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Monodromy of Polygarithms and Cosimplicial Spaces

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Monodromy of polygarithms and cosimplicial spaces (*)

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The aim of this paper is to investigate the monodromy representation of iterated integrals on $P^1(\mathbb{C})$ minus several points. The knowledge of the monodromy of the dilogarithm is essential in the construction of the Bloch-Wigner function D_2 (see [B]) and its higher analogs (see [R2], [W2], [Z1] and [Z2]. These functions play an important role in algebraic K-theory. In this paper, among other things, we give a construction of analogs of the Bloch Wigner function for any iterated integral on $P^1(\mathbb{C})$ minus several points.

The monodromy of iterated integrals is described by the following result. Let $X = P^1(\mathbb{C})\setminus\{a_1, \dots, a_n, \infty\}$. Let $C[[X_1, \dots, X_n]]^*$ be a group of invertible elements with a constant term equal to 1 in the algebra of formal power series in non-commutative variables $X_1, \dots, X_n$. Let

$$\Lambda_x(z) = 1 + \sum\left(\int_{x,\gamma}^{z} \omega_{i_1}, \dots, \omega_{i_k}\right) X_{i_k} \cdot \dots \cdot X_{i_1} \in C[[X_1, \dots, X_n]]^*$$

where $\omega_i = -\frac{dz}{z-a_i}$, γ is a path in X from x to z and the summation is over all non-commutative monomials.

THEOREM. Let $S_j \in \pi_1(X, x)$ be a loop around a_j. The monodromy of the function $\Lambda_x(z)$ along S_j is given by

$$S_j : \Lambda_x(z) \to \Lambda_x(z) \cdot \Lambda_x(S_j)$$

where

$$\Lambda_x(S_j) = 1 + \sum\left(\int_{S_j} \omega_{i_1}, \dots, \omega_{i_k}\right) X_{i_k} \cdot \dots \cdot X_{i_1}.$$

(*) Ce texte reprend une prépublication de l'I.H.E.S. de Novembre 1991 - IHES/M/91/77

The element $\Lambda_x(S_j)$ is conjugated to $e^{-2\pi i X_j}$.

We point out that monodromy of polygarithms is described in [R1] and [Du].

We shall utilize an idea that $\Lambda_x(S_k)$ and $\int_x^z \omega_{i_1}, \ldots, \omega_{i_k}$ can be view as non-abelian unipotent periods. The numbers $\int_{S_k} \omega_{i_1}, \ldots, \omega_{i_k}$ and $\int_x^z \omega_{i_1}, \ldots, \omega_{i_k}$ are interated integrals of one-forms $\omega_1, \ldots \omega_n$ on elements of $\pi_1(X, x)$ and $[I, 0, 1 ; X, x, z]$ respectively. In the analogy with abelian periods we could expect that $\Lambda_x(S_k)$ and $\int_x^z \omega_{i_1}, \ldots, \omega_{i_k}$ one obtains as horizontal sections of some Gauss-Manin connections. We construct morphism between cosimplicial varieties such that the horizontal sections of the associated Gauss-Manin connections are functions $(x, a_1, \ldots, a_n) \to \Lambda_x(S_k)$ and $z \to L_x(z)$.

Our note is a recalculation and a generalization for $P^1(\mathbb{C})\setminus\{a_1,\ldots,a_{n+1}\}$ of some preliminary results of P. Deligne done for $P^1(\mathbb{C})\setminus\{0,1,\infty\}$ (see [D1]). Perhaps the only original thing is the relation between cosimplicial spaces and polygarithms. We planned to include this material in our chapter of the book on polylogarithms prepared by A.M.S., however because of lack of time we were not able to finish on schedule, so we decided to try to publish it separately.

We would like to acknowledge the influence of various seminaire talks of P. Cartier.

2. Canonical unipotent connection on $P^1(\mathbb{C})\setminus\{a_1, \ldots, a_{n+1}\}$.

We choose two coordinate carts on $P^1(\mathbb{C})$. We shall identify $P^1(\mathbb{C})\setminus\{\infty\}$ with $\mathbb{C}$ by a map $\alpha_1 : (z_1 : z_2) \to z_1/z_2$ and $P^1(\mathbb{C}) \setminus\{o\}$ with $\mathbb{C}$ by a map $\alpha_2 : (z_1 : z_2) \to z_2/z_1$.

Let $X = P^1(\mathbb{C})\setminus\{a_1, \ldots, a_{n+1}\}$. Let $A^*(X)$ be a differential, graded subalgebra of $\Omega^*(X)$ generated by linear combinations with complex coefficients of one-forms $\frac{dz}{z-a_i}$ $i = 1, \ldots, n+1$. It is a trivial observation that $(A^1(X))^* \approx H_1(X, \mathbb{C})$.

The isomorphism is given by the bilinear form

$$\int : A^1(X) \otimes H_1(X, \mathbb{C}) \to \mathbb{C}$$

given by $(\omega, \gamma) \to \int_\gamma \omega$.

Let $L(\pi_1(X, x)) := \lim\limits_{\overleftarrow{N}} \left(\bigoplus_{n=1}^{N} \Gamma^n \pi_1(X, x)/\Gamma^{n+1} \pi_1(X, x) \otimes \mathbb{C} \right)$ be a Lie algebra associated with the lower central series of $\pi_1(X, x)$. We equipped $L(\pi_1(X, x))$ with a group law given by the Baker-Hausdorff formula and a topology given by the inverse limit of finite dimensional complex vector spaces. This topological group we denote by $\pi(X)$. The Lie algebra of $\pi(X)$ is $L(\pi_1(X, x))$.

We shall define a one-form ω_X on X with values in $L(\pi_1(X, x))$ in the following way. We have natural isomorphisms

(2.1) $\quad A^1(X) \otimes H_1(X, \mathbb{C}) \approx A^1(X) \otimes (A^1(X))^* \approx \mathrm{Hom}(A^1(X), A^1(X))$

DEFINITION 2.2. $\omega_X \in A^1(X) \otimes H_1(X, \mathbb{C})$ is the one-form which corresponds to $2\pi i.\mathrm{id}$ under the isomorphism 2.1. (see also [D1] 12.5.5).

We consider ω_X as an element of $A^1(X) \otimes L(\pi_1(X, x))$ because of the identification $H_1(X, \mathbb{C}) \approx (\pi_1(X,x)/\Gamma^2\pi_1(X,x)) \otimes \mathbb{C}$.

Let A_i be a loop around a_i in X and let X_i be the image of A_i in $H_1(X, \mathbb{C})$.

Let us assume that $a_{n+1} = \infty$ then

$$\omega_X = \sum_{i=1}^{n} \frac{dz}{z - a_i} \otimes X_i$$

If $a_i \neq \infty$ for $i = 1, \ldots, n+1$ then

$$\omega_X = \sum_{i=1}^{n} \left[\frac{dz}{z - a_i} - \frac{dz}{z - a_{n+1}} \right] \otimes X_i$$

Let f be a regular map from $X = P^1(\mathbb{C}) \setminus \{a_1, \ldots, a_{n+1}\}$ to $Y = P^1(\mathbb{C}) \setminus \{b_1, \ldots, b_{n+1}\}$. Then f induces maps

$$f^* : A^*(Y) \to A^*(X)$$

and

$$f_\# : \pi_1(X, x) \to \pi_1(Y, f(x)).$$

The map $f_\#$ extends to maps

$$f_* : L(\pi_1(X, x)) \to L(\pi_1(Y, f(x)))$$

and

$$f_* : \pi(X) \to \pi(Y).$$

Let $\mathbb{C}[[H_1(X, \mathbb{C})]]$ be an algebra of non-commutative, formal power series on $H_1(X, \mathbb{C})$. We shall denote it shortly by $\mathbb{C}[[X]]$. Let I be an augmentation ideal of $\mathbb{C}[[X]]$. Then $\mathbb{C}[[X]] / I^n$ is a finite dimensional, complex vector space, $\mathbb{C}[[X]] = \varprojlim_n C[[X]]/ I^n$ and we equipped $C[[X]]$ with a topology of an inverse limite of finite dimensional, complex vector spaces. Let $C[[X]]^*$ be a group of invertible elements in $C[[X]]$. From the discussion given above it follows that $C[[X]]^*$ is a topological group, an inverse limit of finite dimensional, complex Lie groups.

The Lie algebra of Lie elements, possibly of infinite length, in $\mathbb{C}[[X]]$ is naturally identified with $L(\pi_1(X, x))$. After this identification the exponential map

$$\exp : \pi(X) \to \mathbb{C}[[X]]^*$$
$$\exp(w) = e^w = 1 + \frac{w}{1!} + \frac{w^2}{e!} + \ldots$$

is defined. The exponential map is a continous monomorphism of topological groups, whose image is a closed subgroup of $\mathbb{C}[[X]]^*$. The inverse of exp is defined on the subgroup $\exp(\pi(X)) \subset \mathbb{C}[[X]]^*$ and we denote it by log.

