials to the case where the number of equations exceeds the number of unknowns. But even in this case a real difficulty arose when the resolvent had multiple factors. The question is whether to each decomposition of the resolvent there corresponds a decomposition of the divisor system. If the resolvent has no multiple factors, then this is the case. When there are multiple factors, this can fail. Molk gave this example, taken from Kronecker's *Grundzüge* (§21). The divisor system $(x^2 + y, y^2)$ is certainly not irreducible, because the system $(x^2 + y = 0, y^2 = 0)$ contains the system $(x = 0, y = 0)$. However, and this reveals the "*différence essentielle*" [Molk 1885, p. 106] between divisors of rank 2 and those of rank 1, it is easy to see that the system $(x^2 + y, y^2)$ does not decompose into two systems of which one is (x, y). Therefore there are systems which are not decomposable and are not irreducible.

Rather than enlarge the idea of decomposition, and thereby lose its essential character of separation, Molk thereafter restricted his attention to systems which can be obtained by arbitrary composition of irreducible systems of ranks 1 and 2. This led him to describe a general theory of elimination and the decomposition of systems in only two variables, leaving a general theory without recourse to algebraic functions for the future. Molk's general theory of elimination proceeded at such a general level, without recourse to a single example, that it remains obscure. The task is to re-write a system of m equations $G_i(x_1, \ldots, x_n) = 0$ in n variables in one equation (called the resolvent) of the form $R_1 \cdot R_2 \cdots R_n = 0$. This being done, the task then became to analyse each polynomial R_i. The conclusion, which Molk then expressed in geometrical language, was that:

"*Toute variété k-ième, prise dans une variété n-ième, peut être figurée par une variété k-ième, prise dans une variété* $(k+1)$*-ième, — par une variété dont le résolvant est, par suite, de rang un — et cette dernière variété k-ième peut être choisie arbitrairement parmi tous les individus faisant partie d'une classe déterminée*" [*Ibid.*, p. 155].

More work allowed Molk to claim that "$(n+1)$ *équations suffisent toujours et sont en général nécessaires pour isoler* [...] *une variété d'ordre quelconque, prise dans une variété n-ième*" [*Ibid.*, p. 163].

At the end of his paper, Molk made a number of claims about the importance of this type of work. He had shown how to decompose systems of polynomials and explained what was meant by equivalent decompositions (cf. *Grundzüge*, §20). This ultimately allowed Molk to say that the most general domain of rationality possible was indeed what he had called the general domain of rationality (cf. *Grundzüge*, §10).

The concept of content, generalising that of divisibility, was extolled even as it too was found to be insufficient. The concept of greatest common divisors was placed at the basis of the whole work. The theory of elimination was found to have implications for the geometry of higher dimensions.

6. KÖNIG

The work of Gyula (Julius) König, published simultaneously in German and his native Hungarian, has become almost forgotten. The passing reference to it in van der Waerden's essay is not entirely accurate. It runs, in its entirety, as follows: "*Elimination theory was developed by Kronecker and his school*". Werner Burau, in his article in the *Dictionary of scientific biography* on König, however, says that "*König had had very little personal contact with Kronecker*", and indeed König spent his working life in Budapest, having studied at Vienna and Heidelberg. He did not claim any personal acquaintance with Kronecker or reliance on correspondence or letters. Rather, he seems to have set himself the task, as he turned fifty, of writing a useful book sorting out an important topic for which no guide exists. Such a work, if successful, will draw others into the field who will go on to discover better results, simpler and more general methods, and if it does not attain the status of a classic gradually the work will be covered up and forgotten. Such, at all events, was the fate of König's *Einleitung in die allgemeine Theorie der algebraischen Gröszen* [1903]. In view of its importance in its day, it is worth saying a little about König himself.

The man and his work are well described in Szénássy [1992], where he rates a chapter to himself, and much more information, including a whole book on him by Szénássy, is available in the Hungarian literature (which I regret I cannot read). Szénássy called König "*a great man of the nation*" (p. 333) and credits him with establishing Hungarian mathematics as a significant force. This he did as much by his own work as by his magnetic personality and the breadth of his organisational work: training teachers and engineers as well as professional mathematicians, lecturing on everything from pure analysis to economics and history of mathematics. Szénássy writes (p. 241) that Hungarian "*secondary school education benefited for decades from his textbook on algebra*". König helped found the Hungarian Mathematical Society, worked with publishers, and was three times Rector of the Technical University. In research, it was his habit to work on one area of mathematics at a time, publish several papers and then a monograph summarising the field, and then move on. He worked on algebra, then analysis and partial differential equations, and finally on Cantorian set theory, where he is better remembered for his unsuccessful attempt on Cantor's continuum hypothesis than for several smaller but secure contributions (see [Moore 1982, p. 86]).

Szénássy's discussion of König's *Einleitung* is rather brief, and although he points out the debt to Kronecker and the extent of the new material, much of it by König himself, it masks the importance of the book by listing its main topics in unduly modern language. In fact, the book possesses two aspects of interest to us. One is the novel mathematical concepts it introduces; the other is the insights of a sharp critic of the period.

