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Quantum Cohomology of Complete Intersections

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Arnaud Beauville [1]

Introduction

The quantum cohomology algebra of a projective manifold X is the cohomology of X endowed with a different algebra structure, which takes into account the geometry of rational curves in X. This structure has been first defined heuristically by the mathematical physicists [V,W]; a rigorous construction (and proof of the associativity, which is highly non trivial) has been achieved recently by Ruan and Tian [R-T].

When computed e.g. for surfaces, the quantum cohomology looks rather complicated [C-M]. The aim of this note is to show that the situation improves considerably when the dimension becomes high with respect to the degree. Our main result is:

Theorem .– *Let* $\mathrm{X} \subset \mathbf{P}^{n+r}$ *be a smooth complete intersection of degree* $(d_1, \ldots, d_r)$ *and dimension* $n \geq 3$ *, with* $n \geq 2\sum(d_i - 1) - 1$ *. Let* $d = d_1 \ldots d_r$ *and* $\delta = \sum(d_i - 1)$ *. The quantum cohomology algebra* $\mathrm{H}^*(\mathrm{X}, \mathbf{Q})$ *is the algebra generated by the hyperplane class* H *and the primitive cohomology* $\mathrm{H}^n(\mathrm{X}, \mathbf{Q})_\mathrm{o}$ *, with the relations:*

$$\mathrm{H}^{n+1} = d_1^{d_1} \ldots d_r^{d_r}\, \mathrm{H}^\delta \qquad \mathrm{H} \cdot \alpha = 0 \qquad \alpha \cdot \beta = (\alpha \,|\, \beta) \frac{1}{d} (\mathrm{H}^n - d_1^{d_1} \ldots d_r^{d_r}\, \mathrm{H}^{\delta - 1})$$

for $\alpha, \beta \in \mathrm{H}^n(\mathrm{X}, \mathbf{Q})_\mathrm{o}$.

The method applies more generally to a large class of Fano manifolds (see Proposition 1 below). It is actually a straightforward consequence of the definitions – except for the exact value of the coefficient $d_1^{d_1} \ldots d_r^{d_r}$, which requires some standard computations in the cohomology of the Grassmannian. Still I believe that the simplicity of the result is worth noticing.

As the referee pointed out, we get actually more than an abstract presentation of the quantum cohomology algebra by generators and relations. The point is that the powers of the generator H have a simple geometric interpretation: denoting by $\mathrm{H}_p \in \mathrm{H}^{2p}(\mathrm{X}, \mathbf{Z})$ the class of a codimension p linear section, one has for $p \leq n$

$$\mathrm{H}^p = \mathrm{H}_p + (\sum_{i=0}^{k-p} \ell_i)\, \mathrm{H}_{p-k} \ , \qquad \mathrm{H}_p = \mathrm{H}^p - (\sum_{i=0}^{k-p} \ell_i)\, \mathrm{H}^{p-k} \ ,$$

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where $k = n + r + 1 - \sum d_i$ and $d\ell_i$ is the number of lines in X meeting two general linear spaces of codimension $n - i$ and $k + i - 1$ respectively (formula (1.7) and Remark 1 below). This allows to write down explicitely the quantum product in the basis (H_p). We get from this a number of enumerative formulas: for instance we find that the number of conics passing through 2 general points in a hypersurface of degree d and dimension $2d - 3$ is $\frac{1}{2}d!(d-1)!$, while the number of twisted cubics through 3 general points in a hypersurface of degree d and dimension $3d - 6$ is $d!((d-1)!)^2$.

I would like to thank A. Bruno, R. Donagi, G. Ellingsrud and Peng Lu for their useful comments. During the preparation of this paper I had long and vivid discussions with Claude Itzykson, while his health was declining very rapidly – till he died on May 22. I would like to dedicate this paper to his memory.

1. Quantum cohomology of Fano manifolds

I am considering in this paper Fano manifolds with $b_2 = 1$, i.e. smooth compact complex manifolds X such that $\mathrm{H}^2(\mathrm{X}, \mathbf{Z})$ is generated by an ample class H and the canonical class $\mathrm{K_X}$ is $-k\mathrm{H}$ for some positive integer k. I will use the following properties of the quantum cohomology product on $\mathrm{H}^*(\mathrm{X}, \mathbf{Z})$ or $\mathrm{H}^*(\mathrm{X}, \mathbf{Q})$ (proved in [R-T]):

(1.1) it is invariant under smooth deformations;

(1.2) it is associative, compatible with the grading mod. 2, and anticommutative. It is compatible with the intersection form $(\ |\ )$ on $\mathrm{H}^*(\mathrm{X}, \mathbf{Z})$, i.e. one has $(x \,|\, yz) = (xy \,|\, z)$ for x, y, z in $\mathrm{H}^*(\mathrm{X}, \mathbf{Z})$. The element $1 \in \mathrm{H}^0(\mathrm{X}, \mathbf{Z})$ is still a unit.