Let Lie $C[[X]]^*$ be a Lie algebra of $C[[X]]^*$. We identify $T \in H_1(X, \mathbb{C}) \subset L(\pi_1(X, x))$ with the tangent vector to $C[[X]]^*$ in 1 given by $t \to 1 + tT$. After this identification the one-form ω_X we shall consider as a one-form with values in Lie $C[[X]]^*$. We shall denote it by $\bar{\omega}_X$. The homomorphism exp maps ω_X into $\bar{\omega}_X$.

Let us consider a principal $\pi(X)$-bundle

$$X \times \pi(X) \to X$$

equipped with the integrable connection given by a one-form ω_X, and a principal $C[[X]]^*$-bundle

$$X \times C[[X]]^* \to X$$

equipped with the integrable connection given by a one-form $\bar{\omega}_X$.

LEMMA 2.4. The morphism $\mathrm{id} \times \exp : X \times \pi(X) \to X \times \mathbb{C}[[X]]^*$ over id_X maps horizontal section with respect to ω_X into horizontal section with respect to $\bar{\omega}_X$.

PROOF. This is clear from the fact that exp maps ω_X into $\bar{\omega}_X$.

Further we shall not distinguish forms ω_X and $\bar{\omega}_X$ and we shall denote them by ω_X.

LEMMA 2.5. For any X, let $G(X)$ denotes $C[[X]]^*$ (resp. $\pi(X)$). Let

$$f : Y = P^1(\mathbb{C})\setminus\{b_1,\ldots,b_{n+1}\} \to X = P^1(\mathbb{C})\setminus\{a_1, \ldots, a_{n+1}\}$$

be a regular map. Then we have

2.5.1 $$(f^* \otimes \mathrm{id})(\omega_X) = (\mathrm{id} \otimes f_*)(\omega_Y).$$

This implies that $f \times f_* : Y \times G(Y) \to X \times G(X)$ maps horizontal sections of the bundle $Y \times G(Y) \to Y$ into horizontal sections along f of the bundle $X \times G(X) \to X$.

PROOF. The formula 2.5.1 follows directly from the definition 2.2. The rest is a consequence of 2.5.1.

3. Horizontal sections.

Let $X = P^1(\mathbb{C}) \setminus \{a_1, a_2, \dots, a_{n+1}\}$. A horizontal section of the principal $\pi(X)$-bundle (resp. $C[[X]]^*$ -bundle) $X \times \pi(X) \to X$(resp. $X \times C[[X]]^* \to X$) with respect to the connection ω_X such that in $x \in X$ the horizontal section is equal to $(x, 0)$ (resp. $(x, 1)$ we shall denote by $(z, L_x(z))$ (resp. $(z, \Lambda_x(z))$).

Now we shall give an explicit construction of $\Lambda_x(z)$. Let A_i be a loop around a_i in X and let X_i be its class in $H_1(X, \mathbb{C})$. Then $C[[X]]^* = C[[X_1, \dots, X_n]]^*$.

Let us set

$$\omega_i = -\left[\frac{dz}{z - a_i} - \frac{dz}{z - a_{n+1}}\right] \qquad i = 1,\dots,n$$

if $a_i \neq \infty$ for all i.

If one $a_i = \infty$ then we assume that $a_{n+1} = \infty$ and we set

$$\omega_i = -\frac{dz}{z - a_i} \qquad i = 1,\dots,n .$$

Let us define

$$\Lambda_x(\alpha_1,\dots,\alpha_k)(z) := \int_{x,\gamma}^{z} \omega_{\alpha_k}, \dots, \omega_{\alpha_1}$$

where $\alpha_i \in \{1, 2, \dots, n\}$.

LEMMA 3.1 . The application

$$X \ni z \to \left(z,\, 1 + \sum \Lambda_x\left(\alpha_1,\ldots,\alpha_k\right)(z)\, X_{\alpha_1}, \ldots, X_{\alpha_k}\right)$$

is horizontal with respect to the connection ω_X and it coincides with the map $z \to (z, \Lambda_x(z))$. (The summation is over all noncommutative monomials in $X_1, \ldots, X_n$).

The proof of this result is a straightforward calculation of horizontal liftings.

Now we shall define another family of horizontal liftings. The base point x we replace by a tangent vector at a_i. This is our attempt to understand §15 p. 156 in [D1] which we continue in section 5.

Let us assume that $a_{n+1} = \infty$. Let $x_0 \in X$ and let $\delta : [0, 1] \ni t \to a_i + t(x_0 - a_i)$ be an interval joining a_i and x and not passing through ∞. Let γ be a path from a_i to z tangent to δ at a_i. We suppose that in a small neighbourhood of a_i the path γ coincides with δ. Observe that $v = x_0 - a_i$ can be canonically identified with the tangent vector to $P^1(\mathbb{C})$ at the point a_i.

We set

$$\Lambda_{a_{i,v}}\left(\alpha_k,\ldots,\alpha_1\right)(z) := \int_{a_i,\gamma}^{z} \omega_{\alpha_1}, \ldots, \omega_{\alpha_k} \quad \text{if } \alpha_1 \neq i$$

and

$$\Lambda_{a_{i,v}}\left(\alpha_k,\ldots,\alpha_{\ell+1}, \alpha_\ell, \ldots, \alpha_1\right)(z) :=$$

$$= \lim_{\varepsilon \to a_i} \int_{\varepsilon,\gamma_\varepsilon}^{z} \left(\int_{\gamma_\varepsilon \circ (\delta_\varepsilon)^{-1}}^{z} \omega_{\alpha_1}, \ldots, \omega_{\alpha_\ell}\right) \omega_{\alpha_{\ell+1}}, \ldots, \omega_{\alpha_k}$$

where γ_ε is a part of γ from ε to z, δ_ε is a part of δ from ε to x_0, $\alpha_1 = \ldots = \alpha_\ell = i$ and $\alpha_{\ell+n} \neq i$.

LEMMA 3.2 . The integrals $\Lambda_{a_{i,v}}(\alpha_k, \ldots, \alpha_1)(z)$ exist and they are analytic, multivalued functions on $X = P^1(\mathbb{C})\backslash\{a_1,\ldots,a_{n+1}\}$.

PROOF. Assume that $\alpha_t \neq i$ for $t \leq \ell$ but $\alpha_{\ell+1} = i$. Then $g(z) := \int_{a_i}^{z} \omega_{\alpha_1}, \ldots, \omega_{\alpha_\ell}$ is an analytic multivalued function on $X \cup \{a_i\}$ which vanishes at a_i. Hence the integral $g_1(z) := \int_{a_i}^{z} g(z) \cdot \frac{-dz}{z - a_i}$ exists and the function $g_1(z)$ is analytic, multivalued on $X \cup \{a_i\}$ and it vanishes at a_i. Hence by induction we get that $\Lambda_{a_{i,v}}(\alpha_k, \ldots, \alpha_1)(z)$ exists and is analytic, multivalued on $X \cup \{a_i\}$.

Assume now that $\alpha_t = i$ for $t \leq \ell$ and $\alpha_{\ell+1} \neq i$. Without loss of generality we can assume that $a_i = 0$ and $x_0 = 1$.

Observe that $\lim\limits_{\varepsilon \to 0} \int_{\varepsilon, \gamma_\varepsilon \circ (\delta_\varepsilon)^{-1}}^{z} z^n (\log z)^m = z^{n+1}\left(\sum_{i=0}^{m} \beta_i (\log z)^{m-i}\right)$ where β_i are rational numbers. The function $z^q(\log z)^p$ is analytic, multivalued on X, continuous on any sufficiently small cone with a vertex in a_i and it vanishes in a_i. The function $\frac{1}{z - a_j}$ $(j \neq i)$ is bounded on any sufficiently small neighbourhood of a_i. Hence the integral

$$g(z) := \lim_{\varepsilon \to a_i} \int_{\varepsilon, \gamma_\varepsilon}^{z} \left(\int_{\gamma_\varepsilon \circ (\delta_\varepsilon)^{-1}} \frac{dz}{z - a_i}, \ldots, \frac{dz}{z - a_i} \right) \frac{dz}{z - a_j}, \ldots, \frac{dz}{z - a_{i_k}}$$

is analytic, multivalued on X, continuous on any sufficiently small cone with a vertex in a_i and it vanishes in a_i.

Let us set

3.2.1 $$\Lambda_{a_i v}(z) := 1 + \sum \Lambda_{a_i, v}(\alpha_k, \ldots, \alpha_1)(z) X_{\alpha_k}, \ldots, X_{\alpha_1}.$$

LEMMA 3.3. The map

$$X \ni z \to \left(z, \Lambda_{a_i v}(z)\right) \in X \times C[[X]]^*$$

is horizontal with respect to the connection ω_X.

The proof is the direct calculation of horizontal liftings with given initial conditions.

It rests to define functions $\Lambda_{a_i,v}(z)$ when $a_i = \infty$ or all a_i are different from ∞.

We recall that if

$$f : Y = P^1(\mathbb{C})\backslash\{b_1, \ldots, b_{m+1}\} \to X = P^1(\mathbb{C})\backslash\{a_1, \ldots, a_{n+1}\}$$

is a regular map then

$$\begin{array}{ccc} X \times C[[X]]^* & \xrightarrow{f \times f_*} & Y \times C[[Y]]^* \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

maps horizontal sections of the first bundle into horizontal sections along f of the second bundle (Lemma 2.5). We shall use this fact to construct a section $\Lambda_{a_i,v}(z)$ if $a_i = \infty$ or all points $a_1, \ldots, a_{n+1}$ are different from ∞.

