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## **Vector Bundles on Riemann Surfaces and Conformal Field Theory**

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# Vector bundles on Riemann surfaces and Conformal Field Theory

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## Introduction

The main character of these lectures is a finite-dimensional vector space, the space of generalized (or non-Abelian) theta functions, which has recently appeared in (at least) three different domains : Conformal Field Theory (CFT), Topological Quantum Field Theory (TQFT), and Algebraic Geometry. The fact that the same space appears in such different frameworks has some fascinating consequences, which have not yet been fully explored. For instance the dimension of this space can be computed by CFT-type methods, while algebraic geometers would have never dreamed of being able to perform such a computation.

In the Kaciveli conference I had focussed (apart from the Algebraic Geometry) on the TQFT point of view. Here I have chosen instead to explain the CFT aspect. The main reason is that there is an excellent account of the TQFT part in the little book [A], which anyone wishing to learn about the subject should read. On the other hand the CFT is the most relevant part for algebraic geometers, and it is not easily accessible in the literature.

This is an introductory survey, intended for mathematicians with little background in Algebraic Geometry or Quantum Field Theory. In the first part I define a rational CFT as a way of associating to each marked Riemann surface a finite-dimensional vector space, so that certain axioms are satisfied. I explain how the dimensions of these spaces can be encoded in a finite-dimensional  $\mathbf{Z}$ -algebra, the *fusion ring* of the theory. Then I consider a particular RCFT, the WZW model, associated to a simple Lie algebra and a positive integer, and I show how the dimensions can be computed in that case.

In the second part I try to explain what is the space of non-abelian theta functions, and why it coincides with the spaces which appear in the WZW model. This allows to give an explicit formula for the dimension of this space. Then I discuss how such a formula can be used in Algebraic Geometry.

I would like to thank the organizers of the Conference for providing such a warm and stimulating atmosphere during the Conference – despite all the material difficulties they had to face.

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(\*) Proceedings de la Conférence "Algebraic and Geometric Methods in Mathematical Physics", Kaciveli (Ukraine), 1993.

### 1. The definition of a RCFT

There are various definitions in the literature of what is (or should be) a Rational Conformal Field Theory (see e.g. [B-K-Z], [F-S], [M-S 1], [S]); unfortunately they do not seem to coincide. In the following I will follow the approach of [F-S], i.e. I will deal only with *compact* algebraic curves.

I suppose given an auxiliary finite set  $\Lambda$ , endowed with an involution  $\lambda \mapsto \lambda^*$  (in practice  $\Lambda$  will be a set of representations of the symmetry algebra of the theory). By a *marked Riemann surface*  $(C, \vec{p}, \vec{\lambda})$  I mean a compact Riemann surface (not necessarily connected)  $C$  with a finite number of distinguished points  $\vec{p} = (p_1, \dots, p_n)$ , each  $p_i$  having attached a “label”  $\lambda_i \in \Lambda$ . Then a RCFT is a functor which associates to any marked Riemann surface  $(C, \vec{p}, \vec{\lambda})$  a finite-dimensional complex vector space  $V_C(\vec{p}, \vec{\lambda})$ , satisfying the following axioms :

**A 0.**  $V_{\mathbf{P}^1}(\emptyset) = \mathbf{C}$  (the symbol  $\emptyset$  means no marked points).

**A 1.** There is a canonical isomorphism

$$V_C(\vec{p}, \vec{\lambda}) \xrightarrow{\sim} V_C(\vec{p}, \vec{\lambda}^*)$$

with  $\vec{\lambda}^* = (\lambda_1^*, \dots, \lambda_n^*)$ .

**A 2.** Let  $(C, \vec{p}, \vec{\lambda})$  be the disjoint union of two marked Riemann surfaces  $(C', \vec{p}', \vec{\lambda}')$  and  $(C'', \vec{p}'', \vec{\lambda}'')$ . Then

$$V_C(\vec{p}, \vec{\lambda}) = V_{C'}(\vec{p}', \vec{\lambda}') \otimes V_{C''}(\vec{p}'', \vec{\lambda}'') .$$

**A 3.** Let  $(C_t)_{t \in D}$  be a holomorphic family of compact Riemann surfaces, parametrized by the unit disk  $D \subset \mathbf{C}$ , with marked points  $p_1(t), \dots, p_n(t)$  depending holomorphically on  $t$  (fig. 1 below). Then for any  $t \in D$  there is a canonical isomorphism

$$V_{C_t}(\vec{p}(t), \vec{\lambda}) \xrightarrow{\sim} V_{C_0}(\vec{p}(0), \vec{\lambda}) .$$

**A 4.** Same picture, but assume now that the “special fibre”  $C_0$  acquires a node  $s$  (fig. 2a and 2b); we assume that the points  $p_i(0)$  stay away from  $s$ . Let  $\tilde{C}_0$  be the *normalization* of  $C_0$ , i.e. the Riemann surface obtained by separating the two branches at  $s$  to get two distinct points  $s'$  and  $s''$ . There is an isomorphism

$$V_{C_t}(\vec{p}(t), \vec{\lambda}) \xrightarrow{\sim} \sum_{\nu \in \Lambda} V_{\tilde{C}_0}(\vec{p}(0), s', s''; \vec{\lambda}, \nu, \nu^*) .$$