(1.3) the product $x \cdot y$ of two homogeneous elements is defined by

$$x \cdot y = (x \cdot y)_0 + (x \cdot y)_1 + \ldots + (x \cdot y)_j + \ldots$$

where $(x \cdot y)_0$ is the ordinary cohomology product, and $(x \cdot y)_j$ is a class of degree $\deg(x) + \deg(y) - 2kj$.

(1.4) Assume that the moduli space $\mathcal{M}_j$ of maps $f : \mathbf{P}^1 \to \mathrm{X}$ of degree j (i.e. such that $\deg f^*\mathrm{H} = j$) has the expected dimension $n + kj$; choose any smooth compactification $\overline{\mathcal{M}}_j$ of $\mathcal{M}_j$ such that the evaluation maps $e_i : \mathcal{M}_j \to \mathrm{X}$ ($0 \le i \le 2$) defined by $e_i(f) = f(i)$ extend to $\overline{\mathcal{M}}_j$. Then the "instanton correction" $(x \cdot y)_j$ is defined by

$$\langle x, y, z\rangle_j := \big((x \cdot y)_j \mid z\big) = \int_{\overline{\mathcal{M}}_j} e_0^*x \,.\, e_1^*y \,.\, e_2^*z \quad .$$

(1.5) If $x, y, z \in \mathrm{H}^*(\mathrm{X}, \mathbf{Z})$ are classes of subvarieties A, B, C of X which are in general position, it follows easily from (1.4) that the triple product $\langle x, y, z\rangle_j$ is the number of curves of degree j meeting A, B and C (counted with multiplicity *abc* if the curve meets A, resp. B, resp. C in a, resp. b, resp. c distinct points).

To avoid confusion I will denote by $H_p \in H^{2p}(X, \mathbf{Z})$ for $0 \le p \le n$ the p-th power of H in the ordinary cohomology, and reserve the notations $x \cdot y$ or x^n ($x, y \in H^*(X, \mathbf{Q})$) exclusively for the quantum product. One has $H_0 = 1$, $H_1 = H$, and H_n is d times the class of a point, where d is (by definition) the degree of X.

The following result is a direct consequence of Property (1.3):

Proposition 1.– *Let X be a projective manifold, of dimension $n \ge 2$, of degree d. Assume:*

(i) *The ordinary cohomology algebra $H^*(X, \mathbf{Q})$ is spanned by H and $H^n(X, \mathbf{Q})$;*

(ii) *One has $K_X = -kH$ with $k > \frac{n}{2}$.*

(iii) *If $n = 2k - 1$, $H^n(X, \mathbf{Q})$ is nonzero.*

(iv) *If $n = 2k - 2$, $\dim H^n(X, \mathbf{Q})_o \neq 1$.*

There exists an integer $\mu(X)$ such that the quantum cohomology algebra $H^*(X, \mathbf{Q})$ is the algebra generated by H and $H^n(X, \mathbf{Q})_o$, with the relations:

$$\text{(R)} \qquad H^{n+1} = \mu(X)\, H^{n+1-k} \qquad H \cdot \alpha = 0 \qquad \alpha \cdot \beta = (\alpha \,|\, \beta) \frac{1}{d} (H^n - \mu(X)\, H^{n-k})$$

for $\alpha, \beta \in H^n(X, \mathbf{Q})_o$.

(Recall that the primitive cohomology $H^n(X, \mathbf{Q})_o$ is by definition equal to $H^n(X, \mathbf{Q})$ if n is odd, and to the orthogonal of $H_{\frac{n}{2}}$ if n is even.)

Let p be an integer, with $-k < p < \frac{n}{2}$. According to (1.3), one has

$$\text{(1.6)} \qquad H \cdot H_{k+p-1} = H_{k+p} + \ell_p\, H_p \,,$$

for some number $\ell_p \in \mathbf{Q}$ (which is zero for $p < 0$). Intersecting both sides with H_{n-p} gives $\ell_p = \frac{1}{d} \langle H, H_{n-p}, H_{k+p-1} \rangle$ (so that $\ell_p = \ell_{n-k+1-p}$).

From (1.6) one obtains inductively, for $-k < p < \frac{n}{2}$,

$$\text{(1.7)} \qquad H_{k+p} = H^{k+p} - \Big(\sum_{i=0}^{p} \ell_i\Big)\, H^p \,.$$

If $n < 2k - 2$, we can apply this with $p = n - k + 1$; since $H_{n+1} = 0$ we obtain

$$\text{(1.8)} \qquad H^{n+1} = \mu(X)\, H^{n+1-k} \qquad \text{with} \quad \mu(X) = \sum_{i=0}^{n+1-k} \ell_i \,.$$

If $n = 2k - 2$, the product $H \cdot H_n$ belongs to $H^n(X, \mathbf{Q})$. We will see below that under the hypothesis (iv) one has for all $\alpha \in H^n(X, \mathbf{Q})_o$ $H \cdot \alpha = 0$, hence $(H \cdot H_n \,|\, \alpha) = (H \cdot \alpha \,|\, H_n) = 0$. Therefore $H \cdot H_n$ is proportional to $H_{\frac{n}{2}}$, which means that (1.6) and (1.7) still hold for $p = k - 1$, yielding again (1.8).