Let $f : Y = P^1(\mathbb{C})\backslash\{b_1, \ldots, b_{n+1}\} \to X = P^1(\mathbb{C})\backslash\{a_1, \ldots, a_{n+1}\}$ be a regular map of degree 1 i. e. $f(z) = \frac{az + b}{cd + d}$. Assume that $b_{n+1} = \infty$ and $a_{n+1} = \infty$. Then we have

$$(**) \qquad \Lambda_{f(b_i),f_*(w)}(f(z)) = f_*(\Lambda_{b_i,w}(z)).$$

If only $b_{n+1} = \infty$ then the left hand side of $(**)$ is still defined by 3.2.1 and it can serve as the definition of the right hand side.

We set

$$3.2.2. \qquad \Lambda_{\infty,v}(z) := f_*(\Lambda_{b_i,w}(f^{-1}(z))$$

where $f(z) = \dfrac{az + b}{cz + d}$ is such $f(b_i) = \infty$, $f_*(w) = v$ and $b_{n+1} = \infty$.

If all a_i are different from ∞ then we choose f such that $b_{n+1} = \infty$ and we set

3.2.3. $$\Lambda_{a_i,v}(z) := f_*(\Lambda_{b_k,w}(f^{-1}(z))$$

where $f_*(b_k) = a_i$ and $f_*(w) = v$.

It is clear that the definitions 3.2.2 and 3.2.3 do not depend on the choice of f.

For the principal bundle

$$X \times \pi(X) \to X$$

equipped with the connection ω_X we set

3.2.4. $$L_{a_i,v}(z) := \log(\Lambda_{a_i,v}(z)).$$

It follows from Lemma 2.4 that $L_{a_i,v}(z)$ is a horizontal section.

4. Generators of $\pi_1(X, x)$

Let $X = P^1(\mathbb{C})\setminus\{a_1,\ldots,a_n, \infty\}$ (we set $a_{n+1} = \infty$) and let $x \in X$. We describe how to choose generators in the fundamental group $\pi_1(X, x)$.

First we choose a family of paths $\Gamma = \{\gamma_i\}_{i=1}^n$ from x to each a_i such that any two paths do not intersect and no path self intersects. The indices are choosen in such a way that when we make a small circle around x in the opposite chockwise direction, starting from γ_1 we meet $\gamma_2, \gamma_3, \ldots, \gamma_n$.

To any such path γ_i we associate an element S_{γ_i} in $\pi_1(X, x)$. We move along γ_i, near a_i we make a small circle around a_i in the opposite clockwise direction and then we return along γ_i to x. This element of $\pi_1(X, x)$ we denote by S_{γ_i}.

To the family of paths $\Gamma = \{\gamma_i\}_{i=1}^{n}$ we associated the sequence $(S_{\gamma_1},\ldots,S_{\gamma_n})$ of elements of $\pi_1(X, x)$. To simplify the notation we shall denote this sequence by $(S_1,\ldots,S_n)$.

The following lemma is obvious.

LEMMA 4.1. The elements $S_1,\ldots,S_n$ are free generators of $\pi_1(X, x)$.

DEFINITION 4.2. The ordered sequence $(S_1,\ldots,S_n)$ of elements of $\pi_1(X, x)$ obtained from the family of paths $\Gamma = \{\gamma_i\}_{i=1}^{n}$ we call a sequence of affine geometric generators of $\pi_1(X, x)$ associated to $\Gamma = \{\gamma_i\}_{i=1}^{n}$.

It is clear that we get different sequences of affine geometric generators if we choose different families of paths. In [P] pages 336-337 one can find suitable pictures. We shall see below how they differ for different choices of Γs.

Let $F(x_1,\ldots,x_n) = F_n$ be a free group on free generators $x_1,\ldots,x_n$. Let $B(x_1,\ldots,x_n) = B_n$ be a subgroup of $\mathrm{Aut}(F_n)$ consisting of automorphisms for which $x_i \to t_i \cdot x_{\mu(i)} \cdot t_i^{-1}$ and $x_1 \cdot \ldots \cdot x_n = t_1 \cdot x_{\mu(1)} \cdot t_1^{-1} \ldots t_n \cdot x_{\mu(n)} \cdot t_n^{-1}$, where $\mu : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is a permutation and $t_i \in F_n$. Let $B_1(x_1,\ldots,x_n) = B_{1,n}$ be the kernel of the obvious map $B_n \to \Sigma_n$.

The group B_n is of course an image of a representation of the n-th braid group as a group of automorphisms of F_n.

LEMMA 4.3. Let $(S_1,\ldots,S_n)$ be a sequence of affine geometric generators of $\pi_1(X, x)$. Then any other sequence of affine geometric generators of $\pi_1(X, x)$ is of the form $(\varphi(S_1),\ldots,(S_n))$ where $\varphi \in B(S_1,\ldots,S_n)$.

PROOF. Let $\alpha_k \in B(S_1,\ldots,S_n)$ be such that $\alpha_k(S_k) = S_{k+1}$, $\alpha_k(S_{k+1}) = S_{k+1}^{-1} \cdot S_k \cdot S_{k+1}$ and $\alpha_k(S_i) = S_i$ for $i \neq k, k+1$. Then $(\alpha_k(S_1), \ldots, \alpha_k(S_n))$ is a sequence of affine geometric generators associated to a family of paths $\Gamma' = \{\gamma'_i\}_{i=1}^{n}$ where $\gamma'_k = \gamma_k$ for

$k \neq i, i+1, \gamma'_i = \gamma_{i+1}$ and γ'_{i+1} is a paths from x to a_i which has a_{i+1} on its right side. The elements $\alpha_1, \ldots, \alpha_{n-1}$ are generators of $B(S_1,\ldots,S_n)$, hence for any $\delta \in B(S_1,\ldots,S_n)$, a sequence of $\delta((S_1),\ldots, \delta(S_n)$ is a sequence of affine geometric generators of $\pi_1(X, x)$.

Observe that for any sequence of affine geometric generators $(T_1,\ldots, T_n)$ we have $T_i = A_i^{-1} \cdot S_{\mu(i)} \cdot A_i$ $(i = 1,\ldots,n)$ where μ is a permutation of the set $\{1,2,\ldots,n\}$ and $T_1 \cdot \ldots \cdot T_n = S_1 \cdot \ldots \cdot S_n$. It follows from the theorem of Artin that there is $\beta \in B(S_1,\ldots, S_n$ such that $\beta(S_i) = T_i$ for $i = 1,\ldots, n$.

Now we shall not distinguish ∞ from any other point of $P^1(\mathbb{C})$. Let $X = P^1(\mathbb{C}) \setminus \{a_1, \ldots a_{n+1}\}$. Let $\Gamma = \{\gamma_i\}_{i=1}^n$ be a family of paths from x to each a_i such that any two paths do not intersect and no path self intersect. The ordering of paths is such that when we making a small circle around x in the opposite clockwise direction starting from γ_1 we meet successively $\gamma_2, \gamma_3, \ldots, \gamma_{n+1}$.

As before to Γ we associate an ordered sequence $(S_{\gamma_1},\ldots,S_{\gamma_{n+1}})$ which for simplicity we denote $(S_1,\ldots,S_{n+1})$.

LEMMA 4.4. The elements $(S_1,\ldots,S_{n+1})$ are generators of $\pi_1(X, x)$ subjet to one relation

$$S_1 \cdot S_2 \cdot \ldots \cdot S_{n+1} = 1.$$

DEFINITION 4.5. The ordered sequence $(S_1,\ldots,S_{n+1})$ we call a sequence of projective geometric generators of $\pi_1(X, x)$ associated to $\Gamma = \{\gamma_i\}_{i=1}^{n+1}$.

Let F^*_{n+1} be a quotient group of F_{n+1} arising from the adjunction of the relation

$$x_1 \cdot x_2 \cdot \ldots \cdot x_{n+1} = 1.$$

Let $B^*(x_1,\ldots,x_{n+1}) = B^*_{n+1}$ be a subgroup of Aut F^*_{n+1} consisting of automorphisms such that

$$x_i \to t_i \cdot x_{\mu(i)} \cdot t_i^{-1} \quad \text{for } i = 1,\dots,n+1$$

where $\mu : \{1, \dots, n+1\} \to \{1,\dots, n + 1\}$ is a permutation. Let $B^*_{1,n+1}$ be the kernel of the obvious map $B^*_{n+1} \to \Sigma_{n+1}$.

Each element of B_{n+1} determines an automorphism of F^*_{n+1}, in fact we have $B^*_{n+1} = B_{n+1}/C(B_{n+1})$ where $C(B_{n+1})$ is the center of B_{n+1}. This implies the following lemma.