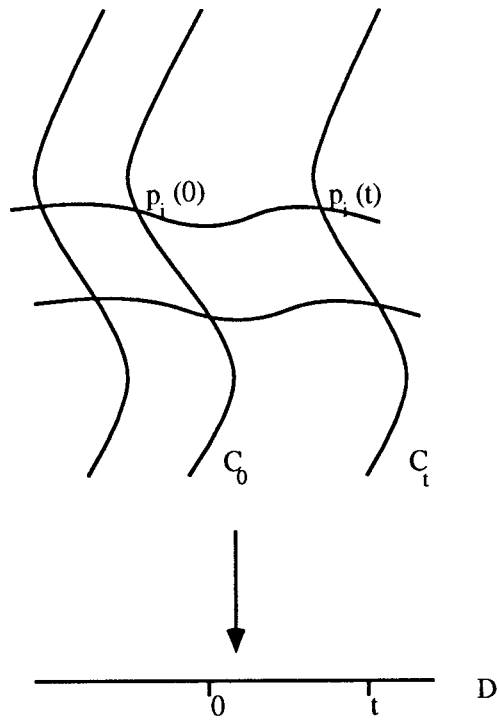


fig. 1

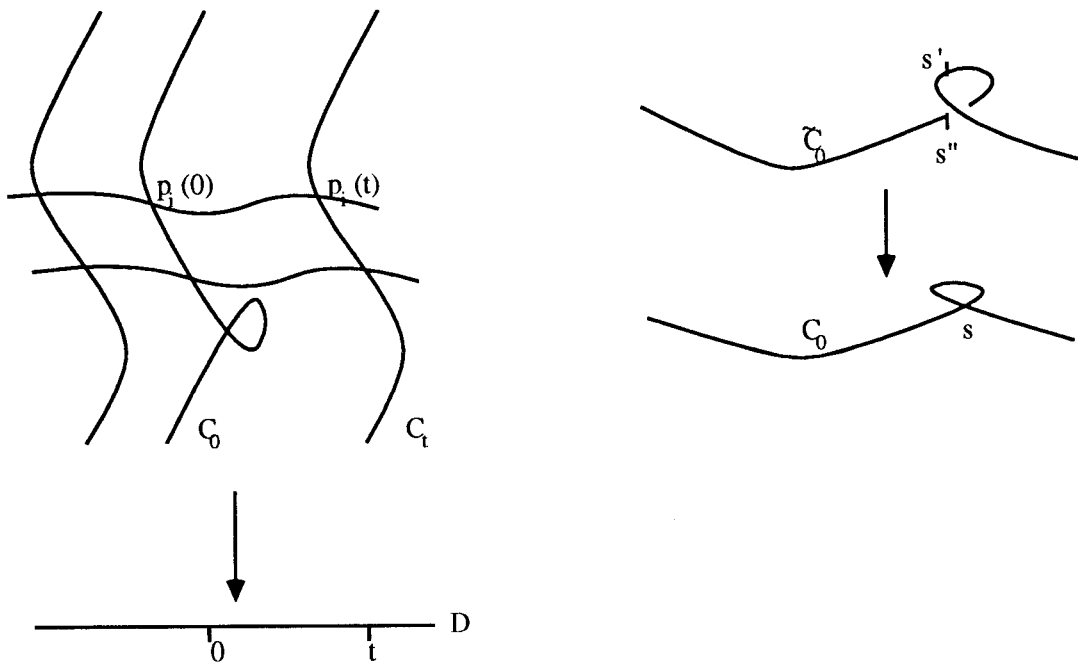


fig. 2a

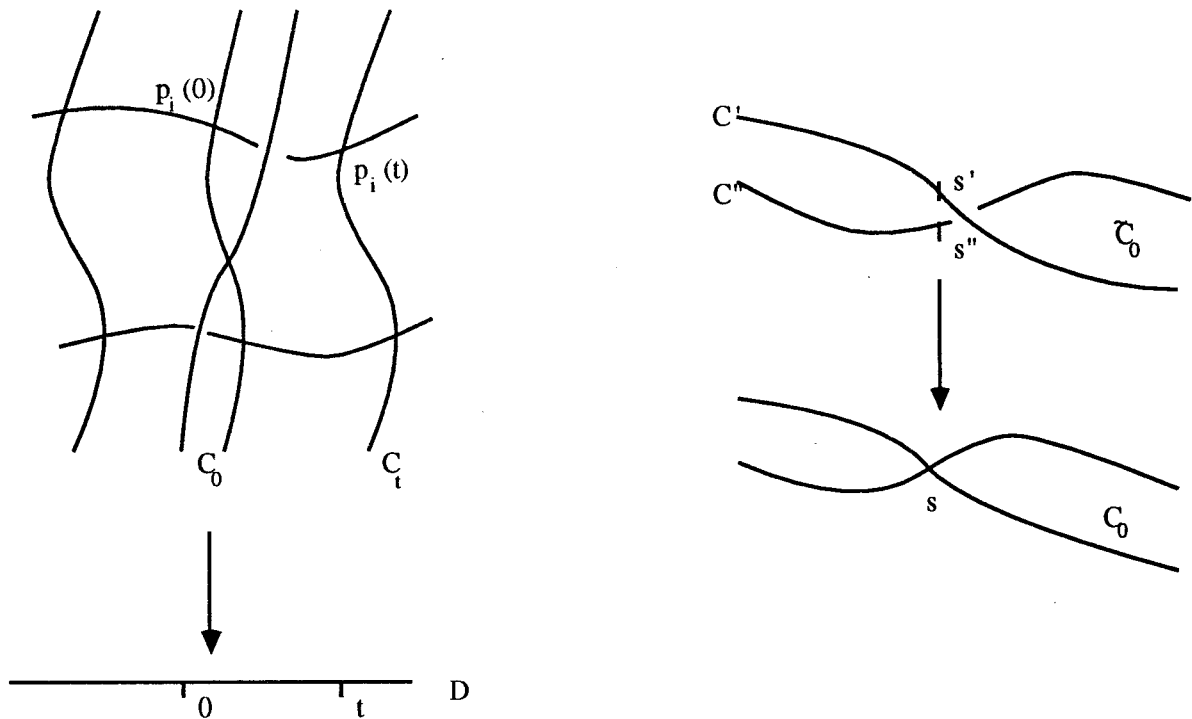


fig. 2b

There are a number of compatibilities that these isomorphisms should satisfy, but we won't need to write them down in this lecture. Let me just mention that they are most easily described in the language of vector bundles over the moduli space of marked Riemann surfaces : for instance **A 3** means that the spaces  $V_C(\vec{p}, \vec{\lambda})$  form a projectively flat vector bundle over the moduli space when  $(C, \vec{p})$  varies.

The physicists usually want the spaces  $V_C(\vec{p}, \vec{\lambda})$  to be *hermitian*, with the above axioms suitably adapted. I will not adopt this point of view here.

## 2. Physical interpretation

In this section I would like to discuss in a very informal and sketchy way why these spaces appear in physics. We are considering a quantum field theory in dimension  $1 + 1$ , so space-time is a surface  $\Sigma$  that we assume to be compact (and oriented). We are given a certain type of geometric objects, that the physicists call *fields* : these may be functions, vector fields, connections on some vector bundle... One of the most basic objects in the theory are the *correlation functions*, which assign to any finite collection of fields  $A_1, \dots, A_n$  located at distinct points  $z_1, \dots, z_n$  on  $\Sigma$  a number  $\langle A_1(z_1) \dots A_n(z_n) \rangle$ . Physically, each field  $A_i$  corresponds to some observable quantity (energy, momentum...); intuitively (and very roughly) we may

think of  $\langle A_1(z_1) \dots A_n(z_n) \rangle$  as the expectation value of the joint measurement of these quantities at the given points.

These correlation functions are usually defined in terms of Feynman integrals, for which no mathematically correct definition is known (in fact what we are trying to do here is to bypass the Feynman integral by formulating its main properties as axioms). These integrals involve a metric on the surface  $\Sigma$ , but if the theory is conformal they actually depend essentially only on the conformal class of the metric – i.e. on a complex structure on  $\Sigma$ , which we see as a point  $m$  in the moduli space of Riemann surfaces.

The symmetry algebra of the theory acts on the space of fields; let me assume that each field  $A_i$  belongs to an irreducible representation  $\lambda_i$  (these are called “primary fields”). From the behaviour of the Feynman integral, the physicists conclude that

$$\langle A_1(z_1) \dots A_n(z_n) \rangle = \langle v_A(\vec{z}, m) | v_A(\vec{z}, m) \rangle$$

where  $v_A(\vec{z}, m)$  is an element of  $V_{\Sigma_m}(\vec{z}, \vec{\lambda})$  which depends holomorphically on  $\vec{z}$  and  $m$  (more precisely,  $v_A$  is a holomorphic section of the projectively flat vector bundle formed by the  $V_{\Sigma_m}(\vec{z}, \vec{\lambda})$ ); here  $\langle | \rangle$  denote the scalar product on the hermitian vector space  $V_{\Sigma_m}(\vec{z}, \vec{\lambda})$ . From the known properties of the correlation functions one may deduce that the spaces  $V_C(\vec{z}, \vec{\lambda})$  must satisfy **A 0** to **A 4** (see [F-S]).

Let me conclude this section with an important warning : in the physical literature the correlation functions are often normalized so that one gets 1 when there are no fields. Here we consider *unnormalized* correlation functions, which means that when no field is inserted we get the *partition function* of the theory – so this is somehow the most important case. We will see later that in algebraic geometry also the corresponding vector spaces  $V_C(\emptyset)$  play a prominent part.

### 3. The fusion ring

In this lecture I will be interested only in the *dimension* of the spaces  $V_C(\vec{p}, \vec{\lambda})$  (this is why I didn’t care to be precise about the isomorphisms involved in the axioms). Observe that as a consequence of **A 3** this dimension does not change when one deforms (holomorphically) the surface and its marked points; therefore it depends only on the *genus*  $g$  of  $C$ , and of the set of labels  $(\lambda_1, \dots, \lambda_n)$  (the order is irrelevant). It is convenient to introduce the monoid  $\mathbf{N}^{(\Lambda)}$  of formal sums  $\lambda_1 + \dots + \lambda_n$  for  $n \geq 0$ ,  $\lambda_1, \dots, \lambda_n \in \Lambda$  (“free monoid generated by  $\Lambda$ ”). For

$x = \lambda_1 + \dots + \lambda_n \in \mathbf{N}^{(\Lambda)}$ , we put

$$N_g(x) := \dim V_C(p_1, \dots, p_n; \lambda_1, \dots, \lambda_n),$$

where  $C$  is any Riemann surface of genus  $g$  with  $n$  arbitrary (distinct) points  $p_1, \dots, p_n$ . So we can view  $N_g$  as a function from  $\mathbf{N}^{(\Lambda)}$  into  $\mathbf{N}$ . Let us write the consequences of our axioms. **A 0** and **A 1** give respectively :

$$(1) \quad N_0(0) = 1 \quad \text{and} \quad N_g(x^*) = N_g(x)$$

(we have extended the involution  $\lambda \mapsto \lambda^*$  to  $\mathbf{N}^{(\Lambda)}$  by linearity).

**A 3** has been already taken into account. As for **A 4**, there are two cases to consider (fig. 2a and 2b). In case a), the normalization  $\tilde{C}_0$  has genus  $g - 1$ , so we get :

$$(2) \quad N_g(x) = \sum_{\lambda \in \Lambda} N_{g-1}(x + \lambda + \lambda^*).$$

In case b),  $\tilde{C}_0$  is the disjoint union of two smooth curves  $C'$  and  $C''$ , of genus  $g'$  and  $g''$  respectively, with  $g' + g'' = g$ ; the curve  $C_0$  is obtained from  $\tilde{C}_0$  by identifying  $s' \in C'$  with  $s'' \in C''$ . Some of the marked points  $(p_i)$  of  $C_0$  lie on  $C'$ , while the others are on  $C''$ ; let  $x' = \sum_{p_i \in C'} \lambda_i$ ,  $x'' = \sum_{p_j \in C''} \lambda_j$ . Using **A 4** and

**A 2** we get

$$(3) \quad N_g(x' + x'') = \sum_{\nu \in \Lambda} N_{g'}(x' + \nu) N_{g''}(x'' + \nu^*).$$

Clearly formula (2) allows to compute all the  $N_g$ 's by induction starting from  $N_0$ , so the problem is to compute the function  $N_0 : \mathbf{N}^{(\Lambda)} \rightarrow \mathbf{N}$ . For the case  $g = 0$  the above relations read

$$(F 0) \quad N_0(0) = 1;$$

$$(F 1) \quad N_0(x^*) = N_0(x) \quad \text{for every } x \in \mathbf{N}^{(\Lambda)};$$

$$(F 2) \quad N_0(x + y) = \sum_{\nu \in \Lambda} N_0(x + \nu) N_0(y + \nu^*) \quad \text{for } x, y \text{ in } \mathbf{N}^{(\Lambda)}.$$

These relations (together with (2)) are called the *fusion rules*. We are now faced with a purely combinatorial problem : can we describe in some simple way all functions satisfying these identities? Here is the elegant solution found by the physicists.