If $n = 2k - 1$, one finds $H \cdot H_n = \ell_k\, H^k + m$ for some integer m. If m is nonzero H is invertible in $H^*(X, \mathbf{Q})$; since $H \cdot H^n(X, \mathbf{Q})$ is zero for degree reasons, this implies $\mathrm{H}^n(\mathrm{X},\mathbf{Q})=0$. Therefore under the hypothesis (iii) we obtain again (1.8).

Let $\alpha\in\mathrm{H}^n(\mathrm{X},\mathbf{Q})_o$; let us prove that $\mathrm{H}\cdot\alpha$ is zero. If $n\neq 2k-2$ this is clear for degree reasons. Assume $n=2k-2$; then $\mathrm{H}\cdot\alpha$ belongs to $\mathrm{H}^0(\mathrm{X},\mathbf{Q})$. If $\alpha\neq 0$, there exists by hypothesis (iv) an element β in $\mathrm{H}^n(\mathrm{X},\mathbf{Q})_o$ not proportional to α; the equality $(\mathrm{H}\cdot\alpha)\beta=(\mathrm{H}\cdot\beta)\alpha$ leads indeed to $\mathrm{H}\cdot\alpha=0$.

Let $\alpha,\beta\in\mathrm{H}^n(\mathrm{X},\mathbf{Q})_o$. By (1.3) and (1.7) there exists a number $q\in\mathbf{Q}$ such that

$$\alpha\cdot\beta=(\alpha\,|\,\beta)\frac{1}{d}\mathrm{H}^n+q\,\mathrm{H}^{n-k}$$

Multiplying by H and using (1.8) yields $q=-(\alpha\,|\,\beta)\dfrac{\mu(\mathrm{X})}{d}$, which gives the last relation (R).

Finally we just have to remark that the $\mathbf{Q}$-algebra spanned by H and $\mathrm{H}^n(\mathrm{X},\mathbf{Q})_o$ with the relations (R) has the same dimension as $\mathrm{H}^*(\mathrm{X},\mathbf{Q})$, so that all relations follow from (R). ■

Remarks.– 1) Assume moreover that the variety of lines contained in X has the expected dimension $n+k-3$, and that H is very ample, i.e. is the class of a hyperplane section of $\mathrm{X}\subset\mathbf{P}^{\mathrm{N}}$. Then according to (1.5) $d\,\ell_p$ is the number of lines in X meeting two general linear spaces of codimension $n-p$ and $k+p-1$ respectively. For instance, ℓ_0 is the number of lines passing through a point in a general linear section of codimension $k-2$ of X.

2) If n is equal to $2k-2$ or $2k-1$, the result of Prop. 1 does not necessarily hold if one assumes only (i) and (ii). Consider for instance a general linear section of codimension 3 of the Grassmannian $\mathbf{G}(2,5)$. This is a Fano threefold of index $k=2$, degree $d=5$, which satisfies the hypotheses (i) and (ii) of the Proposition (but not (iii)). For such a threefold one has by (1.3) $\mathrm{H}\cdot\mathrm{H}_3=\ell_0\mathrm{H}_2+c$, with $c=\frac{1}{d}\langle\mathrm{H},\mathrm{H}_3,\mathrm{H}_3\rangle_2$. From $\mathrm{H}^2=\mathrm{H}_2+\ell_0$ and $\mathrm{H}^3=\mathrm{H}_3+(\ell_0+\ell_1)\mathrm{H}$ (1.7) we deduce

$$\mathrm{H}^4=(2\ell_0+\ell_1)\,\mathrm{H}^2+c-\ell_0^2\,.$$

Easy geometric computations give $\ell_0=3$, $\ell_1=5$, $c=10$, hence $c-\ell_0^2=1\neq 0$.

Now let X be a general linear section of codimension 2 of $\mathbf{G}(2,5)$. This is a Fano fourfold of index $k=3$, which satisfies (i) and (ii). Let c_1 and c_2 be the classes in $\mathrm{H}^*(\mathrm{X},\mathbf{Q})$ of the traces of the special Schubert cycles in $\mathbf{G}(2,5)$ (see § 2 below for the notation). One has $\mathrm{H}=c_1$. A simple computation (using (1.4)) gives $\mathrm{H}\cdot\mathrm{H}_4=5c_2$, from which one can construct a class $\alpha\in\mathrm{H}^4(\mathrm{X},\mathbf{Q})_o$ with $\mathrm{H}\cdot\alpha\neq 0$.

3) Condition (iv) in its current form has been shown to me by A. Bruno. In an earlier version I used a weaker condition (for $n=2k-2$):

(iv)$'$ *the cohomology class of the subvariety of* X *spanned by the lines passing through a general point is proportional to* $\mathrm{H}_{\frac{n}{2}}$.