LEMMA 4.6. Let $(S_1,\dots,S_{n+1})$ be a sequence of projective geometric generators of $\pi_1(X, x)$. Then any other sequence of projective geometric generators of $\pi_1(X, x)$ is of the form $(\varphi(S_1), \dots, \varphi(S_{n+1}))$ where $\varphi \in B^*(S_1,\dots,S_{n+1})$.

Let $X = P^1(\mathbb{C})\setminus\{a_1, a_2, \dots a_{n+1}(\text{resp. } \infty)\}$.

Let $\Gamma = \{\gamma_i\}_{i=1}^{n+1(\text{resp.}n)}$ and $\Delta = \{\delta_i\}_{i=1}^{n+1(\text{resp.}n)}$ be two families of smooth paths without intersections from x to a_i.

We say that the families Γ and Δ are equivalent if one of the following equivalent conditions is satisfied :

i) γ_i and δ_i are homotopic maps in map $([0, 1], 0,1 ; P^1(\mathbb{C})\setminus\{a_1,\dots,\hat{a}_i, \dots, a_{n+1}$ (resp. ∞) ; $x, a_i\}$;

ii) The sequence of projective (resp. affine) geometric generators associated to Γ coincides with the corresponding sequence associated to Δ.

5. Generators of $\pi_1(X, v)$.

Let $X = P^1(\mathbb{C})\setminus\{a_1,\dots,a_{n+1}\}$. Let $\hat{X} := X \cup \bigcup_{i=1}^{n+1} \left(T_{a_i}P^1(\mathbb{C})\setminus\{0\}\right)$. Let $J'(\hat{X})$ be the set of all continous maps from the closed unit interval $[0 ; 1]$ to $P^1(\mathbb{C})$ such that

i) $\varphi((0,1)) \subset X$;

ii) if $\varphi(0) = a_i$ then φ is smooth near a_i and $\dot{\varphi} \neq 0$ and if $\varphi(1) = a_k$ then φ is smooth near a_k and $\dot{\varphi}(1) \neq 0$.

In the sequel we shall identify $\dot{\varphi}(0)$ (resp. $\dot{\varphi}(1)$) with a tangent vector to φ in $T_{a_i}(P^1(\mathbb{C}))$ (resp. $T_{a_k}(P^1(\mathbb{C}))$. This tangent vector we shall denote also by $\dot{\varphi}(0) \in T_{a_i}(P^1(\mathbb{C}))$ (resp. $\dot{\varphi}(1) \in T_{a_k}(P^1(\mathbb{C}))$.

If $\varphi(0) = x \in X$ and $\varphi(1) = y \in X$ then we say that φ is a path from x to y.

If $\varphi(0) = a_i$ (resp. $\varphi(1) = a_k$) then we say that φ is a path from $\dot{\varphi}(0) \in T_{a_i}(P^1(\mathbb{C}))$ (resp.to $-\dot{\varphi}(1) \in T_{a_k}(P^1(\mathbb{C}))$ and we shall write $\varphi(0) = \dot{\varphi}(0)$ (resp. $\varphi(1) = -\dot{\varphi}(1)$).

To $J'(\hat{X})$ we joint all constant maps from $[0, 1]$ to $\hat{X}$ and the resulting set we denote by $J(\hat{X})$.

We shall define a relation of homotopy in the set $J(\hat{X})$. Let $\varphi, \psi \in J(\hat{X})$. If $\varphi(0) = \psi(0) = x \in X$ and $\varphi(1) = \psi(1) = y \in X$ then we say that φ and ψ are homotopic if they are homotopic maps in the space map $([0, 1], 0, 1 ; X, x, y)$.

If $\varphi(0) = \psi(0) = v \in T_{a_i}(P^1(\mathbb{C}))$ and $\varphi(1) = \psi(1) = y \in X$ then we say that φ and ψ are homotopic if there is a homotopy

$H_s \in$ map $([0, 1], 0, 1 ; X \cup \{a_i\}, a_i, y)$ such that

i) $H_s \in J(\hat{X})$ and $H_s(0) = v$ for all $s \in [0, 1]$;

ii) $H_s((0, 1)) \subset X$ for all $s \in [0, 1]$;

iii) $H_0 = \varphi$ and $H_1 = \psi$.

We left to the reader the cases when $\varphi(0) = \psi(0) = x \in X$, $\varphi(1) = \psi(1) = w \in T_{a_k}(P^1(\mathbb{C}))$ and $\varphi(0) = \psi(0) = v \in T_{a_i}(P^1(\mathbb{C}))$, $\varphi(1) = \psi(1) = w \in T_{a_k}(P^1(\mathbb{C}))$.

Let $\varphi \in J(\hat{X})$ be such that $\varphi(0) = \psi(0) = v \in T_{a_i}(P^1(\mathbb{C}))$ and let $\psi \in J(\hat{X})$ be a constant map equal to v. We say that φ and ψ are homotopic if there is a homotopy

$$H_s \in \mathrm{map}([0, 1], 0, 1 ; X \cup \{a_i\}, a_i, a_i)$$

such that

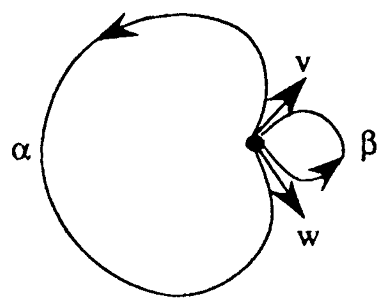
i) $H_s \in J(\hat{X})$ and $H_s(0) = H_s(1) = v$ for all $s \in [0, 1]$;

ii) $H_s((0, 1)) \subset X$ for all $s \in [0, 1]$;

iii) $H_0 = \varphi$ and $H_1(t) = a_i$ for $t \in [0, 1]$.

Observe that $G_t : = H_{1-t}$ defines a homotopy between ψ and φ.

With the definition given above paths $\alpha, \beta \in J\left(\widehat{\mathbb{C}\setminus\{0\}}\right)$ are not homotopic.



We shall write $\varphi \sim \psi$ if φ and ψ are homotopic. The relation $\sim$ is an equivalence relation on the set $J(\hat{X})$. Let $\pi(\hat{X}) : = J(\hat{X})/\sim$ be the set of of equivalence classes.

We define a partial composition in $\pi(\hat{X})$ in the following way. Let $\phi, \psi \in \pi(\hat{X})$ and let $\varphi, \psi, \in J(\hat{X})$ be its representatives.

If $\varphi(1) = \psi(0) = y \in X$ then we set $\psi \circ \phi : = [\psi \circ \varphi]$, the class of $\psi \circ \varphi$ in $\pi(\hat{X})$.

If $\varphi(1) = \psi(0) = v \in T_{a_i}P^1(\mathbb{C})$ then we can assume that φ and ψ coinside near a_i and we define $\psi \circ \phi : = [\psi^{\varepsilon} \cdot \varphi_\eta]$, where $\varphi_{1-\varepsilon} : = \varphi_{|[0,1-\varepsilon]}, \psi_\eta : = \psi_{|[\eta,1]}$ and $\varphi(1 - \varepsilon) = \psi(\eta)$.

The map $pr : J(\hat{X}) \to \hat{X} \times \hat{X}$ $p(\varphi) = (\varphi(0), \varphi(1))$ which associates to a path its beginning $(\varphi(0))$ and its end $(\varphi(1))$ agrees with the relation $\sim$ and it defines $p : \pi(\hat{X}) \to \hat{X} \times \hat{X}$. The partial composition o makes $p = \pi(\hat{X}) \to \hat{X} \times \hat{X}$ into a groupoid over $\hat{X} \times \hat{X}$.

Let $x \in \hat{X}$. We set $\pi_1(X, x) := p^{-1}(x, x)$. It is a fundamental group of X with a base point in $x \in \hat{X}$.

6. Monodromy

Let $X = P^1(\mathbb{C})\setminus\{a_1,\dots,a_{n+1}\}$. Let $x_1, x_2, x_3 \in \hat{X}$ and let $z_0 \in X$. Let γ_i for $i = 1, 2, 3$ be a path belonging to $J(\hat{X})$ from x_i to z_0. Let us set $\gamma_{ij} := \gamma_j^{-1} \circ \gamma_i$.

PROPOSITION 6.1. Let us prolongate each function $\Lambda_{x_i}(z)$ along γ_i to the point z_0. There exists elements $a_{x_j}^{x_i}(\gamma_{ij}) \in C[[X]]^*$ such that

$$\Lambda_{x_i}(z)\cdot a_{x_j}^{x_i}(\gamma_{ij}) = \Lambda_{x_j}(z)$$

for all z in a small neibourhood of z_0. The elements $a_{x_j}^{x_i}(\gamma_{ij})$ satisfy the following relations

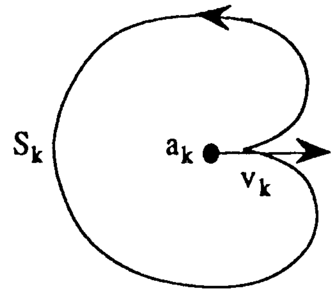
$$a_{x_i}^{x_i}(\gamma_{ii}) = 1,$$

$$a_{x_j}^{x_i}(\gamma_{ij})\cdot a_{x_i}^{x_j}(\gamma_{ji}) = 1,$$

$$a_{x_j}^{x_i}(\gamma_{ij})\cdot a_{x_k}^{x_j}(\gamma_{jk}) = a_{x_k}^{x_i}(\gamma_{ik}).$$

PROOF. The existence of $a_{x_j}^{x_i}(\gamma_{ij})$ follows from the fact that $\Lambda_{x_i}(z)$'s are horizontal sections. The first two relations are obvious. The last relation follows from equalities $\Lambda_{x_i}(z)\cdot a_{x_j}^{x_i}(\gamma_{ij}) = \Lambda_{x_j}(z)$, $\Lambda_{x_j}(z)\cdot a_{x_k}^{x_j}(\gamma_{jk}) = \Lambda_{x_k}(z)$ and $\Lambda_{x_i}(z)\cdot a_{x_k}^{x_i}(\gamma_{ik}) = \Lambda_{x_k}(z)$.