Let me define a *fusion rule* on  $\Lambda$  as a function  $N : \mathbf{N}^{(\Lambda)} \rightarrow \mathbf{Z}$  satisfying (F 0) to (F 2); I will assume moreover that  $N$  takes at least one positive value on  $\Lambda$ . I will also assume that  $N$  is non-degenerate in the sense that for each  $\lambda \in \Lambda$ , there exists an element  $x$  of  $\mathbf{N}^{(\Lambda)}$  such that  $N(\lambda + x) \neq 0$  (otherwise one can forget this  $\lambda$  and consider the restriction of  $N$  to  $\Lambda - \{\lambda\}$ ).

Let us denote by  $\mathcal{F}$  the free abelian group  $\mathbf{Z}^{(\Lambda)}$  generated by  $\Lambda$  ; we will consider  $\Lambda$  as a subset of  $\mathcal{F}$  .

**Proposition 3.1.** – *There exists a one-to-one correspondence between fusion rules on  $\Lambda$  as above and multiplication maps  $\mathcal{F} \otimes_{\mathbf{Z}} \mathcal{F} \rightarrow \mathcal{F}$  with the following properties :*

- $\mathcal{F}$  is a commutative ring, with a unit  $1 \in \Lambda$  ;
- let  $t : \mathcal{F} \rightarrow \mathbf{Z}$  be the  $\mathbf{Z}$ -linear form such that  $t(1) = 1$  ,  $t(\lambda) = 0$  for  $\lambda \in \Lambda$  ,  $\lambda \neq 1$  . Then  $\Lambda$  is an orthonormal basis for the bilinear form  $\langle x | y \rangle := t(xy^*)$  .

The correspondence is as follows : given  $N$  , the multiplication on  $\mathcal{F}$  is defined by

$$(4) \quad \lambda \cdot \mu = \sum_{\nu \in \Lambda} N(\lambda + \mu + \nu^*) \nu .$$

Conversely, starting from the ring  $\mathcal{F}$  , we define  $N$  by

$$N(\lambda_1 + \dots + \lambda_n) = t(\lambda_1 \cdots \lambda_n) .$$

It is not difficult to check that the two constructions are inverse of each other : I refer to [B 3] for a detailed proof. ■

So to each CFT is associated a commutative ring  $\mathcal{F}$  , the *fusion ring* of the theory. It carries a ring involution  $*$  , and a scalar product  $\langle | \rangle$  satisfying  $\langle xz | y \rangle = \langle x | z^*y \rangle$  , with an orthonormal basis containing 1 . The structure of these rings is quite subtle. However, once we extend the scalars from  $\mathbf{Z}$  to  $\mathbf{C}$  , it becomes essentially trivial :

**Lemma 3.2.** – *The ring  $\mathcal{F}_{\mathbf{C}} := \mathcal{F} \otimes_{\mathbf{Z}} \mathbf{C}$  is isomorphic to the product ring  $\mathbf{C}^n$  , with  $n = \text{Card}(\Lambda)$  .*

*Proof :* Extend the bilinear form  $\langle | \rangle$  on  $\mathcal{F}$  to a hermitian scalar product on  $\mathcal{F}_{\mathbf{C}}$  . For any  $x \in \mathcal{F}$  , let  $m_x$  denote the endomorphism  $y \mapsto xy$  of  $\mathcal{F}_{\mathbf{C}}$  . The formula  $\langle yx | z \rangle = \langle y | x^*z \rangle$  implies that the adjoint endomorphism of  $m_x$  is  $m_{x^*}$  ; since the endomorphisms  $m_x$  commute, they are normal, hence diagonalizable, and the  $\mathbf{C}$ -algebra  $\mathcal{F}_{\mathbf{C}}$  is semi-simple. ■

Let  $\Sigma$  be the *spectrum* of  $\mathcal{F}_{\mathbf{C}}$  , that is the (finite) set of characters (= ring homomorphisms)  $\mathcal{F} \rightarrow \mathbf{C}$  . There is a natural homomorphism of  $\mathbf{C}$ -algebras  $\Phi : \mathcal{F}_{\mathbf{C}} \rightarrow \mathbf{C}^{\Sigma}$  mapping  $x \in \mathcal{F}$  to  $(\chi(x))_{\chi \in \Sigma}$  . One can rephrase the lemma in a more intrinsic way by saying that  $\Phi$  is an isomorphism of  $\mathbf{C}$ -algebras.

For any  $x \in \mathcal{F}$  , let  $m_x$  denote the endomorphism  $y \mapsto xy$  of  $\mathcal{F}$  . Then the endomorphism  $\Phi m_x \Phi^{-1}$  of  $\mathbf{C}^{\Sigma}$  is the multiplication by  $\Phi(x)$  ; in the canonical



basis of  $\mathbf{C}^\Sigma$ , it is represented by the diagonal matrix with entries  $(\chi(x))_{x \in \Sigma}$ . This implies in particular  $\text{Tr } m_x = \sum_{\chi \in \Sigma} \chi(x)$ . On the other hand, from the relation

$$\lambda\mu = \sum_{\nu \in \Lambda} N(\lambda + \mu + \nu^*) \nu = \sum_{\nu \in \Lambda} t(\lambda\mu\nu^*) \nu$$

one gets  $\text{Tr } m_\lambda = \sum_{\nu \in \Lambda} t(\lambda\nu\nu^*) = t(\lambda\omega)$ , where  $\omega$  is the element  $\sum_{\lambda \in \Lambda} \lambda\lambda^*$  of  $\mathcal{F}$ . By linearity this gives

$$(5) \quad t(x\omega) = \text{Tr } m_x = \sum_{\chi \in \Sigma} \chi(x)$$

for all  $x \in \mathcal{F}_\mathbf{C}$ . Since  $\chi(\omega) = \sum_{\lambda \in \Lambda} |\chi(\lambda)|^2 > 0$ , the element  $\omega$  is invertible in  $\mathcal{F}_\mathbf{C}$ ; replacing  $x$  by  $x\omega^{-1}$  gives

$$t(x) = \sum_{\chi \in \Sigma} \frac{\chi(x)}{\chi(\omega)}$$

Let us now commute  $N_g$ : from (3) we get by induction on  $g$

$$\begin{aligned} N_g(\lambda_1 + \dots + \lambda_n) &= \sum_{\nu_1, \dots, \nu_g \in \Lambda} N_0(\lambda_1 + \dots + \lambda_n + \nu_1 + \nu_1^* + \dots + \nu_g + \nu_g^*) \\ &= \sum_{\nu_1, \dots, \nu_g \in \Lambda} t(\lambda_1 \dots \lambda_n \nu_1 \nu_1^* \dots \nu_g \nu_g^*) \\ &= t(\lambda_1 \dots \lambda_n \omega^g); \end{aligned}$$

comparing with (5) we obtain

$$N_g(\lambda_1 + \dots + \lambda_n) = \sum_{\chi \in \Sigma} \chi(\lambda_1) \dots \chi(\lambda_n) \chi(\omega)^{g-1}.$$

In conclusion :

**Proposition 3.3.** – *Let  $(C, \vec{p}, \vec{\lambda})$  be a Riemann surface of genus  $g$  with  $n$  marked points. Then for any RCFT*

$$\dim V_C(\vec{p}, \vec{\lambda}) = \sum_{\chi \in \Sigma} \chi(\lambda_1) \dots \chi(\lambda_n) \chi(\omega)^{g-1}$$

where  $\Sigma$  is the set of characters of the fusion ring, and  $\chi(\omega) = \sum_{\lambda \in \Lambda} |\chi(\lambda)|^2$ . ■

Thus we will be able to compute the dimensions of the spaces  $V_C(\vec{p}, \vec{\lambda})$  once we know explicitly the characters of the fusion ring – or equivalently the isomorphism  $\mathcal{F}_\mathbf{C} \xrightarrow{\sim} \mathbf{C}^\Sigma$ .

#### 4. The Verlinde conjecture

The physicists use an equivalent, but slightly different formulation of the Proposition. We have seen in lemma 3.2 that the endomorphisms  $m_x$  ( $x \in \mathcal{F}_\mathbf{C}$ )

form a commutative subalgebra of  $\text{End}(\mathcal{F}_{\mathbf{C}})$ , stable under adjunction. Such an algebra is diagonalizable in an orthonormal basis; in other words, there exists a unitary matrix  $S = (S_{\lambda\mu})_{\lambda,\mu \in \Lambda}$  such that the matrix  $\Delta_x := Sm_x S^{-1}$  is diagonal for every  $x \in \mathcal{F}$  (here we still use the notation  $m_x$  for the matrix of the endomorphism  $m_x$  in the basis  $\Lambda$ ). The physicists use to say that the matrix  $S$  “diagonalizes the fusion rules”.