Using (1.4) one shows that this cohomology class is equal to $\frac{1}{d}\,\mathrm{H}\cdot\mathrm{H}_n$, so that (iv)$'$ is essentially equivalent to the result of prop. 1 (in particular, (iv) implies (iv)$'$).

2. Complete intersections

Let X be a smooth complete intersection in $\mathbf{P}^{n+r}$ of degree $(d_1,\ldots,d_r)$ and dimension $n\geq 3$, with $n\geq 2\sum(d_i-1)-1$. To prove the theorem, we can assume in view of (1.1) that X is general; then the variety of lines (resp. conics, resp. twisted cubics) contained in X has the expected dimension: see for instance [E-S], where the proof (given for the case of twisted cubics) adapts immediately to the easier cases of lines and conics. Let us check that the hypotheses of Proposition 1 are satisfied. Condition (i) holds by the weak Lefschetz theorem. One has $\mathrm{K_X}=-k\mathrm{H}$, with $k=n+1-\sum(d_i-1)$; therefore the inequality on n ensures that (ii) holds. The space $\mathrm{H}^n(\mathrm{X},\mathbf{Q})$ is nonzero except for odd-dimensional quadrics [D], so condition (iii) holds as well. Finally if $\mathrm{H}^n(\mathrm{X},\mathbf{Q})$ is of dimension 2 for n even, it is of type $(\frac{n}{2},\frac{n}{2})$; by [D] this is possible only for even-dimensional quadrics, which gives (iv).

Therefore the quantum cohomology of X is given by Proposition 1; to achieve the proof of the Theorem it remains to compute the number $\mu(\mathrm{X})=\sum\ell_p$. Recall that $d\ell_p$ is the number of lines in X meeting two general linear spaces of codimension $n-p$ and $k+p-1$ respectively (Remark 1). This number has been computed by Libgober [L]; I will give here a different proof.

Let V be a complex vector space, of dimension N ; let us denote by $\mathrm{G}=\mathbf{G}(2,\mathrm{V})$ the Grassmannian of lines in the projective space $\mathbf{P}(\mathrm{V})$ [1]. On G we have a tautological exact sequence

$$0\to \mathrm{S}\longrightarrow \mathcal{O}_\mathrm{G}\otimes_\mathbf{C}\mathrm{V}\longrightarrow \mathrm{Q}\to 0\ ,$$

where the sub- and quotient bundles S and Q are of rank 2 and $\mathrm{N}-2$ respectively.

The Chern classes $c_1,\ldots,c_{\mathrm{N}-2}$ of Q are represented by the *special Schubert cycles*:

$$c_p=\mathrm{cl}\,\{\ell\in\mathrm{G}\mid \ell\cap\mathrm{H}_{p+1}\neq\emptyset\}$$

where H_{p+1} is a fixed linear subspace of $\mathbf{P}(\mathrm{V})$ of codimension $p+1$. In particular, the subvariety of lines in G meeting two general linear spaces of codimension $p+1$ and $q+1$ has cohomology class $c_p\,c_q$.

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1 We use the naive convention, i.e. $\mathbf{P}(\mathrm{V})$ is the variety of lines in V .

Let $f \in \mathrm{S}^d\mathrm{V}^*$ be a homogeneous polynomial of degree d on $\mathbf{P}(\mathrm{V})$. It defines by restriction a global section $\overline{f}$ of $\mathrm{S}^d\mathrm{S}^*$, which vanishes exactly at the points of G where the corresponding line is contained in the hypersurface $f=0$. In other words, the subvariety of lines contained in this hypersurface is the zero locus of $\overline{f} \in \mathrm{H}^0(\mathrm{G}, \mathrm{S}^d\mathrm{S}^*)$. If f is general enough, it has the expected codimension $d+1$, and therefore its cohomology class is the top Chern class $c_{d+1}(\mathrm{S}^d\mathrm{S}^*)$. Hence the cohomology class of the variety of lines contained in our complete intersection X is $c_{d_1+1}(\mathrm{S}^{d_1}\mathrm{S}^*)\ldots c_{d_r+1}(\mathrm{S}^{d_r}\mathrm{S}^*)$. Therefore we find

$$\ell_p = \frac{1}{d}\int_{\mathrm{G}} c_{d_1+1}(\mathrm{S}^{d_1}\mathrm{S}^*)\ldots c_{d_r+1}(\mathrm{S}^{d_r}\mathrm{S}^*)\, c_{n-1-p}\, c_{k-2+p}$$

(recall that $k = n+r+1-\sum d_i$).