PROPOSITION 6.2. Let $v_k \in T_{a_k}P^1(\mathbb{C})\setminus\{0\}$. Let S_k be a loop around a_k based at v_k (see picture)



The monodromy of Λ_{v_k} along S_k is given by

$$S_k : \Lambda_{v_k} \to \Lambda_{v_k}(z)\cdot e^{-2\pi i X_k}$$

PROOF. The monodromy of $\Lambda_{v_k}(k^m)(z) := \Lambda_{v_k}(k, k, \dots, k)(z)$ along S_k is given by $S_k : \Lambda_{v_k}(k^m)(z) \to \Lambda_{v_k}(k^m)(z) + \sum_{\ell=1}^{m} \Lambda_{v_k}\left(k^{m-\ell}\right)(z)\frac{(-2\pi i)^\ell}{\ell!}$. This implies that the monodromy of $\Lambda_{v_k}(\alpha_1, \dots, \alpha_p, k^m)(z)$ along S_k is given by $S_k : \Lambda_{v_k}(\alpha_1, \dots, \alpha_p, k^m)(z) \to \Lambda_{v_k}(\alpha_1, \dots, \alpha_p, k^m) + \sum_{\ell=1}^{m} \Lambda_{v_k}\left(a_1, \dots, a_p, k^{m-\ell}\right)(z)\frac{(-2\pi i)^\ell}{\ell!}$.

Hence it follows the formula for the monodromy of $\Lambda_{v_k}(z)$ along S_k.

Let $x \in \hat{X}$. Let us choose $v_i \in T_{a_i}P^1(\mathbb{C})\setminus\{0\}$ for $i = 1, 2, \dots, n+1$. Let $(S_1, \dots, S_{n+1})$ be a sequence of projective generators of $\pi_1(X, x)$ associated to $\Gamma = \{\gamma_i\}_{i=1}^{n+1}$ where Γ is a family of paths in $J'(\hat{X})$ from x to v_i for $i = 1, 2, \dots, n+1$.

THEOREM 6.3. The monodromy of the function $\Lambda_x(z)$ along the loop S_k is given by

$$S_k : \Lambda_x(z) \to \Lambda_x(z)\cdot a^x_{v_k}(\gamma_k)\cdot e^{-2\pi i X_k}\cdot(a^x_{v_k}(\gamma_k))^{-1}.$$

PROOF. It follows from Proposition 6.1 that

$$(*_1) \qquad \Lambda_x(z)\cdot a^x_{v_k}(\gamma_k) = \Lambda_{v_k}(z)$$

for z in the small neibourhood of some $\gamma(z_0)$. This equality is preserved after the monodromy transformation along S_k, hence we have

$$(*_2) \qquad (\Lambda_x(z))^{S_k}\cdot a^x_{v_k}(\gamma_k) = (\Lambda_{v_k}(z))^{S_k},$$

where $(\ )^{S_k}$ denotes the function $(\ \ )$ after the monodromy transformation along S_k. It follows from Proposition 6.2 that

$$(*_3) \qquad (\Lambda_{v_k}(z))^{S_k} = \Lambda_{v_k}(z)\cdot e^{-2\pi i X_k}.$$

If we substitute $(*_3)$ in $(*_2)$ and then substitute $(*_1)$ for $\Lambda_{v_k}(z)$ we get the formula for $\Lambda_x(z))^{S_k}$.

COROLLARY 6.4. The monodromy of the function $L_x(z)$ along the loop S_k is given by

$$S_k : L_x(z) \to L_k(z)\cdot\alpha^x_{v_x}(\gamma_k)\cdot(-\ 2\pi i X_k).\alpha^x_{v_k}(\gamma_k)^{-1}$$

where $\alpha^x_{v_k}(\gamma_k) = \log(\alpha^x_{v_k}(\gamma_k))$ (and where $\log : P(X) \to \pi(X)$).

PROOF. The corollary follows immediately from Lemma 2.4.

DEFINITION 6.5. The homomorphism $\theta_x = \theta_x(\omega_x) : \pi_1(X, x) \to \pi(X)$ given by $\theta_x(S_k) := \alpha^x_{v_k}(\gamma_k)\cdot(-\ 2\pi i X_k)\cdot\alpha^x_{v_k}(\gamma_k)^{-1}$ we call the monodromy homomorphism of the form ω_X at x.

Observe that $\theta_x(S_k)$ does not depend on γ_k or v_k. The homomorphism θ_x depends only on x and ω_x.

Now we shall try to calculate coefficients of elements $a_y^x(\gamma)$. Let us set

$$a_y^x(\gamma) = 1 + \sum a_y^x(\gamma, \alpha_1, \ldots, \alpha_n) X_{\alpha_1} \cdot \ldots \cdot X_{\alpha_n}.$$

The equality

$$\Lambda_x(z) \cdot a_y^x(\gamma) = \Lambda_y(z)$$

implies

$$\Lambda_x(\alpha_1, \ldots, \alpha_n)(z) + \sum_{p=1}^{n-1} \Lambda_x(\alpha_1, \ldots, \alpha_p)(z)\, a_y^x(\gamma, \alpha_{p+1}, \ldots, \alpha_n) + a_y^x(\gamma, \alpha_1, \ldots, \alpha_n)$$
$$= \Lambda_y(\gamma, \alpha_1, \ldots, \alpha_n)(z).$$

Hence we get the following recursive formula

$$(*_4) \qquad a_y^x(\gamma, \alpha_1, \ldots, \alpha_n)$$
$$= -\Lambda_x(\alpha_1, \ldots, \alpha_n)(y) - \sum_{p=1}^{n-1} \Lambda_x(\alpha_1, \ldots, \alpha_p)(y) \cdot a_y^x(\gamma, \alpha_{p+1}, \ldots, \alpha_n)$$

for $y \in X$ and

$$(*'_4) \qquad a_y^x(\gamma, \alpha_1, \ldots, \alpha_n)$$
$$= \lim_{z \to a_i} \left(-\Lambda_x(\alpha_1, \ldots, \alpha_n)(z) - \sum_{p=1}^{n-1} \Lambda_x(\alpha_1, \ldots, \alpha_p)(z) \cdot a_y^x(\gamma, \alpha_{p+1}, \ldots, \alpha_n) + \Lambda_y(\alpha_1, \ldots, \alpha_n)(z) \right)$$

if $y = v_i \in T_{a_i}P^1(\mathbb{C}) \setminus \{0\}$.

PROPOSITION 6.6. Let $v \in T_{a_i}P^1(\mathbb{C}) \setminus \{0\}$, $w \in T_{a_j}P^1(\mathbb{C}) \setminus \{0\}$ and let $v = a_j - a_i$ and $w = a_i - a_j$. Let γ be an interval belonging to $J(\hat{X})$ from v to w. Then we have

$$a_w^v(\gamma\,;\,i^n,j) = (-1)^{n+1}\zeta(n+1).$$

PROOF. It follows from $(*'_4)$ that $a_w^v(\gamma\,;\,i^n,j) = -\Lambda_v(i^n,j)(a_j)$ because $\Lambda_v(i^p)(a_j) = 0$.

We have

$$\Lambda_v\left(i^n, j\right)\left(a_j\right) = \int_\gamma -\frac{dz}{z-a_j}, -\frac{dz}{z-a_i}, \ldots, -\frac{dz}{z-a_i} = (-1)^n \int_0^1 -\frac{dz}{z-1}, \frac{dz}{z}, \ldots, \frac{dz}{z} = (-1)^n \zeta\,(n+1)$$

Hence $a_w^v(\gamma, i^n, j) = (-1)^{n+1}\,\zeta(n+1)$.

PROPOSITION 6.7. Let $v, w \in T_{a_k} P^1(C)\setminus\{0\}$ and let γ be a path belonging to $J'(\hat{X})$ from v to w contained in the contractible cone with the vertex at a_k. Then we have

$$a_w^v(\gamma) = e^{\left(\int_v^w -\frac{dz}{z-a_k}\right)X_k}.$$

where the integration path is γ.

PROOF. All coefficient of $\Lambda_v(z)$ but those at $(X_k)^n$, vanish at a_i. The part of $\Lambda_v(z)$ which involve only powers of X_k is equal to $e^{\left(\int_v^z -\frac{dz}{z-a_k}\right)X_k}$. This implies the formula for $a_w^v(\gamma)$.