Fix such a matrix  $S$ . For  $\lambda \in \Lambda$ ,  $x \in \mathcal{F}$ , let  $\chi_\lambda(x)$  be the diagonal coefficient  $(\Delta_x)_{\lambda\lambda}$ . Clearly  $\chi_\lambda$  is a character of  $\mathcal{F}$ , and we get in this way all the characters. So *the choice of the matrix  $S$  provides a bijection  $\Lambda \xrightarrow{\sim} \Sigma$* . Moreover the characters  $\chi_\lambda$  have a simple expression in terms of  $S$ : the equality  $Sm_\mu = \Delta_\mu S$ , for  $\mu \in \Lambda$ , is equivalent to

$$\sum_{\nu} S_{\lambda\nu} N(\mu + \rho + \nu^*) = \chi_\lambda(\mu) S_{\lambda\rho}$$

for every  $\lambda, \rho \in \Lambda$ . Take  $\rho = 1$ ; from (4) we get  $N(1 + \mu + \nu^*) = \delta_{\mu\nu}$ , hence

$$\chi_\lambda(\mu) = \frac{S_{\lambda\mu}}{S_{\lambda 1}}.$$

Let us express Proposition 3.1 in terms of  $S$ . Replacing  $S$  by  $DS$ , where  $D$  is a diagonal unitary matrix, we can suppose that the numbers  $S_{\lambda 1}$  are real positive. Since  $S$  is unitary we have

$$\chi_\lambda(\omega) = \sum_{\nu} |\chi_\lambda(\nu)|^2 = \sum_{\lambda} \frac{|S_{\lambda\nu}|^2}{S_{\lambda 1}^2} = \frac{1}{S_{\lambda 1}^2},$$

and therefore

$$N_g(\lambda_1 + \dots + \lambda_p) = \sum_{\nu} \frac{S_{\nu\lambda_1} \dots S_{\nu\lambda_p}}{S_{\nu 1}^{2g-2+p}};$$

this is the formulation usually found in the physics literature.

Let me now explain the original Verlinde conjecture. I have to be sketchy here because I have not formulated precisely the rules that the isomorphisms which appear in the axioms **A 0** to **A 4** should obey.

Let  $E$  be an elliptic curve, which we write as the quotient of  $\mathbf{C}$  by a lattice  $\mathbf{Z} + \mathbf{Z}\tau$ , with  $\tau \in \mathbf{H}$  (Poincaré upper half-plane). In this way  $\mathbf{H}$  parametrizes a (universal) family of elliptic curves. Since for each  $\gamma \in \text{SL}_2(\mathbf{Z})$  the curves corresponding to  $\gamma\tau$  is isomorphic to  $E$ , axiom **A 3** provides an action of  $\text{SL}_2(\mathbf{Z})$  on  $V_E(\emptyset)$ . This action should be linear (or at least projective), and unitary for the natural hermitian metric of  $V_E(\emptyset)$ .

On the other hand, let us degenerate  $E$  into  $\mathbf{P}^1$  with 2 points  $p, p^*$  identified. Axiom **A 4** gives an isomorphism  $V_E(\emptyset) \xrightarrow{\sim} \bigoplus_{\lambda \in \Lambda} V_{\mathbf{P}^1}(p, p^*; \lambda, \lambda^*)$ , which again must be unitary. We know that  $V_{\mathbf{P}^1}(p, p^*; \lambda, \lambda^*)$  is one-dimensional; actually, because of **A 4** it should have a canonical generator, so we get a unitary

isomorphism  $V_E(\emptyset) \xrightarrow{\sim} \mathcal{F}_C$ . Putting things together we obtain a *unitary action of  $SL_2(\mathbf{Z})$  onto  $\mathcal{F}_C$* . This action can usually be written explicitly : for instance when the symmetry algebra is a Kac-Moody algebra (as in the WZW model that we will study below), it corresponds to the usual action of  $SL_2(\mathbf{Z})$  on the characters of the representations parametrized by  $\Lambda$ . In any case, the conjecture is :

**Verlinde's conjecture.** – *The matrix  $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  acting on  $\mathcal{F}_C$  diagonalizes the fusion rules.*

I must say the current status of the conjecture is not clear to me. A proof appears in [M-S 2], but there seems to be some doubt among the experts. Moreover it is not obvious that the axioms of a RCFT given in [M-S 1,2] coincide with ours.

## 5. The WZW model

Of course the above analysis is interesting only if we can exhibit examples of theories satisfying our axioms. A basic example for the physicists is the Wess-Zumino-Witten (WZW) model. It is usually defined through a Feynman integral; in our framework, the rigorous construction of these models and the proof that they satisfy axioms **A 0** to **A 4** have been carried out in the beautiful paper [T-U-Y].

The WZW model is associated to a simple complex Lie algebra  $\mathfrak{g}$  and a positive integer  $\ell$  (the level). We choose a Cartan subalgebra  $\mathfrak{h} \subset \mathfrak{g}$ . Recall that the irreducible finite-dimensional representations of  $\mathfrak{g}$  are parametrized by certain linear forms on  $\mathfrak{h}$  called the *dominant weights* (in the case  $\mathfrak{g} = \mathfrak{sl}_r(\mathbf{C})$ , we take for  $\mathfrak{h}$  the subspace of diagonal matrices; the dominant weights are the linear combinations  $\sum_{i=1}^{r-1} n_i \varepsilon_i$  where  $\varepsilon_i$  is the linear form  $H \mapsto H_{ii}$  and the  $n_i$ 's are integers satisfying  $n_1 \geq n_2 \geq \dots \geq n_{r-1}$ ). We denote by  $P_+$  the set of dominant weights; for  $\lambda \in P_+$ , we let  $V_\lambda$  be the corresponding representation. We define the *level* of  $V_\lambda$  as the integer  $\langle \lambda, \theta^\vee \rangle$ , where  $\theta^\vee$  is the coroot associated to the highest root of  $(\mathfrak{g}, \mathfrak{h})$  – for  $\mathfrak{g} = \mathfrak{sl}_r(\mathbf{C})$  and  $\lambda = \sum n_i \varepsilon_i$  as above, the level is  $n_1$ .

The set  $P_\ell$  of dominant weights of level  $\leq \ell$  is finite; this will be our auxiliary set  $\Lambda$ . For  $\lambda \in P_\ell$ , the dominant weight  $\lambda^*$  associated to the dual representation of  $V_\lambda$  still belongs to  $P_\ell$ ; this defines the involution on  $P_\ell$ .

To define the spaces  $V_C(\vec{p}, \vec{\lambda})$  for a connected Riemann surface  $C$ , we choose an auxiliary point  $q \in C$  distinct from the  $p_i$ 's, and a local coordinate  $z$  at  $q$  (the construction will be independent of these choices). We denote by  $A_C$  the algebra of regular functions on  $C - q$  – that is, functions which are holomorphic in  $C - q$  and meromorphic at  $q$ . We endow  $\mathfrak{g} \otimes A_C$  with the obvious Lie algebra structure given by  $[X \otimes f, Y \otimes g] = [X, Y] \otimes fg$ . We will define below a natural representation  $\mathcal{H}_\ell$

of  $\mathfrak{g} \otimes A_C$  ; on the other hand,  $\mathfrak{g} \otimes A_C$  acts on each  $V_{\lambda_i}$  by  $(X \otimes f) \cdot v = f(p_i)Xv$ , hence on the tensor product  $V_{\vec{\lambda}} := V_{\lambda_1} \otimes \dots \otimes V_{\lambda_n}$ . We put

$$V_C(\vec{p}, \vec{\lambda}) := \text{Hom}_{\mathfrak{g} \otimes A_C}(\mathcal{H}_\ell, V_{\vec{\lambda}}).$$

To explain what is  $\mathcal{H}_\ell$ , let me first recall the definition of the affine Lie algebra  $\widehat{\mathfrak{g}}$  associated to  $\mathfrak{g}$  (I refer to [K] for the few facts I will use about Kac-Moody algebras; the reader may take them as a black box). Let  $\mathbf{C}((z))$  denote the field of meromorphic (formal) Laurent series in  $z$ ; we put  $\widehat{\mathfrak{g}} = (\mathfrak{g} \otimes \mathbf{C}((z))) \oplus \mathbf{C}c$ , the bracket of two elements of  $\mathfrak{g} \otimes \mathbf{C}((z))$  being given by

$$(6) \quad [X \otimes f, Y \otimes g] = [X, Y] \otimes fg + c \cdot (X | Y) \text{Res}_0(g df),$$

where  $( | )$  is the normalized Killing form ( $(A | B) = \text{Tr} AB$  for  $\mathfrak{g} = \mathfrak{sl}_r(\mathbf{C})$ ).

Kac-Moody theory tells us that  $\widehat{\mathfrak{g}}$  admits a unique irreducible representation  $\mathcal{H}_\ell$ , called the *basic representation* of level  $\ell$ , with the following properties :

- a) The central element  $c$  acts by multiplication by  $\ell$  ;
- b) There exists a non-zero vector  $v$  in  $\mathcal{H}_\ell$  annihilated by  $\mathfrak{g} \otimes \mathbf{C}[[z]]$ .