We will compute this number using the Chern classes $x = c_1(\mathrm{S}^*)$, $y = c_2(\mathrm{S}^*)$, or rather the virtual classes α, β such that $x = \alpha+\beta$, $y = \alpha\beta$. The Schubert cycles c_p are then given by

$$\begin{aligned} 1 + c_1 + \ldots + c_{\mathrm{N}-2} = (1-x+y)^{-1} &= (1-\alpha)^{-1}(1-\beta)^{-1} \\ &= \frac{1}{\alpha-\beta}\left(\frac{\alpha}{1-\alpha} - \frac{\beta}{1-\beta}\right), \end{aligned}$$

hence $c_p = \dfrac{\alpha^{p+1}-\beta^{p+1}}{\alpha-\beta}$; the Chern class $c_{d+1}(\mathrm{S}^d\mathrm{S}^*)$ is equal to $\displaystyle\prod_{j=0}^{d}(j\alpha + (d-j)\beta)$. To integrate we use the following lemma:

Lemma.– *Let* $\mathrm{P} \in \mathbf{C}[\alpha,\beta]$ *be a symmetric homogeneous polynomial of degree* $2(\mathrm{N}-2)$ *(so that* $\mathrm{P}(\alpha,\beta)$ *is a polynomial of maximum degree in the Chern classes* x *and* y*). Then* $\int_{\mathrm{G}}\mathrm{P}(\alpha,\beta)$ *is the coefficient of* $\alpha^{\mathrm{N}-1}\beta^{\mathrm{N}-1}$ *in* $-\frac{1}{2}(\alpha-\beta)^2\,\mathrm{P}(\alpha,\beta)$.

This is probably well-known; let me give a quick proof for the sake of completeness. Put $c_p = \dfrac{\alpha^{p+1}-\beta^{p+1}}{\alpha-\beta}$ for *all* p. The (usual!) cohomology algebra of G is the algebra of symmetric polynomials in α, β, modulo the ideal generated by $c_{\mathrm{N}-1}$ and c_{N} [G]. Consider the linear form which associates to a symmetric polynomial $\mathrm{P}(\alpha,\beta)$ the coefficient of $\alpha^{\mathrm{N}-1}\beta^{\mathrm{N}-1}$ in $-\frac{1}{2}(\alpha-\beta)^2\,\mathrm{P}(\alpha,\beta)$. It vanishes on the ideal $(c_{\mathrm{N}-1}, c_{\mathrm{N}})$ and on the polynomials of degree $< 2\mathrm{N}-4$, hence factors through a linear form $\ell : \mathrm{H}^{2\mathrm{N}-4}(\mathrm{G},\mathbf{Q}) \to \mathbf{Q}$, necessarily proportional to $\int_{\mathrm{G}}$. Let us evaluate these two forms on the polynomial $c_{\mathrm{N}-2}^2$. One has $(\alpha-\beta)^2\,c_{\mathrm{N}-2}^2 = (\alpha^{\mathrm{N}-1}-\beta^{\mathrm{N}-1})^2$, hence $\ell(c_{\mathrm{N}-2}^2) = 1$; on the other hand $\int_{\mathrm{G}} c_{\mathrm{N}-2}^2$ is the number of lines in $\mathbf{P}(\mathrm{V})$ through 2 points, that is 1. This proves the lemma.

Let us apply the lemma to the polynomial $\mathrm{F}(\alpha,\beta)\,c_{n-1-p}\,c_{k-2+p}$, where $\mathrm{F}(\alpha,\beta) = \displaystyle\sum_{j=1}^{e-1} a_j\,\alpha^j\beta^{e-j}$ is a symmetric homogeneous polynomial of degree $e :=$

$\sum(d_i+1)$. One has

$$\begin{aligned}(\alpha-\beta)^2\, c_{n-1-p}\, c_{k-2+p} &= (\alpha^{n-p}-\beta^{n-p})\,(\alpha^{k-1+p}-\beta^{k-1+p})\\ &= \alpha^{n+k-1}+\beta^{n+k-1}-\alpha^{n-p}\beta^{k-1+p}-\alpha^{k-1+p}\beta^{n-p}\ .\end{aligned}$$

Since $\mathrm{N}=n+r+1$, the coefficient of $\alpha^{\mathrm{N}-1}\beta^{\mathrm{N}-1}$ in $(\alpha-\beta)^2\,\mathrm{F}(\alpha,\beta)\,c_{n-1-p}\,c_{k-2+p}$ is $2a_{r-k+1}-2a_{r+p}$; if moreover $\mathrm{F}(\alpha,\beta)$ is divisible by $(\alpha\beta)^r$, the first coefficient is zero (recall that $k>\frac{n}{2}\geq 1$). Applying this to the polynomial $\mathrm{F}(\alpha,\beta)=c_{d_1+1}(\mathsf{S}^{d_1}\mathsf{S}^*)\ldots c_{d_r+1}(\mathsf{S}^{d_r}\mathsf{S}^*)$ we get $\ell_p=a_{r+p}$, that is

$$\sum_{p=0}^{n+1-k}\ell_p\,\alpha^{r+p}\beta^{e-r-p}=\frac{1}{d}\prod_{i=1}^{r}\prod_{j=0}^{d_i}(j\alpha+(d_i-j)\beta)\ . \tag{2.1}$$

Taking $\alpha=\beta=1$ gives $\mu(\mathrm{X})=\sum\ell_p=\prod_{i=1}^{r}d_i^{d_i}$, which achieves the proof of the Theorem. Note that Libgober's formula (2.1) gives explicit expressions for the ℓ_p 's, for instance

$$\ell_0=\prod_{i=1}^{r}d_i! \tag{2.2}$$

$$\ell_1=\prod_{i=1}^{r}d_i!\ \Big(\sum_{\substack{1\leq i\leq r\\ 1\leq j<d_i}}\frac{d_i-j}{j}\Big)\ , \tag{2.3}$$

and so on.