7. Non-abelian, unipotent periods.

Let $y^2 = x(x-1)(x-t)$ be a family of elliptic curves over $P^1(\mathbb{C})\setminus\{0, 1, \infty\}$. The forms $\frac{dx}{y}$ and $\frac{xdx}{y}$ generate $H^1_{DR}(E_t)$. We consider two integrals

$$I_1(t) = \int_{\gamma_t} \frac{dx}{y} \quad , \quad I_2(t) = \int_{\gamma_t} \frac{xdx}{y}$$

where γ_t is a cycle on E_t which varies continuously with t. Then $I_1(t)$ and $I_2(t)$ satisfy a certain system, call it (*), of differential equations with regular singular points (see [Ph]).

In modern terms a linear system of differential equations is a connection on a vector fiber bundle. Here it is another construction of this system of differential equaitons. Let $E \xrightarrow{\pi} X$ be the family $y^2 = x(x - 1)(x - t)$ and the projection $\pi(x, y, t) = t \in X = P^1(\mathbb{C})\setminus\{0, 1, \infty\}$.

By the construction of Katz and Oda (see [K0]) $H^1_{DR}(R\,\pi_*\,\Omega^*_{E/X})$ is equipped with an integrable connection ∇ (the Gauss-Manin connection) whose horizontal sections are given by $R^1\,\pi_*\,\mathbb{C}$. The integrable connection

$$(**) \qquad (H^1_{DR}\,(R\,\pi_*\,\Omega^*_{E/X}),\,\nabla)$$

has only regular singular points. The monodromy representations associated to linear systems of differential equations (*) and (**) are isomorphic. Hence the differential equations (*) and (**) are isomorphic and $I_1(t)$, $I_2(t)$ are also solutions (horizontal sections) of (**).

Our aim is to show that polylogarithms and more generally iterated integrals are horizontal sections of a Gauss-Manin connection associated to a morphism $X^\bullet \to S^\bullet$ between cosimplicial algebraic varieties. It seems to us that one cannot use a morphism between algebraic varieties to construct a system of differential equations whose solutions are polygarithms.

Let X be a projective line minus several points. Without loss of generality we can assume that $X = P^1(\mathbb{C})\setminus\{a_1, \ldots, a_n, \infty\}$. Let $X_i = \left(\frac{dz}{z - a_i}\right)^*$ $i = 1, \ldots, n$ and let $P(X) = C[[X_1, \ldots, X_n]]^*$ be the group of invertible elements whose constant terms are equal to 1 in the algebra of formal power series. We recall that a principal $P(X)$-fibre bundle

$$X \times P(X) \to X$$

we equipped with an integrable connection given by the form ω_X. Let $D(X)$ be a vector space generated by linear functionals on $C[[X_1, \ldots, X_n]]$ which associate to any element of $C[[X_1, \ldots, X_n]]$ its coefficient at a given monomial. The element $g \in P(X)$ acts on

$C[[X_1, \ldots, X_n]]$ by the left multiplication by g^{-1}, hence it acts also on $D(X)$. The connection ω_X induces a linear connection δ_X on the associated vector bundle.

$$X \times D(X) \to X.$$

The monodromy of ω_X along a path α is given by the right multiplication by the element

$$\left(1 + \sum \left(\int_\alpha \frac{-dz}{z - a_{i_k}}, \ldots, \frac{-dz}{z - a_{i_1}}\right) X_{i_1} \ldots X_{i_k}\right) =$$

$$\left(1 + \sum \left(\int_{\alpha^{-1}} \frac{dz}{z - a_{i_1}}, \ldots, \frac{dz}{z - a_{i_k}}\right) X_{i_1} \ldots X_{i_k}\right) =$$

$$\left(1 + \sum \left(\int_\alpha \frac{dz}{z - a_{i_1}}, \ldots, \frac{dz}{z - a_{i_k}}\right) X_{i_1} \ldots X_{i_k}\right)^{-1}$$

Hence the monodromy of δ_X is given by the left multiplication by the element $\left(1 + \sum \left(\int_\alpha \frac{dz}{z - a_{i_1}}, \ldots, \frac{dz}{z - a_{i_k}}\right) X_{i_1} \ldots X_{i_k}\right)$. The image of the functional $\left(X_{i_1} \ldots X_{i_n}\right)^*$ - the coefficient at $X_{i_1} \ldots X_{i_n}$ - along the path α is given by

$$(*_1) \qquad \left(X_{i_1} \ldots X_{i_n}\right)^* \xrightarrow{\alpha} \sum_{k=0}^{n} \left(\int_\alpha \frac{dz}{z - a_{i_1}}, \ldots, \frac{dz}{z - a_{i_k}}\right) \left(X_{i_{k+1}} \ldots X_{i_n}\right)^*$$

Now we shall construct a morphism between cosimplicial algebraic varieties such that the monodromy representation of the Gauss-Manin connection associated to this morphism will coincide with the representation $(*_1)$.

We shall omit all details on cosimplicial algebraic varieties and Gauss-Manin connections associated to morphisms between cosimplicial varieties which the reader can find in [W1].

Let X be a smooth algebraic variety over the field k of characteristic zero. The inclusion of simplicial sets

$$\partial\Delta[1] \hookrightarrow \Delta[1]$$

induces a morphism of cosimplicial varieties

$$p^{\bullet} : X^{\Delta[1]} \to \partial X^{\Delta[1]}$$

Imitating the construction of Katz and Oda (see [KO]) we equipped the sheaves $H^i(t\, R\, p^{\bullet}_* \, \Omega^{*}_{X^{\Delta[1]}/\partial X^{\Delta[1]}})$ with the integrable connection d_R. We shall calculate the monodromy representation of this connection for $X = P^1(\mathbb{C})\backslash\{a_1,\ldots,a_n\}$ and $i = 0$. The fibre of $H^i(t\, R\, p^{\bullet}_* \, \Omega^{*}_{X^{\Delta[1]}/\partial X^{\Delta[1]}})$ over the point $(a, b) \in X \times X$ is equal to $H^i_{DR}(p^{\bullet -1}(a, b))$. Let $A^*(X^n) \subset \Omega^*(X^n)$ be a subcomplex of $\Omega^*(X^n)$ generated by 1 and products $\frac{dz_{i_1}}{z_{i_1} - a_{\alpha_1}} \wedge \ldots \wedge \frac{dz_{i_k}}{z_{i_k} - a_{\alpha_k}}$ for $k \le n$. The inclusion $A^*(X^n) \to \Omega^*(X^n)$ is a quasi-isomorphism. Moreover the complexes $A^*(X^n)$ are preserved by coboundaries and codegeneracies of $p^{\bullet -1}(a, b)$. This implies that

$$H^i_{DR}(p^{\bullet -1}(a, b)) = H^i(\mathrm{Tot}(A^*(X^n)).$$

Observe that $H^0\left(\mathrm{Tot}\left(\bigoplus_i A^*\left(X^n\right)\right)\right) = \bigoplus_{n=0}^{\infty} A^n\left(X^n\right)$. The inclusion $A^*(X^n) \to \Omega^*(X^n)$ is a quasi-isomorphism. Moreover complexes $A^*(X^n)$ are preserved by coboundaries and codegeneracies of $(p^{\bullet -1}(a, b))$. This implies that

$$H^i_{DR}(p^{\bullet -1}(a, b)) = H^i(\mathrm{Tot}\, A^*(X^n)).$$

Observe that $H^0_{DR}\left(\mathrm{Tot}\, A^*\left(X^n\right)\right) = \bigoplus_{n=0}^{\infty} A^n\left(X^n\right)$. Let $\omega = \omega_1 \wedge \ldots \wedge \omega_n \in A^n(X^n) \subset H^0\left(\mathrm{Tot}\left(\bigoplus_i A^*\left(X^i\right)\right)\right) = H^0_{DR}(p^{\bullet -1}(a, b))$ where $\omega_i = \frac{dz_i}{z_i - a_{k_i}}$. We shall extend the element ω to a section of $H^0_{DR}(t\ R\ p^{\bullet}_{*}\ \Omega^{*}_{X^{\Delta[1]}/\partial_{X^{\Delta[1]}}})$ in some open neibourhood of (a, b). Let

$$\theta = \sum_{0 \le k \le n-\ell \le n} \left(\int_a \overline{\omega}_1,\dots,\overline{\omega}_k\right)\omega_{k+1}\wedge\dots\wedge\omega_{n-\ell}\left((-1)^{\ell}\right)\int_b\left(\overline{\omega}_n,\overline{\omega}_{n-1},\dots,\overline{\omega}_{n-\ell+1}\right)\in$$

$$\bigoplus_{0 \le k \le n-\ell \le n} \Omega^{n-k-\ell}\left(X \times X^{n-k-\ell} \times X\right).$$

where z_i in ω_i (resp z_{n-i} in ω_{n-i}) becomes z_0 in $\overline{\omega}_i$ (resp. z_{n+1} in $\overline{\omega}_{n-i}$