From the standpoint of the early history of field theory, König's book introduced some useful terminology and made some interesting distinctions. He based his account on the twin concepts of an orthoid domain and a holoid domain. An orthoid domain corresponds exactly to our concepts of a field (of characteristic zero) and a holoid domain to our (commutative) ring with a unit 1 such that no sum of the form $1+1+\cdots+1$ vanishes; Szénássy incorrectly glosses a holoid domain as an integral domain. König gave no rationale for the terms; presumably he had in mind the Greek roots *holo* for whole or entire and *ortho* for straight or right. He advocated the terms holoid and orthoid to express general properties of domains, by analogy with the integers and rational numbers.

As König saw it, a field, a *Körper* in Dedekind's terminology, is an orthoid domain (certain vaguenesses in Dedekind's definitions, and certain methodological differences, aside). But a field or orthoid domain is not the same concept as Kronecker's domain of rationality. König argued first that Kronecker's natural domains were obtained by taking any finite set of μ elements from any holoid or orthoid domain, and forming the function field in them (so non-trivial relations may exist among these generators). This gave an orthoid domain that, he said, Kronecker called a domain of rationality. It might be that just one element was chosen, and it was equal to 1, in which case the absolute domain of rationality was obtained (i.e. the rational numbers). If the μ quantities were completely undetermined the resulting domain was the natural domain of rationality in μ indeterminates. It follows that every domain of rationality is an algebraic extension of a natural domain of rationality. His proof was to pick x and form polynomials in x with at least one non-zero coefficient. Either none vanish or one at least does. In the first case, the domain of rationality is (x), i.e. $\mathbf{Q}(x)$. In the second case, x is an algebraic number and (x), i.e. $\mathbf{Q}(x)$ is an algebraic extension. The proof for any number of elements x follows by induction.

Conversely, if all the elements of an orthoid domain can be written in the form $r_1\omega_1 + \cdots + r_n\omega_n$, where the r_i belong to a natural domain of rationality $D = (x_1, \ldots, x_m)$ then the orthoid domain is a domain of rationality. For this, he first showed that every ω in the orthoid domain satisfies a polynomial equation with coefficients in the natural domain of rationality D. In particular, this is true of the quantities $\omega_1, \ldots, \omega_n$ so D and the domain of rationality in which $x_1, \ldots, x_m, \omega_1, \ldots, \omega_n$ are adjoined to **Q** coincide.

The smallest number n of elements ω_i is therefore the order of the domain of rationality thought of as an algebraic extension of its underlying natural domain of rationality. The elements $\omega_1, \dots, \omega_n$ themselves form a basis for the domain of rationality. But an orthoid domain is not necessarily a domain of rationality. For example, the domain of all algebraic numbers is an orthoid domain that is not a domain of rationality. The domains of real and of complex numbers are likewise orthoid but not, König seems to suggest, domains of rationality.

König found much to criticise. The original papers were very hard to read and remained restricted to a small circle of readers. They had therefore, he said, failed in their principal purpose and so he had set himself the task of popularising the spirit of Kronecker's method. He was pleased to offer an elementary proof of Kronecker's fundamental theorem. From this he deduced a generalisation of the concept of resultant to what he called the *Resultantform* which enabled him to deal with multiplicities in systems of equations. Geometrically, he gave a general account of Noether's fundamental theorem in n-dimensional space, which he connected to results of Hilbert. Arithmetically, he showed how to decompose algebraic integers in terms of prime ideals.

Kronecker's fundamental theorem is the result he proved after so much effort in 1883, and which Molk had then reproved in much the same way, claiming however that it was elementary. König introduced it for two polynomials in one variable: $f(x) = \sum a_i x^{m-i}$ and $g(x) = \sum b_i x^{n-i}$, whose product is

$$f(x)g(x) = \sum c_i x^{m+n-i}.$$

The theorem claims that there are identities connecting the products $a_i b_j$ and homogeneous linear expressions in the c_k. Similarly in general, there are identities connecting the products of the coefficients of any number of polynomials and the coefficients of the product. König went on to offer a truly elementary (and for that matter simple) proof of this result. Since Edwards says that Kronecker's paper remains obscure to him, and since he then explains just why it is so significant, it is worth digressing to explain what is going on. The matter is discussed in Edwards [1990, part 0].

It is a famous result due to Gauss that if the coefficients of the product $f(x)g(x)$ of two monic polynomials are all integers, then so are all the coefficients of the polynomials themselves, $f(x)$ and $g(x)$. This can be generalised: if the coefficients of the product $f(x)g(x)$ are all algebraic integers, then so are all the coefficients of $f(x)$ and $g(x)$. Dedekind proved that if the coefficients of a product $f(x)g(x)$ are all integers, then the product of any coefficient of $f(x)$ and any coefficient of $g(x)$ is an integer (the constituents need no longer be monic). This he then generalised: if the coefficients of the product $f(x)g(x)$ are all algebraic integers, then the product of any coefficient of $f(x)$ and any coefficient of $g(x)$ is an algebraic integer. He published it in 1892, and it became known as his Prague theorem (because of its place of publication). Unquestionably he did not know that it was a consequence of Kronecker's theorem published in 1883. Either he had not read that paper or we have further evidence that it was obscure.