Let  $U^-$  be the subalgebra of  $\text{End}(\mathcal{H}_\ell)$  spanned by the elements  $X \otimes z^{-p}$  with  $p \geq 1$  ; let  $X_\theta \in \mathfrak{g}$  be an eigenvector for the adjoint action of  $\mathfrak{h}$  w.r.t. the highest root  $\theta$  (for  $\mathfrak{g} = \mathfrak{sl}_r(\mathbf{C})$ ,  $X_\theta$  is the elementary matrix  $E_{1r}$ ). Then

- c) As a  $U^-$ -module,  $\mathcal{H}_\ell$  is spanned by the vector  $v$ , with the only relation  $(X_\theta \otimes z^{-1})^{\ell+1}v = 0$ .

Let us go back to our situation. By associating to each function  $f \in A_C$  its Laurent expansion at  $q$ , we get an embedding  $A_C \hookrightarrow \mathbf{C}((z))$ , hence also an embedding of Lie algebras  $\mathfrak{g} \otimes A_C \hookrightarrow \mathfrak{g} \otimes \mathbf{C}((z))$ . The Residue theorem and formula (6) imply that  $\mathfrak{g} \otimes A_C$  is also a Lie subalgebra of  $\widehat{\mathfrak{g}}$ , hence  $\mathfrak{g} \otimes A_C$  acts on  $\mathcal{H}_\ell$  as required.

Let me now state the main result of [T-U-Y] :

**Proposition 5.1.** — *The spaces  $V_C(\vec{p}, \vec{\lambda}) = \text{Hom}_{\mathfrak{g} \otimes A_C}(\mathcal{H}_\ell, V_{\vec{\lambda}})$  satisfy the axioms A 0 to A 4, and therefore define a RCFT. ■*

We will denote by  $\mathcal{R}_\ell(\mathfrak{g})$  the corresponding fusion ring. What can we say about this ring? The spaces  $V_C(\vec{p}, \vec{\lambda})$  are quite difficult to compute in general, but the situation is simpler when  $C = \mathbf{P}^1$  : the ring  $A_C$  is just the polynomial ring  $\mathbf{C}[z^{-1}]$ , so  $V_{\mathbf{P}^1}(\vec{p}, \vec{\lambda})$  is the space of maps  $\mathcal{H}_\ell \rightarrow V_{\vec{\lambda}}$  which are both  $\mathfrak{g}$ -linear and  $U^-$ -linear. By property c) above such a map is determined by the image  $v'$  of  $v$ , with the only relations  $\mathfrak{g} \cdot v' = 0$  and  $(X_\theta \otimes z^{-1})^{\ell+1} \cdot v' = 0$ . Therefore :

**Proposition 5.2.** —  *$V_{\mathbf{P}^1}(\vec{p}, \vec{\lambda})$  is the subspace of elements of  $V_{\vec{\lambda}}$  which are annihilated by  $\mathfrak{g}$  and by  $(X_\theta \otimes z^{-1})^{\ell+1}$ . ■*

To explain the significance of this result, consider the situation when  $\ell \rightarrow \infty$ . The set  $P_\ell$  becomes the (infinite) set  $P_+$  parametrizing all irreducible (finite-dimensional) representations of  $\mathfrak{g}$ . The condition of annihilation by  $(X_\theta \otimes z^{-1})^{\ell+1}$  is always satisfied for  $\ell$  large enough, since the action of  $X_\theta$  on any representation is nilpotent. So the limit space  $V_{\mathbf{P}_+}^{(\infty)}(\vec{p}, \vec{\lambda})$  is simply the  $\mathfrak{g}$ -invariant subspace of  $V_{\vec{\lambda}}$ . In particular, we find

$$N(\lambda + \mu + \nu^*) = \dim \operatorname{Hom}_{\mathfrak{g}}(V_\nu, V_\lambda \otimes V_\mu)$$

Write  $V_\lambda \otimes V_\mu$  as a sum of irreducible representations  $V_\rho$  (possibly with multiplicities). By Schur's lemma  $\operatorname{Hom}_{\mathfrak{g}}(V_\nu, V_\rho)$  is 0 for  $\nu \neq \rho$ , and  $\mathbf{C}$  for  $\nu = \rho$ . Hence  $N(\lambda + \mu + \nu^*)$  is (in the limit) the multiplicity of  $V_\nu$  as an irreducible component of  $V_\lambda \otimes V_\mu$ . In other words, the limit fusion ring  $\mathcal{F}_\infty$  is the *representation ring*  $\mathcal{R}(\mathfrak{g})$  of  $\mathfrak{g}$ : by definition, this is the free abelian group with basis  $([V_\lambda])_{\lambda \in P_+}$  and with multiplication rule

$$[V_\lambda] \cdot [V_\mu] = [V_\lambda \otimes V_\mu] := \sum_{\nu} N_{\lambda\mu}^{\nu} [V_\nu] \quad \text{where} \quad V_\lambda \otimes V_\mu = \bigoplus_{\nu \in P_+} N_{\lambda\mu}^{\nu} V_\nu.$$

For finite  $\ell$ , we only get  $N(\lambda + \mu + \nu^*) \leq N_{\lambda\mu}^{\nu}$ . Hence the product  $[V_\lambda] \cdot [V_\mu]$  in the fusion ring  $\mathcal{R}_\ell(\mathfrak{g})$  is the class of a  $\mathfrak{g}$ -module  $V_\lambda \circ V_\mu$  which appears as a quotient (or a submodule) of  $V_\lambda \otimes V_\mu$ . We have thus defined a kind of “skew tensor product” for representations of level  $\leq \ell$ , which unlike the usual tensor product is still of level  $\leq \ell$ . Finding a more natural definition of this product, e.g. through the theory of quantum groups, is a very interesting question which is apparently still open; such a definition should provide a better proof of the proposition below.

We see in particular that the natural inclusion of  $\mathcal{R}_\ell(\mathfrak{g})$  into  $\mathcal{R}(\mathfrak{g})$  is *not* a ring homomorphism. It turns out that  $\mathcal{R}_\ell(\mathfrak{g})$  can be viewed as a *quotient* of  $\mathcal{R}(\mathfrak{g})$ :  
**Proposition 5.3.** — *There is a natural ring homomorphism  $\pi : \mathcal{R}(\mathfrak{g}) \rightarrow \mathcal{R}_\ell(\mathfrak{g})$  such that  $\pi([V_\lambda]) = [V_\lambda]$  for each  $\lambda \in P_\ell$ .*

The proof (see [F]) follows from a case by case combinatorial analysis. It is easy for the Lie algebras  $\mathfrak{sl}_r$  or  $\mathfrak{sp}_{2r}$ , more involved for the other classical Lie algebras; to my knowledge it does not even exist for some exceptional Lie algebras. Hopefully a more conceptual proof would follow from a better definition of the product in  $\mathcal{R}_\ell(\mathfrak{g})$  as mentioned above. ■

From Proposition 5.3 it is not difficult to write down explicitly the characters of  $\mathcal{R}_\ell(\mathfrak{g})$ : they correspond to those characters of  $\mathcal{R}(\mathfrak{g})$  which factor through  $\pi$ . I refer to [B 3] for the general case, which involves some Lie theory. Let me give the simplest possible example, namely the case  $\mathfrak{g} = \mathfrak{sl}_2(\mathbf{C})$ .

## 6. An example : $\mathfrak{g} = \mathfrak{sl}_2(\mathbf{C})$

In this section we take  $\mathfrak{g} = \mathfrak{sl}_2(\mathbf{C})$ ; we denote by  $S^p$  the  $p$ -th symmetric product of the standard representation  $\mathbf{C}^2$  of  $\mathfrak{g}$ . The  $S^p$ 's for  $p \geq 0$  form all irreducible representations of  $\mathfrak{g}$ . The tensor product of two such representations is given by the Clebsch-Gordan rule

$$S^p \otimes S^q = S^{p+q} \oplus S^{p+q-2} \oplus \dots \oplus S^{p-q} \quad \text{for } p \geq q .$$

The level of the representation  $S^p$  is  $p$  (with the notation of § 5, the highest weight is  $p\varepsilon_1$ ), so  $\mathcal{R}_\ell(\mathfrak{g})$  is the free  $\mathbf{Z}$ -module with basis  $\{S^0, \dots, S^\ell\}$ . Working out Proposition 5.2 in that case gives the following rule for the product :

$$\begin{aligned} S^p \odot S^q &= S^p \otimes S^q && \text{if } p+q \leq \ell \\ &= S^{2\ell-p-q} \oplus S^{2\ell-p-q-2} \oplus \dots \oplus S^{p-q} && \text{if } p \geq q, p+q \geq \ell . \end{aligned}$$

From this it is an easy exercise to check that *the fusion ring  $\mathcal{R}_\ell(\mathfrak{g})$  is the quotient of  $\mathcal{R}(\mathfrak{g})$  by the ideal generated by  $[S^{\ell+1}]$ .*