One checks that θ is a section of $H^0(t\ R\ p^{\bullet}_{*}\ \Omega^{*}_{X^{\Delta[1]}/\partial_{X^{\Delta[1]}}})$ over some open neibourhood of $(a, b) \in X \times X$. Moreover using the definition of the connection $d_{\mathbb{C}}$ as a d_1 differential of a certain spectral sequence (see [W1]) one verifies that θ is a horizontal section of the connection $d_{\mathbb{C}}$. Let $(\alpha, \beta) \in \pi_1(X \times X, (a, b)) = \pi_1(X, a) \times \pi_1(X, b)$. Moving ω horizontally along the loop (α, β) we get

$$\sum_{0 \le k \le n-l \le n} \left(\int_\alpha \overline{\omega}_1,\dots,\overline{\omega}_k\right)\omega_{k+1}\wedge\dots\wedge\omega_{n-\ell}\left(\left((-1)^{\ell}\right)\int_\beta\left(\overline{\omega}_n,\dots,\overline{\omega}_{n-\ell+1}\right)\right) =$$

$$\sum_{0 \le k \le n-l \le n} \left(\int_\alpha \overline{\omega}_1,\dots,\overline{\omega}_k\right)\omega_{k+1}\wedge\dots\wedge\omega_{n-\ell}\left(\left((-1)^{\ell}\right)\int_{\beta^{-1}}\left(\overline{\omega}_{n-\ell+1},\dots,\overline{\omega}_n\right)\right).$$

If we restrict the sheaf and the connection to $X \times \{b\}$ then the image of ω along the path α is given by

$$(**_1) \qquad \omega \xrightarrow{\alpha} \sum_{k=0} \left(\int_\alpha \overline{\omega}_1,\dots,\overline{\omega}_k\right)\omega_{k+1}\wedge\dots\wedge\omega_n .$$

THEOREM 7.1. The vector bundle $X \times D(X) \to X$ equipped with the connection δ_X and the sheaf $H^0(t\ R\ p^{\bullet}_{*}\ \Omega^{*}_{X^{\Delta[1]}/\partial_{X^{\Delta[1]}}})_{|X\times\{b\}}$ equipped with the connection $d_{\mathbb{C}}$ have only regular singular points and they are isomorphics objects in the category of vector bundles equipped with integrable connections.

PROOF : One can directly check that the connection δ_X have only regular singular points. It follows from [W1] that the connection $d_{\mathbb{C}}$ have only regular singular points. In fact the connection $d_{\mathbb{C}}$ is a succesive extension of trivial connections, so it has only regular singular points. The isomorphism from $D(X)$ to $H^0_{DR}\left(\mathrm{Tot}\, A^*\left(X^n\right)\right) = \bigoplus_{n=0}^{\infty} A^n\left(X^n\right)$ is given by $(X_{i_1} \dots X_{i_n})^* \to \frac{dz_1}{z_1 - a_{i_1}} \wedge \dots \wedge \frac{dz_n}{z_n - a_{i_n}}$. This isomorphism gives an isomorphism of representations $(*_1)$ and $(**_1)$. The bundles equipped with connections δ_X and $d_{\mathbb{C}}$ have isomorphic monodromy representations hence they are isomorphic.

8. Unipotent differential equations

Let X be a smooth, analytic variety. The connection ∇ on the trivial vector bundle $(\mathcal{O}_X)^n$ is called trivial if $\nabla(f_1, \dots, f_n) = (df_1, \dots, df_n)$.

DEFINITION 8.1. An unipotent differential equation on X is a vector bundle $\mathcal{V}$ equipped with an integrable connection ∇ and a filtration $\{\mathcal{V}_i\}_{i=0}^n$ ($\mathcal{V}_0 = 0$, $\mathcal{V}_n = \mathcal{V}$) compatible with the connection ∇ such that the connection induced by ∇ on the associated graded vector bundle $\mathrm{Gr}\,\mathcal{V} = \bigoplus_{i=1}^{n} \mathcal{V}_i/\mathcal{V}_{i-1}$ is trivial.

Examples. Let $X = \mathbb{C}\setminus\{0,1\}$. The system of differential equations on X given by

$$f_0' = 0\,,\; f_1' = f_0 \frac{1}{z}\;,\; f_2' = f_0 \frac{1}{z-1}\,,\; f_3' = -f_2 \frac{1}{z}$$

is an unipotent differential equation on X.

Let E be an elliptic curve and let $X = E\setminus\{0\}$. Then

$$f_0' = 0\,,\; f_1' = f_0\;,\; f_2' = f_0\, \wp(z)\,,\; f_3' = f_2$$

is an unipotent differential equation on X.

Let $X = P^1(\mathbb{C})\setminus\{a_1, \dots, a_{n+1}\}$. Then the principal fibre bundle $X \times \pi(X) \xrightarrow{P} X$ is equipped with the integrable connection given by the one form ω_X. If $\rho : \pi(X) \to GL(V)$ is an algebraic representation in a finite dimensional complex vector space V then the associated vector bundle $\mathcal{V}_\rho := ((X \times \pi(X)) \underset{\pi(X)}{\times} V \xrightarrow{P_\rho} X)$ is equipped with an integrable connection ∇_ρ induced by the connection ω_X. Moreover the representation ρ is unipotent (see [SGA] Corollaire 3.4) therefore the pair $(\mathcal{V}_\rho, \nabla_\rho)$ is a unipotent differential equation on X.

The main result of this section is the following theorem.

THEOREM 8.2. Let $X = P^1(\mathbb{C})\setminus\{a_1, \dots, a_{n+1}\}$.

The following categories are equivalent :

- the category of unipotent differential equations on X (denoted by UDE(X)), and
- the category of algebraic representation of $\pi(X)$ in finite dimensional complex vector spaces (denoted by Rep $\pi(X)$).

The equivalence is given by the correspondence

$$(\rho : \pi(X) \to GL(V)) \rightsquigarrow (\mathcal{V}_\rho, \nabla_\rho)$$

on objects and

$$(f : V \to W) \rightsquigarrow ((X \times \pi(X)) \underset{\pi(X)}{\times} V \to (X \times \pi(X)) \underset{\pi(X)}{\times} W)$$

or morphism.

Before we shall prove the theorem we shall need another definition.

DEFINITION 8.3. Let π be a discrete group. We say that a representation of π in a finite dimensional vector space V is unipotent if its image is contained in upper triangular matrices with 1's on a diagonal with respect to some base of V.

LEMMA 8.4. For $X = P^1(\mathbb{C})\setminus\{a_1, \dots, a_{n+1}\}$ the homomorphism $\theta_X : \pi_1(X, x) \to \pi(X)$ (see Definition 6.5) induces an equivalence of categories ;

- category of algebraic representations of $\pi(X)$ in finite dimensional complex vector spaces and
- category of complex, finite dimensional unipotent representations of $\pi_1(X, x)$. (This last category we denote by U Rep$(\pi_1(X, x))$.)

The lemma follows from the fact that θ_X is a pro-nilpotent completion of $\pi_1(X, x)$.

PROOF OF THE THEOREM. We have a diagram of categories

$$\begin{array}{ccc} \mathrm{Rep}\big(\pi(X)\big) & \xrightarrow{\quad\alpha\quad} & \mathrm{UDE}(X) \\ \beta\downarrow & & \gamma\downarrow \\ \mathrm{U\,Rep}\big(\pi_1(X,x)\big) & \xrightarrow{\text{identity}} & \mathrm{U\,Rep}\big(\pi_1(X,x)\big) \end{array}$$

where α is the functor from Theorem 7.3 and β is the functor from Lemma 8.4. The functor γ associates to a unipotent differential equation its monodromy representation at a point x. It follows from Lemma 8.4 that β is an equivalence of categories. It follows from [D2] Theorem 2.17 and Corollaire 1.4 that γ is an equivalence of categories. The monodromy representation at x of the unipotent differential equation $\alpha(\rho : \pi(X) \to GL(V))$ is the homomorphism $\pi_1(X, x) \xrightarrow{\theta_X} \pi(X) \xrightarrow{\rho} GL(V)$. Therefore the diagram commutes. Hence α is an equivalence of categories.

9. Principal $\pi(X)$-fibrations.

In section 6 we have shown that the connection form ω_X on $X = P^1(\mathbb{C})\setminus\{a_1, \ldots, a_{n+1}\}$ give rise to a monodromy homomorphism

$$\theta_x : \pi_1(X, x) \to \pi(X).$$

The evident question appears, which is in fact a variant of the classical one. Which homomorphisms $\pi_1(X, x) \to \pi(X)$ can be realized as a monodromy homomorphism of an integrable connection on a principal $\pi(X)$-bundle over X ?