What Kronecker, Molk, and König all proved, in their different ways, is that the modular system generated by the products $a_i b_j$ and the c_k are equivalent in an extended sense of the term due to Kronecker. Let us call this result the ABC theorem. What Landsberg [1899, p. 313] pointed out is that Hilbert's *Nullstellensatz* shows that the concepts of equivalence for systems of equations and for the corresponding modular systems are not exactly the same. Moreover, this problem had already been spotted and dealt with by Kronecker, and Kronecker's discussion is exactly the theorem we are discussing. Let the module generated by the products $a_i b_j$ denoted AB, and that generated by the c_k denoted C. Then certainly C is divisible by AB. Conversely, every $a_i b_j$ is the root of polynomial equation $v^m + g_1 v^{m-1} + \cdots + g_m = 0$ whose coefficients are divisible by successive powers of C (g_i by C^i). Kronecker called such a function v divisible by C in an extended sense, and proclaimed the equivalence of the modular systems AB and C in this sense. In a short paper of 1895, Hurwitz deduced that if the a_i and b_j are algebraic integers and the c_k are divisible by an algebraic integer ω, then so is every product $a_i b_j$. In modern terms this is a theorem about the integral dependence of ideals.

König's book is at its most geometrical in chapter 7 (Linear Diophantine problems, general theorems and algebraic theory). He posed the situation this way. Given a (possibly strict) holoid domain $\mathbf{A}$, and a form domain over it $[\mathbf{A}, x_1, \ldots, x_m]$, let F_{ij} ($i = 1, \ldots, k$; $j = 0, \ldots, \ell$) be forms in that domain. The problem is to find forms $X_1, \ldots, X_\ell$ in the given domain so that the following k linear Diophantine equations in the ℓ unknowns are satisfied

$$F_{i_1}X_1 + F_{i_2}X_2 + \cdots + F_{i_\ell}X_\ell = F_{i_0} \qquad (i = 1, \ldots, k).$$

Two cases had to be distinguished: when **A** is orthoid; and when **A** is the rational integers (corresponding to an algebraic and an arithmetic formulation of the problem). This does not sound remotely geometrical; rather it is a refinement of the general thrust of the book, which is elimination theory. The turn to geometry came 40 pages into the chapter, in §12 "Der verallgemeinerte Noethersche Satz" (The generalised Noether theorem).

The theorem applied to such situations as these. Let F_1, F_2, F_3 be polynomials in variables x, y, z. Assume that the equations $F_i = 0$ each separately define an algebraic surface in 3-space, and that these surfaces meet in r points $P_1 = (\xi_{11}, \xi_{12}, \xi_{13})$, $\ldots$, $P_r = (\xi_{r1}, \xi_{r2}, \xi_{r3})$. Necessary and sufficient conditions for a given polynomial Φ in x, y, z to belong to the divisor system defined by F_1, F_2, F_3 were then given in terms of the existence of forms H_{ij} such that $\Phi - \sum_{j=1}^{3} H_{ij}F_i$ vanishes to suitable orders at the points $P_1, \ldots, P_r$. The generalisation of this theorem applies to m polynomials in m variables. This was the first time such a generalisation had been proved.

König then observed that the necessary conditions for an arbitrary form Φ to belong to a modular system $(F_1, F_2, \ldots, F_k)$ is that Φ vanishes at the common zeros of the F_i, but that this is not sufficient. Instead, as Hilbert had shown, if Φ vanishes at those points, then some power of it belongs to the modular system. This is Hilbert's *Nullstellensatz.* König then showed how to replace Hilbert's "*rather complicated*" proof and establish the theorem as a simple corollary of the elimination theory developed in his book.

How Noether's theorem can fail

When the number of variables and the number of polynomials is the same, the context is that of the principal class, in Kroneckerian terminology. This numerical coincidence is far from being trivial. To see that Noether's theorem is a delicate one, it is advisable to have a clear case of how it can fail. The simplest case is provided by a rational quartic curve in space. Let the curve, C, be defined as follows

$$C = \{[s^4, s^3t, st^3, t^4] : s, t \in \mathbf{C}\}.$$

Then C meets any hyperplane, H, in $\mathbf{CP}^3$ in four points. If the theorem were to be true, it would assert that any hypersurface through these four points is a linear combination of $H = 0$ and the equations of the surfaces defining C. There is indeed a quadric, Q, through C; it is $x_0x_3 - x_1x_2$. And there are several cubic surfaces, for example $x_1^3 - x_0^2x_2$. But there are infinitely many quadrics through the four points which are not a linear combination of Q, H, and these cubic surfaces. For more details about this example, see Eisenbud [1994, p. 466]. Note, at least, that the failure of Noether's theorem in this case is complete; there is no prospect that a valid theorem can be obtained generalising Noether's theorem to the present case.