3. Application I: enumerative formulas

Let X be a smooth projective manifold satifying the hypotheses of Proposition 1; it follows from that Proposition that all the triple products $\langle \mathrm{H}_p,\mathrm{H}_q,\mathrm{H}_r\rangle_i$ can be computed in terms of the integers ℓ_p . If the variety of lines, conics or twisted cubics in X has the expected dimension, this gives some nice enumerative formulas which we are going to describe.

Let p,q,r be positive integers $\leq n$ such that $p+q+r=n+k$; we arrange them so that $p\leq q\leq r$. Since $2k>n$ by hypothesis this implies $p<k$ and $k\leq p+q<2k$ (hence $q\geq k$). Therefore

$$\mathrm{H}_p\cdot\mathrm{H}_q=\mathrm{H}^p\cdot(\mathrm{H}^q-(\sum_{i=0}^{q-k}\ell_i)\,\mathrm{H}^{q-k})=(\mu(\mathrm{X})-\sum_{i=0}^{q-k}\ell_i)\ \mathrm{H}_{p+q-k}\ ,$$

hence $$\langle \mathrm{H}_p, \mathrm{H}_q, \mathrm{H}_r\rangle_1 = (\mathrm{H}_p\cdot \mathrm{H}_q \,|\, \mathrm{H}_r) = d\,(\mu(\mathrm{X}) - \sum_{i=0}^{q-k} \ell_i)\ .$$

Using the equalities $\ell_i = \ell_{n+1-k-i}$ and the convention $\ell_i = 0$ for $i > n+1-k$, we find

Proposition 2 .– *Assume that the variety of lines contained in* X *has the expected dimension* $n+k-3$. *Let* p,q,r *be positive integers such that* $p \le q \le r \le n$ *and* $p+q+r = n+2k$. *The number of lines in* X *meeting three general linear spaces of codimension* p , q *and* r *respectively in* $\mathbf{P}^{n+r}$ *is* $d\sum_{i=0}^{n-q} \ell_i$.

Actually this could also be obtained by a computation in the Grassmannian as in § 2. This is probably also the case for the next results, though the computation would be much more involved.

Let us look at conics. Let p,q,r be positive integers such that $p+q+r = n+2k$; as above we assume $p \le q \le r \le n$. Moreover we will assume $k < n$, which excludes only the trivial case of quadrics [K-O]. This implies $p < k$ and therefore $2k \le p+q < 3k$. We have as before $\mathrm{H}_p\cdot\mathrm{H}_q = (\sum_{i=0}^{n-q}\ell_i)\ \mathrm{H}^{p+q-k}$; since $\mathrm{H}^{p+q-k} = \mathrm{H}_{p+q-k} + (\sum_{j=0}^{n-r}\ell_j)\,\mathrm{H}_{n-r}$, we obtain
$$\langle \mathrm{H}_p, \mathrm{H}_q, \mathrm{H}_r\rangle_2 = (\mathrm{H}_p\cdot \mathrm{H}_q \,|\, \mathrm{H}_r) = d\,(\sum_{i=0}^{n-q}\ell_i)\ (\sum_{j=0}^{n-r}\ell_j)\ .$$

Proposition 3 .– *Assume that* X *is not a quadric, and that the variety of conics contained in* X *has the expected dimension* $n+2k-3$. *Let* p,q,r *be positive integers such that* $p\le q\le r\le n$ *and* $p+q+r = n+2k$. *The number of conics in* X *meeting three general linear spaces of codimension* p , q *and* r *respectively in* $\mathbf{P}^{n+r}$ *is* $d\,(\sum_{i=0}^{n-q}\ell_i)\,(\sum_{j=0}^{n-r}\ell_i)$.

This has to be taken with a grain of salt in the case $p=1$, $q=r=n$, because every hyperplane meets a conic twice, so the above number must be divided by 2 . Since H_n is d times the class of a point we find that the number of conics in X through 2 general points is $\frac{\ell_0^2}{2d}$, where ℓ_0 is the number of lines through a general point in the intersection of X with a general linear space of codimension $k-2$. For complete intersections formula (2.2) gives:

Corollary .– *Let* X *be a smooth complete intersection of degree* $(d_1,\ldots,d_r)$ *in* $\mathbf{P}^{n+r}$, *with* $n = 2\sum(d_i-1)-1$. *The number of conics in* X *passing through* 2 *general points is* $\dfrac{1}{2d}\prod_{i=1}^{r}(d_i!)^2$.