THEOREM 9.1.

a) Let $\psi : \pi(X) \to \pi(X)$ be an isomorphism of complex, affine groups. Then the homomorphism $\psi \circ \theta_x : \pi_1(X, x) \to \pi(X)$ can be realized as a monodromy homomorphism of an integrable connection on a principal $\pi(X)$-bundle over X.

b) Let $\omega = \sum_{i=1}^{n} \frac{dz}{z - a_i} \otimes \omega_i$ be a one form on X with values in the Lie algebra $\pi(X)$ such that $\omega_1, \ldots, \omega_n$ generate the Lie algebra $\pi(X)$. Then the monodromy homomorphism $\theta(\omega) : \pi_1(X, x) \to \pi(X)$ of an integrable connection given by ω on a principal $\pi(X)$-bundle $X \times \pi(X) \to X$ is equal to

$$\psi_\omega \circ \theta_x : \pi_1(X, x) \to \pi(X)$$

where $\psi_\omega : \pi(X) \to \pi(X)$ is an isomorphism such that $\psi_\omega(X_i) = \omega_i$.

c) Let $\chi : \pi_1(X, x) \to \pi(X)$ be such that the induced map $(\pi_1(X, x)/\Gamma^2\pi_1(X, x)) \otimes \mathbb{C} \to \pi(X)/\Gamma^2\pi(X)$ is an isomorphism. Then there is a one form ω on X with values in the Lie algebra $\pi(X)$ such that the monodromy homomorphism of the integrable connection given by ω on a principal $\pi(X)$-bundle $X \times \pi(X) \to X$ is equal to χ.

PROOF. Let $X \times \pi(X) \to X$ be equipped with the connection given by ω_X. Then the monodromy homomorphism of the associated bundle $(X \times \pi(X)) \underset{\pi(X)}{\times} \pi(X) \to X$, where $\pi(X)$ acts on $\pi(X)$ by ψ, is given by the composition $\psi \circ \theta_x$.

To show point b) let us observe that the map $id \times \psi_\omega : X \times \pi(X) \to X \times \pi(X)$ satisfies $(id \otimes \psi_\omega)\omega_X = \omega$. This implies that the map $id \times \psi_\omega$ maps horizontal sections with respect to ω_X into horizontal sections with respect to ω. Hence it follows that $\theta(\omega) = \psi_\omega \circ \theta_x$.

To show point c) observe that there exists a homomorphism $\psi : \pi(X) \to \pi(X)$ such that $\chi = \psi \circ \theta_x$. One defines $\omega = \sum_{i=1}^{n} \frac{dz}{z - a_i} \otimes \psi(X_i)$. Now point c) follows from point b).

10. Generalized Bloch-Wigner functions

The function $Li_2(z)$ is multivalued. Wigner observed that the function

$$D_2(z) = \mathrm{Im}\, Li_2(z) + \mathrm{Log}\, |z| \cdot \arg(1 - z)$$

is sigle valued, real analytic on $P^1(\mathbb{C})\setminus\{0, 1, \infty\}$. D. Ramakrishnan constructed analogs of D_2 for functions $Li_n(z)$ (see [R2]). Various analogs of the function D_2 appeared in [W2], [Z1] and [Z2]. Very elegant construction of higher analogs of D_2 is given in [D3].

Our aim it to construct analogs of the Bloch-Wigner function D_2 for any iterated integral $\int_x^z \frac{dz}{z - a_i}, \ldots, \frac{dz}{z - a_n}$. We shall try to generalize the construction from [D3].

Let $X = P^1(\mathbb{C})\setminus\{a_1, \ldots, a_n, \infty\}$. The principal $P(X)$-bundle $X \times P(X) \to X$ is equipped with the connection ω_X. The monodromy homomorphism $\theta_x : \pi_1(X, x) \to \pi(X)$ is given by $\theta_x(S_k) = \alpha_k \cdot e^{-2\pi i X_k} \cdot \alpha_k^{-1}$. (We recall that $x \in \hat{X}$, $S_1, \ldots, S_n$ is a sequence of geometric generators such that S_k is a loop around a_k, $X_k \in H_1(X, \mathbb{C})$ is the class of S_k and $\alpha_k \in P(X)$).

Let $\psi_x : \pi_1(X, x) \to P(X)$ be given by $\psi_x(S_k) = \bar{\alpha}_k \cdot e^{-2\pi i X_k} \cdot (\bar{\alpha}_k)^{-1}$ ($\bar{\ }$) denotes the complex conjugate of ( )). Let τ_x be a one-form on X with values in the Lie algebra of $P(X)$ such that its monodromy homomorphism is ψ_x. The existence of such τ_x follows from Theorem 8.1. Let $\mathcal{L}_x(z)$ be a horizontal section starting at x of the bundle $X \times P(X) \to X$ equipped with the connection τ_x . We recall that $\Lambda_x(z)$ is a horizontal section with respect to the connection ω_X.

PROPOSITION 10.1. The function

$$X \ni z \to P_x(z) := \Lambda_x(z)\left(\overline{\mathcal{L}_x(z)}\right)^{-1}$$

is single valued on X. The real and the imaginary part of any coordinate of $P_x(z)$ is real analytic on X.

PROOF. After the monodromy transformation along the loop S_k the function $P_x(z)$ changes into

$$\Lambda_x(z)\cdot\overline{\alpha_k}\cdot e^{2\pi i X_k}\cdot\alpha_k^{-1}\cdot\left(\overline{\text{Ł}_x(z)\cdot\overline{\alpha_k}\cdot e^{2\pi i X_k}\cdot\alpha_k^{-1}}\right)^{-1}=$$

$$\Lambda_x(z)\cdot\alpha_k\cdot e^{2\pi i X_k}\cdot\alpha_k^{-1}\cdot\alpha_k\cdot e^{2\pi i X_k}\cdot\alpha_k^{-1}\cdot\left(\overline{\text{Ł}_x(z)}\right)^{-1}=\Lambda_x(z)\left(\overline{\text{Ł}_x(z)}\right)^{-1}.$$

The second part of the proposition is clear.

Now let $X = P^1(\mathbb{C})\setminus\{0, 1, \infty\}$. We choose a base U, V of $H_1(X, \mathbb{C})$ given by loops around 0 and 1. Let $v \in T_0P^1(\mathbb{C})$ be a vector in $\mathbb{C}$ from 0 to 1. We calculate coordinates of $P_v(z)$ at U^nV. Let us denote by $\mathcal{D}_{n+1}(z)$ the coordinate of $P_v(z)$ at U^nV.

PROPOSITION 10.2. We have

$$\mathcal{D}_{n+1}(z)=(-1)^n\,\mathrm{Li}_{n+1}(z)+\overline{\mathrm{Li}_{n+1}(z)}+\sum_{k=1}^{n}\frac{1}{k!}(-1)^k(2\log|z|)^k\,\overline{\mathrm{Li}_{n+1}(z)}.$$

PROOF. Let $P'(X)$ be a group of series of the form $e^{aU}+\sum_{n=0}^{\infty} b_n U^n V$ with a multiplication given by the following formula

$$\left(e^{aU}+\sum_{n=0}^{\infty} b_n U^n V\right)\left(e^{a'U}+\sum_{n=0}^{\infty} b_n' U^n V\right):=$$

$$e^{(a+a')U}+\sum_{n=0}^{\infty}\left(b_n+b_n'+\sum_{i=1}^{n} b_{n-i}'\frac{a^i}{i!}\right)U^nV.$$

The principal $P'(X)$-bundle $X \times P'(X) \to X$ we equipped with the connection ω_X. The horizontal sections at $v \in T_0P^1(\mathbb{C})$ are given by

$$e^{(-\log z)U}+\sum_{n=0}^{\infty}(-1)^n\,\mathrm{Li}_{n+1}(z)\,U^nV=:\mathrm{Li}(z).$$

Observe that the monodromy of $\mathrm{Li}(z)$ around U and V is given by

$$U:\mathrm{Li}(z)\to\mathrm{Li}(z)\cdot e^{(-2\pi i)U},\quad V:\mathrm{Li}(z)\to\mathrm{Li}(z)\cdot(-2\pi i V).$$

Hence the image of the form τ_X under the homomorphism $P(X) \to P'(X)$ is $\tau'_X = -\frac{dz}{z} \otimes U - \frac{dz}{z-1} \otimes V$. The horizontal sections at $v \in T_0P^1(\mathbb{C})$ of the principal $P'(X)$-bundle $X \times P'(X) \to X$ equipped with the connection τ'_X are given by

$$e^{(\log z)U} + \sum_{n=0}^{\infty} (-1)\ \mathrm{Li}_{n+1}(z)\, U^n V =: \mathcal{L}(z).$$

After a calculation we get

$$\mathrm{Li}(z).\ \overline{(\mathcal{L}(z))}^{-1} = e^{\left(-\log z - \overline{\log z}\right)U} +$$

$$\sum_{n=0}^{\infty} \left((-1)^n\ \mathrm{Li}_{n+1} + \overline{\mathrm{Li}_{n+1}} + \sum_{k=1}^{n} \frac{1}{k!} \left(-\log z - \overline{\log z}\right)^k\ \overline{\mathrm{Li}_{n+1-k}(z)} \right) U^n V .$$

Observe that for $n + 1$ even the function $\mathrm{Im}(\mathcal{D}_{n+1}(z))$ is an analog of the Bloch-Wigner function. For example we have $\mathrm{Im}(\mathcal{D}_2(z)) = -2D_2(z)$. If $n + 1$ is odd then the function $\mathrm{Re}(\mathcal{D}_{n+1}(z))$ is an analog of the Bloch-Wigner function.

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