Compare the situation with the other type of quartic curve in space, the elliptic curve, C', of genus 1 obtained as the intersection of two quadrics, Q_1 and Q_2, say. As before, a hyperplane, H', in $\mathbf{CP}^3$ meets C in four points. Noether's theorem asserts that any hypersurface through these four points is a linear combination of $H' = 0$ and the equations of the surfaces defining C, which can be taken to be $Q_1 = 0$ and $Q_2 = 0$, say. It is now true that any conic in the hyperplane through those four points is a linear combination of $Q_1 \cap H'$ and $Q_2 \cap H'$. It follows that any quadric through the four points is a linear combination of Q_1, Q_2 and H'. It follows then that Noether's theorem is true in this case.

7. LASKER

In 1905 Emanuel Lasker published his important paper "Zur Theorie der Moduln und Ideale". He is much better remembered today for having been the World Chess Champion for 26 years, from 1894, and indeed this paper was submitted to *Mathematische Annalen* from New York, where he had gone to play in a tournament. In the paper he introduced the concept of a primary ideal, and established that every ideal is an intersection of primary ideals in an essentially unique way. Zariski and Samuel explain that a primary ideal is the analogue of a power of a prime, so this theorem is the analogue of the unique factorisation theorem for integers. Lasker accomplished much else besides (for example, he gave a proof of König's generalisation of Noether's theorem).

Lasker was a student of M. Noether's, so it is not surprising that his article is full of interesting historical comments. Lasker saw two approaches deriving from Kummer's original work. One was Dedekind-Hurwitz, the other Kronecker-Weber-König. Enlarging the frame of reference to include geometry, Lasker added the work of Cayley and above all Salmon, as made rigorous by Noether in his fundamental theorem. He gave quite an extensive list of accounts of this result, culminating in König's, and of its number theoretical analogue as formulated by Hensel, Hancock and Landsberg. However, he said, the important applications Noether had deduced for his fundamental theorem for geometry, algebraic functions, and abelian integrals, which had shed a powerful light on the significance of the idea of modules, had waited 20 years for the next important advance. This was Hilbert's work, which he summarised in four theorems (two versions of the basis theorem, the finiteness of syzygies, and the Hilbert polynomial).

8. HADAMARD AND KÜRSCHÁK

József Kürschák was a Hungarian mathematician educated in Budapest. He organised seminars with König, and König thanked him for his help during the writing of his *Einleitung*. He also shared something of his mentor's breadth of interests and his influence. One sign of this is a prize competition organised for school leavers in mathematics and physics, which is named after him. In the early 1890s he came into contact with Hadamard because of his study of the relation between the simple pole of a power series and its coefficients. This stimulated Hadamard to investigate what conditions on a power series yield particular types of singularity. One supposes, in the absence of evidence, that it was Jules Molk who encouraged them to expand upon Landsberg's article in the *Encyklopädie der mathematischen Wissenschaften* when he came to organise the French version for his *Encyclopédie*. At all events, it is a much larger article than the German original, and its comments provide an interesting view of how all this material was regarded in 1910–1911. It is not clear, however, what impact this article had; references to it are hard to find. Let us take it, then, as a snapshot of the times.

The first part was published on 30 August 1910. The title is interesting in itself: “Propriétés générales des corps et des variétés algébriques”. Landsberg’s had been “IB1c Algebraic varieties. IC5 Arithmetic theory of algebraic quantities”, so it is clear that the editors had not been sure where to place it, because IB is devoted to algebra and IC to number theory. Generally speaking, Hadamard and Kürschák kept Kronecker’s approach at a distance, setting off material specifically of that nature in square brackets [...]. They reviewed the proliferating terminology carefully: The term “known quantities” had given way to Dedekind’s “field”, which was the same thing as König’s orthoid domain. Kronecker’s domain of rationality was also a field, more precisely a finite or algebraic extension of the rationals (the terms “finite” and “algebraic” were regarded as synonyms by Hadamard and Kürschák). There were two sorts of field: number fields and function fields, but every field contained the field of rational numbers, which they denoted R. Hadamard and Kürschák regarded the simplest function field as the function field in n variables, which they denoted indifferently R. The coefficients were to be unrestricted numerically, which means complex numbers. If the coefficients were restricted in any way, say to be rational, they wrote the field R_{R}. They admitted this was back to front from Kronecker’s approach.

What they called finite or algebraic fields in the strict sense of the word were simple algebraic extensions of any of the fields just defined, to wit, the fields that Kronecker had called derived domains of rationality, reserving the term natural for just the fields in the paragraph above. They also presented the concept of a field in purely formal terms (analogous to that of a group, they said). A commutative group is a field, they said, when it also admits an associative multiplication with a multiplicative identity, and when every element that is not a divisor of zero has a unique multiplicative inverse. Weber’s definition was more restrictive, they observed: the multiplication must be commutative, and there are no zero divisors. König’s orthoid domains satisfied all these conditions and were of characteristic 0. What he called a hyperorthoid domain dropped the condition about divisors of 0, as, for example, the domain $\{a + bx\colon\ a, b \in \mathbf{C},\ x^2 = 0\}$. König called a domain pseudoorthoid if it had no divisors of 0, but was not of characteristic 0. They arose, for example, by taking numbers modulo a prime. So pseudoorthoid domains were also fields in Weber's sense of the term. Hadamard and Kürschák settled on the definition of field that agreed with König's orthoid domain. What is notable is that at this stage in the paper they were unable to take on board Steinitz's paper of 1910. They could only do that at the end of the paper, printed in the next fascicle and published on 15 February 1911.