Example .– Let X be a cubic threefold, P, Q two general points in X , L, M two general lines. We find that there are 6 conics in X through P and Q – a fact that can easily be checked geometrically (the line $\langle P, Q\rangle$ meets X along a third point R ; conics in X through P and Q are in one-to-one correspondence with lines through R). Similarly from Proposition 3 we find 14 conics through P meeting L and M .

The computation for twisted cubics is very similar. Let p, q, r be positive integers with $p \leq q \leq r \leq n$ and $p+q+r = n+3k$. Since $2k > n$ this implies $p \geq k$ and $p+q > k+n$. We have

$$\begin{aligned} \mathrm{H}_p \cdot \mathrm{H}_q &= (\mathrm{H}^p - (\sum_{i=0}^{p-k} \ell_i)\, \mathrm{H}^{p-k}) \cdot (\mathrm{H}^q - (\sum_{j=0}^{q-k} \ell_j)\, \mathrm{H}^{q-k}) \\ &= \big(\mu(\mathrm{X})^2 - \mu(\mathrm{X}) \sum_{j=0}^{q-k} \ell_j - \mu(\mathrm{X}) \sum_{i=0}^{p-k} \ell_i + (\sum_{i=0}^{p-k} \ell_i)(\sum_{j=0}^{q-k} \ell_j)\big)\, \mathrm{H}^{p+q-2k} \\ &= \big(\mu(\mathrm{X}) - \sum_{i=0}^{p-k} \ell_i)(\mu(\mathrm{X}) - \sum_{j=0}^{q-k} \ell_j)\big) \ \big(\mathrm{H}_{p+q-2k} + (\sum_{m=0}^{p+q-3k} \ell_m)\, \mathrm{H}_{p+q-3k}\big) \end{aligned}$$

Reasoning as above we get:

Proposition 4 .– *Assume that the variety of twisted cubics contained in* X *has the expected dimension* $n+3k-3$. *Let* p, q, r *be positive integers such that* $p \leq q \leq r \leq n$ *and* $p+q+r = n+3k$. *The number of twisted cubics in* X *meeting three general linear spaces of codimension* p , q *and* r *respectively in* $\mathbf{P}^{n+r}$ *is*

$$d\, (\sum_{i=0}^{n-p} \ell_i)\ (\sum_{j=0}^{n-q} \ell_j)\ (\sum_{m=0}^{n-r} \ell_m)\ .$$

In particular:

Corollary .– *Let* X *be a smooth complete intersection of degree* $(d_1, \ldots, d_r)$ *in* $\mathbf{P}^{n+r}$, *with* $n = 3\sum(d_i - 1) - 3$. *Then the number of twisted cubics in* X *passing through* 3 *general points is* $\frac{1}{d^2} \prod (d_i!)^3$.

Example .– Going back to our cubic threefold we find that the number of twisted cubics through 3 general points is 24 ; this can be checked geometrically, as shown to me by S. Verra.

4. Application II: the primitive cohomology

So far we have only considered the subalgebra of $\mathrm{H}^*(\mathrm{X}, \mathbf{Z})$ generated by H . In this last section I would like to look at the remaining part. Because of the relations (R), the only interesting triple product which appears is $\langle \mathrm{H}_k, \alpha, \beta\rangle_1$ for $\alpha, \beta \in \mathrm{H}^n(\mathrm{X}, \mathbf{Z})_o$. Since $\mathrm{H}_k \cdot \alpha = (\mathrm{H}^k - \ell_0)\cdot \alpha = -\ell_0 \alpha$, we get

$$\langle \mathrm{H}_k, \alpha, \beta\rangle = (\mathrm{H}_k \cdot \alpha \,|\, \beta) = -\ell_0\, (\alpha \,|\, \beta) \ . \tag{4.1}$$

Let us translate this geometrically using (1.4). We suppose given a smooth subvariety Y of X, of codimension k and degree d_{Y}, such that the variety Γ of lines in X meeting Y is smooth, of dimension $n-2$. For instance we can take for Y a general linear section of codimension k in X; if $k = n-1$, we can take for Y a line. The correspondence

$$\begin{array}{ccc} \mathrm{R} & \xrightarrow{q} & \mathrm{X} \\ {\scriptstyle p}\big\downarrow & & \\ \Gamma & & \end{array} \qquad\qquad \text{with } \mathrm{R} := \{(\mathrm{L}, x) \in \Gamma \times \mathrm{X} \mid x \in \mathrm{L}\}$$

gives rise to a homomorphism $\varphi = p_* q^* : \mathrm{H}^n(\mathrm{X}, \mathbf{Z}) \to \mathrm{H}^{n-2}(\Gamma, \mathbf{Z})$. By definition this is a morphism of Hodge structures, i.e. $\varphi_{\mathbf{C}}$ maps $\mathrm{H}^{p,q}(\mathrm{X})$ into $\mathrm{H}^{p-1,q-1}(\Gamma)$ for $p+q=n$.