A convenient way of describing the characters of  $\mathcal{R}(\mathfrak{g})$  is as follows. Let  $a \in \mathbf{C}$ ; for any representation  $V$  of  $\mathfrak{g}$ , put  $\chi_a(V) = \text{Tr } e^{\tilde{a}v}$ , where  $\tilde{a}v$  is the endomorphism of  $V$  defined by the element  $\begin{pmatrix} ia & 0 \\ 0 & -ia \end{pmatrix}$  of  $\mathfrak{g}$ . Then  $\chi_a$  is a character of  $\mathcal{R}(\mathfrak{g})$ , and all characters are obtained in this way. The character  $\chi_a$  factors through  $\mathcal{R}_\ell(\mathfrak{g})$  if and only if it vanishes on  $[S^{\ell+1}]$ ; an easy computation gives

$$\chi_a(S^p) = \frac{\sin(p+1)a}{\sin a} ,$$

so  $\chi_a(S^{\ell+1}) = 0$  iff  $a$  is of the form  $\frac{k\pi}{\ell+2}$  for  $1 \leq k \leq \ell+1$ . In other words, *the characters of the fusion ring  $\mathcal{R}_\ell(\mathfrak{g})$  are the characters  $[V] \mapsto \chi_a(V)$  for  $a = \frac{k\pi}{\ell+2}$ , ( $1 \leq k \leq \ell+1$ ).*

Recall that the formula for  $\dim V_{\mathbf{C}}(\emptyset)$  involves the numbers  $\chi(\omega) = \sum_{p=0}^{\ell} \chi(S^p)^2$

(the involution  $*$  is trivial for  $\mathfrak{sl}_2$ ). Let  $a = \frac{k\pi}{\ell+2}$ ; a simple computation gives

$$\sum_{p=0}^{\ell} \sin^2((p+1)a) = \frac{\ell}{2} + 1 ,$$

hence  $\chi_a(\omega) = (\frac{\ell}{2} + 1) / \sin^2 a$ . Applying Proposition 3.3 we obtain :

**Proposition 6.1.** — *Let  $C$  be a Riemann surface of genus  $g$ . For the RCFT associated to  $\mathfrak{sl}_2(\mathbf{C})$  at level  $\ell$ , one has*

$$\dim V_{\mathbf{C}}(\emptyset) = \left(\frac{\ell}{2} + 1\right)^{g-1} \sum_{k=1}^{\ell+1} \frac{1}{\left(\sin \frac{k\pi}{\ell+2}\right)^{2g-2}} \quad \blacksquare$$

## 7. Classical theta functions : a reminder

This section contains a brief overview, meant for non-specialists, of the classical theory of theta functions. There are plenty of places that the reader wishing to learn more may consult, like [B-La] or [A-C-G-H]; a short and accessible introduction can be found in [C].

Let me start with some generalities on line bundles and their global sections. Let  $X$  be a compact complex manifold and  $L$  a (holomorphic) line bundle on  $X$ . We denote by  $H^0(X, L)$  the space of global holomorphic sections of  $L$ ; it is finite-dimensional. Let  $s \in H^0(X, L)$ . Locally over  $X$  we can write  $s = f\tau$  where  $\tau$  is a nowhere vanishing section and  $f$  is a holomorphic function; we see in this way that  $s$  vanishes along finitely many irreducible hypersurfaces  $D_i$ , possibly with some multiplicities  $m_i$ : we write  $\text{div}(s) = \sum_i m_i D_i$ .

So we have associated to the pair  $(L, s)$  a divisor, that is a (finite) formal combination of irreducible hypersurfaces with integer coefficients; moreover this divisor is effective, which means that the coefficients are non-negative. Conversely, given an effective divisor  $D$ , there exists a unique line bundle  $\mathcal{O}(D)$  on  $X$  and a section  $s \in H^0(X, \mathcal{O}(D))$ , unique up to a scalar, such that  $\text{div}(s) = D$ . We say that  $\mathcal{O}(D)$  is the line bundle associated to  $D$ .

Let us denote by  $\text{Div}(X)$  the group of divisors on  $X$ , and by  $\text{Pic}(X)$  the set of isomorphism classes of line bundles on  $X$ . The tensor product operation defines a group structure on  $\text{Pic}(X)$ , which is called the *Picard group* of  $X$ . The map  $D \mapsto \mathcal{O}(D)$  extends by linearity to a group homomorphism

$$\text{Div}(X) \longrightarrow \text{Pic}(X)$$

which is *surjective* if the manifold  $X$  is projective.

Let me now specialize to the case of a compact Riemann surface  $C$ . Then a divisor is simply a finite sum  $D = \sum_i m_i p_i$ , with  $p_i \in C$ ; we put  $\text{deg}(D) := \sum_i m_i$ . It is easy to see that the homomorphism  $\text{deg} : \text{Div}(C) \longrightarrow \mathbf{Z}$  factors through  $\text{Pic}(C)$ , so we have an exact sequence

$$0 \rightarrow \text{JC} \longrightarrow \text{Pic}(C) \xrightarrow{\text{deg}} \mathbf{Z} \rightarrow 0.$$

The group  $\text{JC}$  which parametrizes line bundles of degree 0 on  $C$  has a natural holomorphic structure; it is called the *Jacobian* of  $C$ , and is certainly the most fundamental object associated to the Riemann surface  $C$ . It is a complex torus, i.e. the quotient of a complex vector space  $V$  by a lattice  $\Gamma$ : in our case we take for  $V$  the dual  $\Omega^*$  of the space of holomorphic 1-forms on  $C$ , and for  $\Gamma$  the homology

$H_1(\mathbf{C}, \mathbf{Z})$ , embedded in  $\Omega^*$  by associating to a loop  $\gamma$  the linear form  $\int_\gamma$  on  $\Omega$  (the fact that this complex torus parametrizes in a natural way the line bundles of degree 0 on  $\mathbf{C}$  is a translation in modern language of the classical Abel-Jacobi theorem).

The complex torus  $\mathbf{JC} = \mathbf{V}/\Gamma$  has the extra property of having a *principal polarization*, that is a hermitian form  $\mathbf{H}$  on  $\mathbf{V}$  whose imaginary part takes integral values on  $\Gamma$  and defines a unimodular alternate form on  $\Gamma$  (here  $\mathbf{H}$  will be the dual form of the hermitian form  $(\alpha, \beta) \mapsto \int_{\mathbf{C}} \bar{\alpha} \wedge \beta$  on  $\Omega$ ; the integrality property follows from Poincaré duality).

What makes a polarization interesting is that it allows to define beautiful functions on  $\mathbf{V}$ . By the maximum principle we cannot expect any interesting holomorphic function on  $\mathbf{V}$  periodic with respect to  $\Gamma$ , but we can look for *quasi-periodic* functions, namely those which satisfy

$$(7) \quad \theta(z + \gamma) = e_\gamma(z) \theta(z) \quad \text{for all } z \in \mathbf{V}, \gamma \in \Gamma .$$

for a certain system of nowhere vanishing functions  $(e_\gamma)_{\gamma \in \Gamma}$  on  $\mathbf{V}$ . In order for (7) to have solutions this system must necessarily satisfy

$$(8) \quad e_{\gamma+\delta}(z) = e_\gamma(z + \delta) e_\delta(z) .$$

For a general lattice  $\Gamma \subset \mathbf{V}$  (8) will have only uninteresting solutions. However, if we have a (principal) polarization  $\mathbf{H}$ , we can take

$$(9) \quad e_\gamma(z) = \varepsilon(\gamma)^k e^{k\pi\mathbf{H}(\gamma, z + \frac{\gamma}{2})}$$

where  $k$  is a positive integer, and  $\varepsilon : \Gamma \rightarrow \mathbf{C}^*$  is any map satisfying  $\varepsilon(\gamma + \delta) = \varepsilon(\gamma)\varepsilon(\delta)e^{i\pi\Im m \mathbf{H}(\gamma, \delta)}$  (the particular choice of  $\varepsilon$  is essentially irrelevant, since one passes from one choice to another by a translation  $z \mapsto z + a$ ). Then (7) has solutions, which are called *theta functions of order  $k$* ; they form (for a fixed  $\varepsilon$ ) a vector space of dimension  $k^g$ . These functions have a simple explicit description as convergent series, at the same time they encode a large part of the geometry of the torus.

The theta functions can be naturally interpreted as sections of a line bundle on  $\mathbf{V}/\Gamma$ . To explain this, notice first that any system of functions  $(e_\gamma)$  satisfying (8) defines a natural action of  $\Gamma$  onto  $\mathbf{V} \times \mathbf{C}$  by

$$\gamma \cdot (z, t) = (z + \gamma, e_\gamma(z)t) .$$

This action is free, linear in the fibres, and it makes the projection  $\pi : \mathbf{V} \times \mathbf{C} \rightarrow \mathbf{V}/\Gamma$  equivariant. Let us denote by  $\mathcal{L}_e$  the quotient variety  $(\mathbf{V} \times \mathbf{C})/\Gamma$ . We have a



commutative diagram

$$\begin{array}{ccc}
 V \times \mathbf{C} & \longrightarrow & \mathcal{L}_e \\
 \pi \downarrow & & \downarrow \bar{\pi} \\
 V & \longrightarrow & V/\Gamma
 \end{array}$$

and  $\mathcal{L}_e$  is (via  $\bar{\pi}$ ) a line bundle over  $V/\Gamma$ . The sections of this line bundle correspond in a one-to-one way to the sections of  $\pi$  which are  $\Gamma$ -equivariant; but the condition for a section  $z \mapsto (z, \theta(z))$  to be equivariant is exactly (7). In other words, solutions of (7) with respect to the system  $(e_\gamma)$  correspond in a natural one-to-one way to holomorphic sections of  $\mathcal{L}_e$ . In particular, let us consider the system  $(e_\gamma)$  given by (9) with  $k = 1$ , for a fixed  $\varepsilon$ ; let us denote simply by  $\mathcal{L}$  the corresponding line bundle  $\mathcal{L}_e$ . One checks at once that the system  $(e_\gamma^k)$  corresponds to the line bundle  $\mathcal{L}^k$ ; hence *theta functions of order  $k$  correspond in a natural way to holomorphic sections of  $\mathcal{L}^k$ .*

The case  $k = 1$  is particularly important. In that case the line bundle  $\mathcal{L}$  has only one non-zero section (up to a scalar), whose divisor is therefore canonically defined up to translation: it is called the *theta divisor* of the torus.