A holoid domain satisfies all the defining conditions of an orthoid domain except those relating to division. From a holoid domain one can always form an orthoid domain — its field of fractions. The algebraic integers form a holoid domain. The algebraic integers in a field K likewise form a holoid domain, as do subdomains generated over the rational integers by a finite set of algebraic integers. Such domains were called *Art* or *Species* by Kronecker, *Ordnung* by Dedekind, and rings or integral domains by Hilbert.

For the purposes of this paper I shall concentrate on the geometrical side of the article. When a field of algebraic functions is considered as an algebraic extension of a general field R, it is identical with the function field attached to an algebraic variety. The analogy breaks down when the concept of algebraic integer is introduced. Let us, as Hadamard and Kürschák did, restrict our attention to algebraic curves. In function theory, an algebraic function is integral when it is finite at every finite point, but as it stands this concept makes no sense in algebraic function theory, because birational transformations can interchange finite and infinite points. Instead, the concept of a divisor takes the leading role, and for this they referred to the book by Hensel and Landsberg (the definition is the modern one). Prime divisors correspond to points on the algebraic curve or Riemann surface. In due course they reached the Riemann-Roch theorem and the theorem on the reduction of singularities by birational transformations, all following Hensel and Landsberg.

There followed nine sections due to Kürschák on topics related to Hensel's p-adic numbers. After this, in the second fascicle they plunged into the theory of modular systems, and in due course Hilbert's invariant theory and the ABC theorem. Hadamard and Kürschák show that Hilbert's third theorem implies Noether's fundamental theorem (the $AF + BG$ theorem).

The second fascicle, published on 15 February 1911, is also of interest to us because by now the authors had had time to take on board Steinitz's

"Algebraische Theorie der Körper". This lies outside my theme, but it is clear from the clarity of the exposition and the new generality of expression just why Steinitz's paper had the foundational effect that it did. I take this opportunity to make another parenthetical remark. Kürschák himself went on to make an important contribution to the theory of fields which was promptly taken up by Ostrowski. It is he who extended Hensel's idea of forming the p-adic numbers to create the theory of fields with real valuations. He announced his ideas at the International Congress of Mathematicians in Cambridge in 1912 (the lecture was actually delivered by Hadamard) and published them in his paper [Ostrowski 1913].

9. MACAULAY

The English mathematician F.S. Macaulay's Cambridge Tract *The algebraic theory of modular systems* [1916] was the first significant work on algebraic geometry in the spirit of Hilbert and Kronecker to be written in Britain. Macaulay had graduated from Cambridge in 1882, and from 1885 to 1911 taught mathematics at one of England's more ambitious schools, St Paul's School in London (J.E. Littlewood was one of his pupils). During his time as a teacher he published steadily, and was accordingly invited to address the third International Congress of Mathematicians at Heidelberg in 1904, where he spoke on his work generalising the Brill-Noether theory of the intersections of plane curves to higher dimensional varieties. This may be the only time a school teacher has been invited to present a paper at the Congresses. After the War he settled in Cambridge, and was a frequent attender at Baker's Saturday tea parties, the first regular mathematical seminars organised at the university.

It is clear from Baker's somewhat perfunctory obituary of Macaulay [Baker 1938] that he was by no means convinced that Macaulay's work was truly geometry:

"*In conversation he was wont to regard the subject as a geometrical one; but ... many important differences ... do in fact disclose themselves .... It is doubtless true that the geometrical approach takes for granted many algebraic results of which the formal proof is at present incomplete ... gratitude is due to such as Macaulay, who seek to supply the evidence in a purely algebraic form. But up to now an intimate knowledge of the geometrical aspect seems a necessary part of the algebraist's equipment*".

This sense of distance between Baker and Macaulay may explain van der Waerden's reminiscence:

"*Most important work on the theory of polynomial ideals was done by ... Macauly* [sic]*, a schoolmaster who lived near Cambridge, England, but who was nearly unknown to the Cambridge mathematicians when I visited Cambridge in 1933. I guess the importance of Macaulay's work was known only in Göttingen*" [van der Waerden 1971, p. 172].

In fact, Macaulay's early career does not exactly conform to Baker's few comments. They are indeed, as Baker said, "*for the most part concerned with the knotty problem of the multiple points, and the intersections of, plane algebraic curves*". This was Brill and Noether territory. The whole burden of Macaulay's papers is to extend the Brill-Noether theorem to the general setting and to prove, in that context, the most fundamental theorem in the subject, the Riemann-Roch theorem. This theorem, which Macaulay saw geometrically, relates dimensions of families of curves through a given set of points to the nature of the points on the curve.