Proposition 5.– *One has* $(\varphi(\alpha) \,|\, \varphi(\beta)) = -\dfrac{\ell_0 d_{\mathrm{Y}}}{d}\, (\alpha \,|\, \beta)$ *for* $\alpha, \beta \in \mathrm{H}^n(\mathrm{X}, \mathbf{Z})_o$.

Let us choose a desingularization F of the variety of lines in X; as above it has the expected dimension $n+k-3$. We denote by U the natural family of lines above F and by $q : \mathrm{U} \to \mathrm{X}$ the natural map. The moduli space $\mathcal{M}_1$ of degree 1 maps $\mathbf{P}^1 \to \mathrm{X}$ has a natural smooth compactification, namely $\overline{\mathcal{M}}_1 = \mathrm{U} \times_{\mathrm{F}} \mathrm{U} \times_{\mathrm{F}} \mathrm{U}$; the map $e_i : \overline{\mathcal{M}}_1 \to \mathrm{X}$ $(0 \le i \le 2)$ is obtained by composing the projection p_{i+1} with q. The inverse image of Y under e_0 is then identified with the fibered product $\mathrm{R} \times_\Gamma \mathrm{R}$, in such a way that the evaluation map $e_i : \mathrm{R} \times_\Gamma \mathrm{R} \to \mathrm{X}$ is $q \circ p_i$. (1.4) yields

$$\langle \mathrm{Y}, \alpha, \beta \rangle_1 = \int_{\mathrm{R} \times_\Gamma \mathrm{R}} e_1^* \alpha \ e_2^* \beta \ .$$

Since R is a $\mathbf{P}^1$-bundle over Γ and the class $q^*\mathrm{H}$ is transversal to the fibres, the map $\lambda : \mathrm{H}^{n-2}(\Gamma, \mathbf{Z}) \oplus \mathrm{H}^n(\Gamma, \mathbf{Z}) \longrightarrow \mathrm{H}^n(\mathrm{R}, \mathbf{Z})$ given by $\lambda(\gamma, \delta) = p^*\gamma \,.\, q^*\mathrm{H} + p^*\delta$ is an isomorphism, which satisfies $p_*\lambda(\gamma, \delta) = \gamma$. Let us write

$$q^*\alpha = p^*\varphi(\alpha) \,.\, q^*\mathrm{H} + p^*\alpha' \qquad , \qquad q^*\beta = p^*\varphi(\beta) \,.\, q^*\mathrm{H} + p^*\beta' \ .$$

Let $\pi = p \circ p_1 = p \circ p_2$ be the projection of $\mathrm{R} \times_\Gamma \mathrm{R}$ onto Γ. One has

$$e_i^*\alpha = p_i^* q^* \alpha = \pi^*\varphi(\alpha)\, p_i^* q^* \mathrm{H} + \pi^*\alpha' \quad \text{, and similarly for } e_2^*\beta \ .$$

For degree reasons the last terms disappear in the product $e_1^*\alpha \,.\, e_2^*\beta$, and we get

$$\langle \mathrm{Y}, \alpha, \beta\rangle_1 = (\varphi(\alpha) \,|\, \varphi(\beta)) \int_{\mathrm{L}\times\mathrm{L}} p_1^* q^* \mathrm{H} \,.\, p_2^* q^* \mathrm{H} \;,$$

where L is a general line intersecting Y. The value of the integral is obviously 1; since the cohomology class of Y is $\dfrac{d_{\mathrm{Y}}}{d}\mathrm{H}_k$, the result follows from (4.1).

Example.– Let us go back to our favorite example, the cubic threefold, taking for Y a generic line in X. Then Γ is a smooth curve; the map $\varphi : \mathrm{H}^3(\mathrm{X}, \mathbf{Z}) \to \mathrm{H}^1(\Gamma, \mathbf{Z})$ gives rise to a morphism $\Phi : \mathrm{JX} \to \mathrm{J}\Gamma$, where $\mathrm{J}\Gamma$ is the Jacobian of Γ and JX the *intermediate Jacobian* of X (see e.g. [C-G]); the formula $(\varphi(\alpha) \,|\, \varphi(\beta)) = -2\,(\alpha \,|\, \beta)$ for $\alpha, \beta \in \mathrm{H}^3(\mathrm{X}, \mathbf{Z})$ given by Proposition 5 means that *the principal polarization of* $\mathrm{J}\Gamma$ *induces twice the principal polarization of* JX. One deduces easily from this that *the intermediate Jacobian* JX *is isomorphic* (as a principally polarized Abelian variety) *to the Prym variety associated to* Γ *and the natural involution of* Γ which maps a line L to the third line cut down on X by the 2-plane spanned by Y and L – a fundamental fact for the geometry of the cubic threefold, due to Mumford (see Appendix C of [C-G]).

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