All I have said so far applies to any complex torus with a principal polarization. A special feature in the case of the Jacobian of a curve  $C$  is a simple geometric interpretation of the theta divisor. Recall that  $JC$  parametrizes line bundles of degree 0 on  $C$ . Fix a line bundle  $M$  of degree  $g - 1$  on  $C$  and put

$$\Theta_M := \{L \in JC \mid H^0(C, L \otimes M) \neq 0\}.$$

Then  $\Theta_M$  is a *theta divisor* on  $JC$  (Riemann's theorem). So in this case we can define the theta divisor either as a geometric locus, or by an equation given by an explicit power series. This interplay between the analysis and the geometry of theta functions gives rise to one of the most beautiful chapters of Algebraic Geometry; I have to refer e.g. to [A-C-G-H] or [B-La] for an introductory account.

## 8. Non-abelian theta functions

Theta functions play such a prominent role in the theory of Riemann surfaces that it is natural to look for generalizations. In the influential paper [W], A. Weil observes that topologically  $JC$  is just the space of 1-dimensional unitary representations of  $\pi_1(C)$ , i.e.  $\text{Hom}(\pi_1(C), \mathbf{S}^1)$ ; he proposes as a natural generalization the space of equivalence classes of  $r$ -dimensional unitary representations of  $\pi_1(C)$ . It is

only much later than a celebrated theorem of Narasimhan and Seshadri provided this space with a natural complex structure (depending on the complex structure of  $\mathbb{C}$ ) : this analytic space  $\mathcal{U}_{\mathbb{C}}(r)$  is a projective variety, which parametrizes holomorphic vector bundles of rank  $r$  and degree 0 on  $\mathbb{C}$  (the degree of a rank  $r$  vector bundle  $E$  is defined as the degree of the line bundle  $\wedge^r E$ ). Actually a new phenomenon occurs in rank  $> 1$  : in order to make the above assertion correct, and also to obtain a reasonable moduli space, one must exclude some degenerate vector bundles, and consider only those which are *semi-stable*, i.e. which do not contain subbundles of degree  $> 0$ .

The variety  $\mathcal{U}_{\mathbb{C}}(r)$  is, up to a finite étale covering, a product of  $\text{JC}$  with the subvariety  $\mathcal{SU}_{\mathbb{C}}(r)$  parametrizing semi-stable vector bundles of rank  $r$  with trivial determinant ; since we know pretty well the Jacobian part, it is more convenient to study  $\mathcal{SU}_{\mathbb{C}}(r)$ , which is somehow, together with  $\text{JC}$ , the primitive building block.

So we now have projective varieties  $\mathcal{SU}_{\mathbb{C}}(r)$  which by all means constitute natural non-abelian generalizations of the Jacobian. What should be the generalization of theta functions, however, is not so clear : we do not know what should replace the presentation of  $\text{JC}$  as  $V/\Gamma$ . The varieties  $\mathcal{SU}_{\mathbb{C}}(r)$  are simply connected, so we cannot define quasi-periodic functions. But we can still look at line bundles on  $\mathcal{SU}_{\mathbb{C}}(r)$  and their global sections. The classification of line bundles on  $\mathcal{SU}_{\mathbb{C}}(r)$  turns out to be very simple. Note that the geometric definition of the theta divisor extends in a natural way to the higher rank case : for any line bundle  $M \in \mathcal{J}^{g-1}(X)$ , define

$$\Theta_M = \{E \in \mathcal{SU}_X(r) \mid H^0(X, E \otimes M) \neq 0\} .$$

This turns out to be a divisor on  $\mathcal{SU}_X(r)$ . The associated line bundle  $\mathcal{L} := \mathcal{O}(\Theta_M)$ , called the determinant bundle, does not depend on the choice of  $M$ . It is in fact canonical, because of the following result [D-N] :

**Proposition 8.1.** — *Any line bundle on  $\mathcal{SU}_{\mathbb{C}}(r)$  is a power of  $\mathcal{L}$ . ■*

By analogy with the rank one case, the global sections of the line bundles  $\mathcal{L}^k$  are sometimes called *generalized* (or *non-abelian*) *theta functions*. The link between these spaces and Conformal Field Theory is provided by the following result ([F], [B-L]) :

**Theorem 8.2.** — *The space  $H^0(\mathcal{SU}_{\mathbb{C}}(r), \mathcal{L}^\ell)$  of  $\ell^{\text{th}}$  order generalized theta functions is naturally isomorphic to the space  $V_{\mathbb{C}}(\emptyset)$  associated to the Lie algebra  $\mathfrak{sl}_r(\mathbb{C})$  and the level  $\ell$ .*

Recall the definition of  $V_{\mathbb{C}}(\emptyset)$  : we choose a point  $q \in \mathbb{C}$  and let  $A_{\mathbb{C}}$  be the algebra of regular functions on  $\mathbb{C} - q$  ; then  $V_{\mathbb{C}}(\emptyset)$  is the subspace of the dual  $\mathcal{H}_\ell^*$  annihilated by the Lie algebra  $\mathfrak{sl}_r(A_{\mathbb{C}})$ .

Let me give a very sketchy idea of the proof in [B-L].

1) The key point is that a vector bundle with trivial determinant is *algebraically trivial* over  $C - q$  (*Hint*: show that such a bundle has always a nowhere vanishing section, and use induction on the rank). We consider triples  $(E, \rho, \sigma)$  where  $E$  is a vector bundle on  $C$ ,  $\rho$  a trivialization of  $E$  over  $C - q$  and  $\sigma$  a trivialization of  $E$  in an open disk  $D$  centered at  $q$ . Over  $D - q$  these two trivializations differ by a holomorphic map  $D - q \rightarrow GL_r(\mathbf{C})$  which is meromorphic at  $q$ , that is given by a Laurent series  $\gamma(z) \in GL_r(\mathbf{C}((z)))$ . Conversely given such a matrix  $\gamma(z)$  one can use it to glue together the trivial bundles on  $C - q$  and  $D$  and recover the triple  $(E, \rho, \sigma)$ . Since we want  $\gamma(z)$  in  $SL_r(\mathbf{C}((z)))$  we impose moreover that  $\wedge^r \rho$  and  $\wedge^r \sigma$  coincide over  $D - q$ . This gives a bijection of the set of triples  $(E, \rho, \sigma)$  (up to isomorphism) onto  $SL_r(\mathbf{C}((z)))$ .

2) To get rid of the the trivializations, we have to mod out by the automorphism group of the trivial bundle over  $D$  and  $C - q$ . We get the following diagram :

$$\begin{array}{ccc}
 \{E, \rho, \sigma\} & \longleftrightarrow & SL_r(\mathbf{C}((z))) \\
 \downarrow & & \downarrow \\
 \{E, \rho\} & \longleftrightarrow & \mathcal{Q} := SL_r(\mathbf{C}((z)))/SL_r(\mathbf{C}[[z]]) \\
 \downarrow & & \downarrow \pi \\
 \{E\} & \longleftrightarrow & SL_r(A_C) \backslash SL_r(\mathbf{C}((z)))/SL_r(\mathbf{C}[[z]]) .
 \end{array}$$

So the set of isomorphism classes of vector bundles on  $C$  with trivial determinant appears in one-to-one correspondence with the set of double classes  $SL_r(A_C) \backslash SL_r(\mathbf{C}((z)))/SL_r(\mathbf{C}[[z]])$ . With some technical work one shows that this bijection is actually an isomorphism between algebro-geometric objects. The appropriate objects here are slightly more complicated than algebraic varieties : the quotient  $\mathcal{Q} = SL_r(\mathbf{C}((z)))/SL_r(\mathbf{C}[[z]])$  is an *ind-variety*, i.e. the (infinite-dimensional) direct limit of an increasing sequence of projective varieties; the double coset space  $SL_r(A_C) \backslash \mathcal{Q}$  is isomorphic to the *algebraic stack* of rank  $r$  vector bundles with trivial determinant. For simplicity I will ignore these technical difficulties and just pretend that I have a quotient map of algebraic varieties  $\pi : \mathcal{Q} \rightarrow SU_C(r)$ . We want to describe the pull back  $\pi^* \mathcal{L}$  of our determinant line bundle to  $\mathcal{Q}$ .