Macaulay's criticisms were trenchant, his remedies well argued, and contrary to Baker's remarks, his knowledge of contemporary literature extensive. One significant source was Castelnuovo, and Macaulay particularly acknowledged the influence of Charlotte Scott, with whom he enjoyed quite an extensive co-operation. When he comes to the vexed topic of the Brill-Noether theorem his references were numerous and his comments about them sharp. In Macaulay's [1900, p. 382n], for example, Brill and Noether themselves are judged to have given an incomplete proof and that for only the simple case, which was taken over uncritically in Clebsch-Lindemann but reworked correctly by Picard and Simart in the second volume of their treatise [1906]. (A modern mathematician would not exonerate even the later authors.) Throughout these papers, Macaulay refers to the problem as a geometrical one requiring a geometrical solution. So the subtleties of multiple points are to be unravelled, he suggested, by formulating the concept of an infinitesimal curve (whose branches, so to speak, exemplify the nature of the singularity). In this way he sought to re-present the conditions imposed on a curve by the requirement that it pass through a given multiple point as equivalent conditions imposed by a set of distinct points.

What might be called Macaulay's geometrical period ended in 1905 with his Heidelberg address. It was on the subject of generalising the Brill and Noether theorem to varieties defined by k polynomials in n variables, and set out the problem carefully in the plane before sketching how a generalisation might proceed. The paper, although tentative, represents an advance on König's work, by entering the region where Noether's theorem may fail. In a footnote to the published paper he tells us that Brill and Noether informed him at the Congress of the recently published book by the Hungarian mathematician J. König popularising Kronecker's much more algebraic approach, and he offered a few favourable comments on the work. Indeed, it seems to have made a deep impression on him, for, a few minor papers aside, and for whatever reason, Macaulay fell silent in 1905 and when he began again in 1913 he wrote in the language of what, following Kronecker and König he called modular systems. The first paper in this style is in the *Mathematische Annalen* [1913], which is a direct response to Lasker's. What Lasker had shown could in principle be done, Macaulay showed how to do in practise. His second is his Cambridge Tract [1916].[13]

In the Tract, a modular system (or module of polynomials) was defined by Macaulay to be a set of polynomials in n variables such that the sum of any polynomials in the set is again in the set and the product of any polynomial in the set with any polynomial in the n variables is again in the set. This would make a modular system an ideal in a polynomial ring on n variables, but Macaulay makes further stipulations which complicate the matter, and into which we need not enter. He recognised Hilbert's basis theorem as the main theorem in the subject, but the main detailed influence he acknowledged was that of Kronecker, as represented by J. König in his book of 1903.

In the introduction to the book he makes a number of interesting comments: "*The object of the algebraic theory is to discover those general properties of a module which will afford a means of answering the question whether a given polynomial is a member of a given module or not.*" (This was the subject of his paper for the *Mathematische Annalen.*) "*Such a question in its simpler aspect is of importance in Geometry and in its general aspect is of importance in Algebra.*" It is hard to

[13] There is more to be said about Macaulay; I hope , with help, to return to him at a later date.

know what this distinction implies, but it strikes a familiar chord: "*The theory resembles Geometry in including a great variety of detached and disconnected theorems. As a branch of Algebra it may be regarded as a generalised theory of the solutions of equations in several unknowns*" [Macaulay 1916, p. 2]. This attitude to geometry, that it is disorganised and perhaps even incapable of organisation, is often encountered, although more usually by those who disdain the subject altogether. It is not the complaint that geometry lacks rigour, and can only be made rigorous by importing methods from outside (be they algebraic or analytic). It is the complaint that the subject lacks a grand design.

It may make Macaulay's views a little more precise if we consider the work of his Cambridge contemporaries. There were about a dozen of these people, of whom Baker was the acknowledged leader, and he was a geometer in the Italian style. In the main these people studied interesting curves and interesting surfaces. There was a botanical aspect to their work, as there had been before to Cayley's, and the profusion of special results was accompanied by a number of neat tricks needed to obtain the results. Ingenious arguments to limited ends do not convey the impression that there is a deep underlying pattern to it all. Such work is commonly regarded as routine in any branch of mathematics (and poses its own problems for the historians who come along afterwards). If there is an organising principle that, however subtly and diffidently, shapes the subject and which is occasionally illuminated by papers of real profundity, then matters are different. If Macaulay, looking at Cambridge from his school in London, felt the lack of such a focus this would explain his opinion.

It may also make Macaulay's views a little more precise if we consider the trajectory of his own work. It resembles the fate of Noether's theorem, in which sweeping general claims were forever called into question by awkward doubts. The resolution of singularities of algebraic curves is generally not regarded as having been satisfactorily concluded until the 1930s. Until then most proofs of the theorem and its generalisations were found wanting (see [Bliss 1923]). Macaulay was one of the sterner critics, and his work moved more and more towards algebra. This is the more familiar objection, that rigour in geometry is to be brought about by injections of algebra, but it denies to geometry an underlying organisation of method.

The Tract is, as remarked, a study of ideals in polynomial rings. It opens with a careful account of the theory of resultants and, for use when that method breaks down, of Kronecker's theory of resolvents, as described by König with further corrections by Macaulay. Then comes the theory of modules (i.e. ideals), starting with the Hilbert basis theorem (in König's proof), Lasker's theorem, and the *Nullstellensatz*. Given a set of polynomials F_i in n variables $x_1, \ldots, x_n$, a set of solutions which exist for $x_1, \ldots, x_r$ when the remaining $x_{r+1}, \ldots, x_n$ take arbitrary values Macaulay, following König, called a spread of rank r and dimension $(n - r)$. If there are solutions of rank r and no solutions of rank $< r$, the system of equations $F_i = 0$ and the module generated by the F_i were said to be of rank r. Macaulay observed that a spread defined by a system of equations generally broke up into several irreducible spreads. He used the idea of spreads to give examples of the theorems under discussion, and by means of the method of resolvents to find fault with König and especially Hadamard and Kürschák. He was scathing about König's definition of simple or mixed modules (which was couched in terms of the nature of various partial resolvents) and instead offered a refinement of the geometrical idea that an unmixed module is one whose spreads are all of the same dimension. The refinement made essential use of Lasker's decomposition theorem. This is the origin of the theory of unmixedness, which is the reason Macaulay's name survives in the term Cohen-Macaulay ring. He re-derived Lasker's proof that a module of the principal class is unmixed. He then discussed inverse systems, which appear today in the theory of Gorenstein rings. There is much more to Macaulay's Tract than this sketch suggests, but it is technical and I shall not go into it here. Instead I refer the reader to Paul Roberts' clear introduction to the re-edition of the Tract [Macaulay 1916/1996] and turn instead to Emmy Noether.

10. EMMY NOETHER

As is well known, Emmy was the daughter of Max Noether, and studied mathematics at Erlangen under Paul Gordan, an old family friend, writing her thesis on old-fashioned explicit invariant theory before gravitating to the school of mathematicians around Hilbert at Göttingen. There she distinguished herself with work on differential invariants in the theory of general relativity, as it was being created by Einstein, Hilbert, and Klein. This remarkable work, in such distinguished company, literally made her name — it was the occasion of her discovery of what mathematical physicists call Noether's theorem on conservation laws to this day (see Rowe [*to appear*]). Then her interests shifted back towards questions in the algebraic theory of invariants that Hilbert had done so much to transform from the style of Gordan. The first real signs of this are the papers [Noether 1915, 1919] some cite and others omit in the compulsion to get to the major papers she was soon to write [Noether 1921, 1927].

Since these major works are described in many places, and most recently in [Corry 1996], who however omits the earlier papers, it is reasonable to reverse the balance and describe only the earlier papers. For reasons of space, I discuss only the second of them in detail. The first one is devoted to finding bases for systems of rational functions drawn from the field $K_{n,\rho}$ in n variables, any ρ of which are independent but any $\rho+1$ of which are connected by an algebraic equation. Whenever possible a minimal basis of exactly ρ elements is to be found. She cites Lüroth, who established the existence of minimal bases for $K_{n,1}$, but follows Steinitz's method of proof in this case, Castelnuovo and Enriques for $K_{n,2}$, and observes that there is not, generally, a basis for$K_{n,\rho}$, $\rho \geq 3$. In view of comments I shall make below about Emmy Noether's awareness of contemporary literature, I note that she cites König's *Einleitung* for its extension of Gordan's theorem.

The paper [1919], written for the *Deutsche Mathematiker-Vereinigung* and published in their *Jahresbericht*, is aimed at a gap in the *Encyklopädie der mathematischen Wissenschaften.* It carries the informative title "The arithmetic theory of algebraic functions of a single variable, with respect to the other theories and to the theory of number fields". It opens with the remark:

"*The following report arises from the wish to have something connecting the independent theories of algebraic functions; at the same time it can be read as an extension of the report of Brill and Noether which essentially lacks a treatment of the arithmetic theory.*"[14]

[14] "*Der folgende Bericht ist aus dem Wunsche entstanden, ein verbindendes Glied zwischen den einzelnen Theorien der algebraischen Funktionen zu haben; zugleich kann*

A footnote adds that the report of Brill and Noether did describe Kronecker's theory of the discriminant. In barely twenty pages, she surveyed the transcendental theory of Riemann, the arithmetic theory of Hensel and Landsberg, and the algebraic geometrical theory of Brill and Noether, and indicated the parallels between algebraic numbers and algebraic functions and the corresponding ideal theories. Here she gave a quick comparison of the definitions of a module due to Hilbert and Kronecker, but only in a footnote, and noted Hurwitz's contribution and its treatment by Weber. She compared the arithmetic and geometrical theories, looking at how they captured the idea of invariance differently and at how they treated singular points. Finally she came to compare the Residue theorem with theorems in the theory of ideals.

The residue theorem of Brill and Noether is usually taken as the main result of that theory, from which other consequences follow, such as the Riemann-Roch theorem (see [Gray 1989]). To state it requires recalling three definitions: two sets of points, A and B, on an algebraic curve X are said to be coresidual if there is a third point set R such that there is a curve cutting the curve X in the set $A \cup R$ and another curve cutting the curve X in the set $B \cup R$. The set R is called the residue of the complete intersections $A \cup R$ and $B \cup R$. The residue theorem asserts that concept is well-defined; it does not depend on the set R. A curve is called an adjoint curve of the curve X if it passes $(n-1)$ times through each singular point of X of order n. The residue theorem implies that if two sets of points A and B are coresidual then they are coresidual with respect to adjoint curves (ignoring the behaviour at singular points).