3) On a homogeneous space  $Q = G/H$ , one associates to any character  $\chi : H \rightarrow \mathbf{C}^*$  a line bundle  $L_\chi$  : it is the quotient of the trivial bundle  $G \times \mathbf{C}$  on  $G$  by

the action of  $H$  defined by  $h(g, \lambda) = (gh, \chi(h)\lambda)$ . We apply this to the homogeneous space  $\mathcal{Q} = \mathrm{SL}_r(\mathbf{C}((z))) / \mathrm{SL}_r(\mathbf{C}[[z]])$ . The line bundle  $\pi^*\mathcal{L}$  does *not* admit an action of  $\mathrm{SL}_r(\mathbf{C}((z)))$ , but of a group  $\widehat{\mathrm{SL}}_r(\mathbf{C}((z)))$  which is a central  $\mathbf{C}^*$ -extension of  $\mathrm{SL}_r(\mathbf{C}((z)))$ . This extension splits over the subgroup  $\mathrm{SL}_r(\mathbf{C}[[z]])$ , so that  $\mathcal{Q}$  is isomorphic to  $\widehat{\mathrm{SL}}_r(\mathbf{C}((z))) / (\mathbf{C}^* \times \mathrm{SL}_r(\mathbf{C}[[z]]))$ . Then  $\pi^*\mathcal{L}$  is the line bundle  $L_\chi$ , where  $\chi : \mathbf{C}^* \times \mathrm{SL}_r(\mathbf{C}[[z]]) \longrightarrow \mathbf{C}^*$  is the first projection.

4) A theorem of Kumar and Mathieu provides an isomorphism  $H^0(\mathcal{Q}, L_\chi^\ell) \cong \mathcal{H}_\ell^*$ . It follows that  $H^0(\mathcal{SU}_C(r), \mathcal{L}^\ell)$  can be identified with the subspace of  $\mathcal{H}_\ell^*$  invariant under  $\mathrm{SL}_r(\mathbf{A}_C)$ . This turns out to coincide with the subspace of  $\mathcal{H}_\ell^*$  invariant under the Lie algebra  $\mathfrak{sl}_r(\mathbf{A}_C)$ , which is by definition  $V_C(\emptyset)$ . ■

The theorem can be extended to an arbitrary simple Lie algebra  $\mathfrak{g}$ ; the space  $\mathcal{SU}_C(r)$  must be replaced by the moduli space of principal  $G$ -bundles on  $C$ , where  $G$  is the simply-connected complex Lie group with Lie algebra  $\mathfrak{g}$  (see [F]). More generally, there is an analogous interpretation for the spaces  $V_C(\vec{p}, \vec{\lambda})$ , which has been worked out by C. Pauly (to appear); it involves the moduli spaces of *parabolic bundles* on the curve  $C$ .

## 9. A few examples

The main application of Theorem 8.2 is to give an explicit formula for the dimension of  $H^0(\mathcal{SU}_C(r), \mathcal{L}^\ell)$ . In this final section I would like to explain how this formula may be used in algebraic geometry. I will restrict myself to rank 2 vector bundles, partly for simplicity and partly because we know much more in this case.

Proposition 6.1 gives us a formula for  $h^0(\mathcal{L}^\ell) := \dim H^0(\mathcal{SU}_C(2), \mathcal{L}^\ell)$ . The first values are :

$$h^0(\mathcal{L}) = 2^g \quad , \quad h^0(\mathcal{L}^2) = 2^{g-1}(2^g + 1) \quad , \quad h^0(\mathcal{L}^3) = 2((5 + \sqrt{5})^{g-1} + (5 - \sqrt{5})^{g-1}) \dots$$

The first two of these formulas have nice geometric interpretations. Observe first that there is a natural map  $i : \mathrm{JC} \longrightarrow \mathcal{SU}_C(2)$  which associates to  $L \in \mathrm{JC}$  the vector bundle  $L \oplus L^{-1}$ . It is easy to check that the pull back  $i^*\mathcal{L}$  of the determinant bundle is  $\mathcal{O}(2\Theta)$ .

**Proposition 9.1** [B 1].— *The pull back map  $i^* : H^0(\mathcal{SU}_C(2), \mathcal{L}) \longrightarrow H^0(\mathrm{JC}, \mathcal{O}(2\Theta))$  is an isomorphism.*

This means that theta functions of order 2 extend (uniquely) to the moduli space  $\mathcal{SU}_C(2)$ . From this it is easy for instance to give an explicit basis for the space  $H^0(\mathcal{SU}_C(2), \mathcal{L})$ .

The Proposition is an easy consequence of the formula  $h^0(\mathcal{L}) = 2^g$  – the main part of [B 1] is actually devoted to an *ad hoc* proof of the formula in that particular case.

The next number,  $2^{g-1}(2^g + 1)$ , is well-known from algebraic geometers; it is the number of *even theta-characteristics* on  $C$ , i.e. of line bundles  $\kappa$  such that  $\kappa^2$  is isomorphic to the canonical bundle  $K_C$  and  $\dim H^0(C, \kappa)$  is even. As a matter of fact, we can associate to each even theta-characteristic  $\kappa$  the subset  $D_\kappa \subset SU_C(2)$  consisting of vector bundles  $E$  such that there exists a non-scalar map  $E \rightarrow E \otimes \kappa$ . It turns out that  $D_\kappa$  is the divisor of a section  $d_\kappa$  of  $\mathcal{L}^2$ , and that *the sections  $d_\kappa$  form a basis of  $H^0(SU_C(2), \mathcal{L}^2)$*  [B 2]. The proof uses in a decisive way the formula for  $h^0(\mathcal{L}^2)$ .

I should finally mention that in the rank 2 case there are various proofs of the formula using more classical algebraic geometry – the most illuminating probably appears in [T]. So far none of these proofs has been extended to the higher rank case.

## REFERENCES

- [A] M. ATIYAH : *The Geometry and Physics of Knots*. Cambridge University Press (1990).
- [A-C-G-H] E. ARBARELLO, M. CORNALBA, P. GRIFFITHS, J. HARRIS : *Geometry of Algebraic Curves I*. Springer-Verlag (1985).
- [B 1] A. BEAUVILLE : *Fibrés de rang 2 sur les courbes, fibré déterminant et fonctions thêta*. Bull. Soc. math. France **116**, 431-448 (1988).
- [B 2] A. BEAUVILLE : *Fibrés de rang 2 sur les courbes, fibré déterminant et fonctions thêta II*. Bull. Soc. math. France **119**, 259-291 (1991).
- [B 3] A. BEAUVILLE : *Conformal blocks, Fusion rings and the Verlinde formula*. Proc. of the Conference “Hirzebruch 65”, to appear.
- [B-L] A. BEAUVILLE, Y. LASZLO : *Conformal blocks and generalized theta functions*. Commun. Math. Phys. **164**, 385-419 (1994).
- [B-La] Ch. BIRKENHAKE, H. LANGE : *Complex Abelian Varieties*. Springer-Verlag (1992).
- [B-K-Z] A. BELAVIN, A. POLYAKOV, A. ZAMOLODCHIKOV : *Infinite conformal symmetry in two-dimensional Quantum Field Theory*. Nucl. Phys. B **241**, 333-380 (1984).

- [C] M. CORNALBA : *Complex tori and Jacobians*. Complex Analysis and its applications, vol. II, 39-100; Intern. Atomic Energy, Vienna (1976).
- [D-N] J.M. DREZET, M.S. NARASIMHAN : *Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques*. Invent. math. **97**, 53-94 (1989).
- [F ] G. FALTINGS : *A proof for the Verlinde formula*. J. Algebraic Geometry **3**, 347-374 (1994).
- [F-S] D. FRIEDAN, S. SHENKER : *The analytic Geometry of two-dimensional Conformal Field Theory*. Nucl. Phys. B **281**, 509-545 (1987).
- [K] V. KAC : *Infinite dimensional Lie algebras* (3rd edition). Cambridge University Press (1990).
- [M-S 1] G. MOORE, N. SEIBERG : *Classical and Quantum Conformal Field Theory*. Commun. Math. Phys. **123**, 177-254 (1989).
- [M-S 2] G. MOORE, N. SEIBERG : *Polynomial Equations for Rational Conformal Field Theories*. Phys. Let. B **212**, 451-460 (1988).
- [S] G. SEGAL : *Two dimensional Conformal Field Theory and modular functors*. Proc. Intern. Cong. Math. Phys. Swansea 22-37, A. Hilger (1989).
- [T] M. THADDEUS : *Stable pairs, linear systems and the Verlinde formula*. Inventiones math. **117**, 317-353 (1994).
- [T-U-Y] A. TSUCHIYA, K. UENO, Y. YAMADA : *Conformal field theory on universal family of stable curves with gauge symmetries*. Adv. Studies in Pure Math. **19**, 459-566 (1989).
- [V] E. VERLINDE : *Fusion rules and modular transformations in 2d conformal field theory*. Nuclear Physics **B300** 360-376 (1988).
- [W] A. WEIL : *Généralisation des fonctions abéliennes*. J. de Math. P. et App. **17** (9<sup>ème</sup> sér.), 47-87 (1938).

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