As Hensel and Landsberg had shown, all of this can be written in the language of ideal theory (Dedekind and Weber had preferred the closely related language of divisor classes). To a set of points, A, there corresponds the family of rational functions defined on the curve and vanishing at those points. This family is an ideal in the ring of all integral rational functions defined on the curve. If the sets of points A, B, and R generate ideals, denoted $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{r}$ respectively, then the fact that A and R are coresidual translates as $\boldsymbol{a} \cdot \boldsymbol{r} = (\alpha)$, where α is a rational function on X.

er als eine Ergänzung zu dem Bericht von Brill-Noether angesehen werden, in dem die Besprechung der arithmetischen Theorie im wesentlichen fehlt" [Noether 1919, p. 271].

Similarly $\boldsymbol{b} \cdot \boldsymbol{r} = (\beta)$, where β is another rational function on X, and this means that $\boldsymbol{a}$ and $\boldsymbol{b}$ are equivalent ideals. The converse is also true, and so, said Emmy Noether, "*equivalent ideals and coresidual sets of points are therefore identical ideas.*"[15] She then gave a sketch of the proof of the residue theorem in the language of ideals, and indicated, still following Hensel and Landsberg, how this way of thinking extends to cover the Riemann-Roch theorem. Here she cited papers by R. König [1918, 1919].

One should not place too much significance on this one paper. Noether's work takes off with the papers she was to write next. But some comments can be made. One notes the patchy historical references. Even though a considerable amount of literature is cited there are omissions (Gyula König, Macaulay, and even Lasker). Some, but not all of these were to find their way into the folk memory of Noether's school. One notes the ruthless tendency to cut to the heart of a concept, and the preference for abstract algebra. And one notes the presence of geometry and function theory, in the form of the theory of complex curves, at the start of all of this work. What was done with it, whether it was algebra or geometry, is a topic for another occasion. But it is significant that the consensus was that the case of plane curves was essentially done, even though troublesome points remained, while the case of surfaces and higher dimensional varieties remained obscure. In all those dimensions the neat one-to-one correspondence between varieties and function fields breaks down. Birationally equivalent varieties have isomorphic function fields (regarded as algebras over their ground field). But a birational equivalence class contains algebraic surfaces that are not even homeomorphic (for example, $\mathbf{CP}^1 \times \mathbf{CP}^1$ and $\mathbf{CP}^2$). So studying function fields is weaker than studying (isomorphism classes of) surfaces, or varieties in higher dimensions.

CONCLUSION

The fascination of the union of algebra and geometry is precisely that one does not dominate the other, however much one style may be favoured by individual writers. We have seen that we cannot conclude

[15] "*äquivalente Ideale und koresiduale Punktgruppen sind also identische Begriffe*" [Noether 1919, p. 288].

that in the creation of classical algebraic geometry the questions of central importance were exclusively algebraic. There was a unity that embraced quite specific geometrical questions (in the opinion of Hilbert) and that drew on complex function theory. The central feature, in many people's opinion, was the strong analogy between algebraic number theory and algebraic function theory. This was widely emphasised, not least by Kronecker and his immediate followers, although he himself did not go along with the transcendental function theorists. The same unity was perceived, much more abstractly, by Emmy Noether.

The complexity of the picture is what makes it interesting. Abstract algebra and classical algebraic geometry have their roots in a rich mixture of topics: the algebra is visible, the geometry (and even the function theory) is partially hidden. The role of the minor figures in this story is to bring out how these connections were seen at the time. What Macaulay, König, and even Lasker show us is how strong the Kroneckerian approach was in its day, and yet how cumbersome. It is not possible to say why it faded almost completely from sight. Claims have been made that early invariant theorists, such as Gordan, were too interested in explicit problems, and that later ones, such as Hilbert, were too little interested (see remarks referred to in [Corry 1996] and [Study 1923]). The latter claims founder because Hilbert did address specific questions, but it is plausible that the explicit questions defeated those who tackled them (see the essay by Sturmfels in [Hilbert 1897/1993]), and that mathematicians were relieved to be able to give them up (as the anonymous quotation from Study, quoted in footnote 4 above, suggests). Certainly Emmy Noether was happy to isolate the abstract kernel of algebra, as she saw it. But it is not clear if this was part of a reasoned rejection of the approach previously taken by Kronecker, König, and Macaulay, or the indirect result of a lack of familiarity with their work (which she knew very little about). Perhaps by 1919 it had withered on the vine. But if we cannot, here at any rate, explain why Emmy Noether so splendidly took the route she did (whatever an explanation of such an event might be) we can conclude that it grew out of a rich complex of unresolved questions that were to remain, albeit somewhat hidden, as a vital part of classical algebraic geometry.

Acknowledgements

I would like to thank the organisers of the joint AMS-Benelux Mathematical

Societies' meeting in Antwerp in May 1996 for an opportunity to present an early version of this paper, and David Eisenbud for his help in revising it for publication.

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