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CYCLIC COHOMOLOGY, SUPERSYMMETRY AND KMS STATES THE KMS STATES AS GENERALIZED ELLIPTIC OPERATORS*

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Abstract : We present the result that each Super-KMS "functional" of a $\mathbb{Z}/2$ graded C^*-algebra with respect to a "supersymmetric" one-parameter automorphism group gives rise to an entire cyclic cocycle. We recall a background concerning KMS states and cyclic cohomology (original and entire) in order to place this result in perspective.

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Let me begin this talk at a conference devoted to the interface between physics and geometry by stressing the essential identity of geometry and analysis. The distinction is one of methods. The common theme is manifolds and the pseudodifferential operators attached to their vector bundles.

There are strong indications that one of the main avenues of future mathematical development will lead to generalize geometry-analysis in a double respect :

(i) the passage from finite to infinite dimension

(ii) the more radical passage from usual to "non commutative manifolds" (C^∞ replaced by a non commutative algebra).

Step (i) is already there classically (e.g. loop spaces). Step (ii) is "*quantization*" in a broad sense, as purported by Alain Connes' "non commutative geometry", the actual frame of my talk [1] [2], at the same time non commutative analysis (cf. Fredholm modules) and differential geometry/topology (cf. Chern character and cyclic cohomology). This theory has two versions corresponding to finite, resp. infinite dimension (in non commutative paraphrase !). The first is concerned with *cyclic cohomology* and p-*summable Fredholm modules* [1], the second and more recent with *entire cyclic cohomology and* θ-*summable Fredholm modules* [2][2a].

My specific purpose is to convince you that the *KMS states* (physically : temperature equilibrium states, mathematically the "generalized traces" of the second era of Von Neumann algebras) are appropriate substitutes for the classical elliptic operators in the generalizations (i) and (or) (ii) above. This is the moral I want to draw from the fact [3] that *graded* KMS-*functionals of supersymmetric dynamical* C^*-*systems yield entire cyclic cocycles*. Before going into the matter (and rather than displaying gory technicalities[1]) I shall endeavour to put this result with its claimed interpretation into the broad perspective of Alain Connes' doctrine.

The general philosophy is that *non commutative (associative) algebras* - technically C^*-algebras -*are "non commutative spaces"*. This thought arose from the recognition that function algebras are a general model of commutative algebras (Gelfand) : specifically : each commutative

1 The proofs in [3] and [4] are, to my taste, too much of a verification. A more conceptual proof would be desirable.

C^*-algebra A is isomorphic to $C_0(X)$, the algebra of continuous functions on its "spectrum" X (in the absence of a unit, one has to take the functions vanishing at ∞ ; for a unital algebra A, the spectrum X is compact).[2]. Now, if you suppress the sole commutativity axiom from a system of seven axioms, you pass from model $C_0(X)$ to general C^*-algebras: therefore it is tempting to consider the latter as "non commutative (= quantum) spaces" -whatever this means.

Well, nowadays it means a lot : the "quantum version" of spaces has progressed in four successive stages of growing structure, viz.

1) measure theory
2) topology (topological K-theory)
3) smooth structures
4) riemannian (spin^c) structures

as shown in the table below displaying the quantum substitutes of the classical items. A deep parallelism repeatedly appears as the miraculous fact that quantum proofs are already present in classical proofs which, phrased in the language of algebras, appear not to require commutativity !

Classical	Quantum
1° Measure theory	
Bounded measures	States. Hilbert space representations
$C_0(X)$, X locally compact with countable basis	Separable C^*-algebras.
$(L^\infty(X,\mu)$	Hyperfinite von Neumann algebras completely classified by means of - traces (type I and II) - KMS *states* (type III)
if a group acts : ergodic theory (topological or measure theoretic)	C^*- and W^*-systems (obtained from actions of groups or more generally Ocneanu paragroups).
2° Topological K-theory	
$K^0(X)$, $K^1(X)$ (X locally compact) classification of vector bundles	$K_1(A)$ $K_0(A)$ (Aa C^*-algebra) classification of finite projective modules
Bott periodicity	
3° Smooth structures	
Finite, resp. ∞-dimensional manifolds	Cyclic, resp. entire cyclic cohomology of non commutative algebras
Elliptic operators	Fredholm modules, p-summable, resp. θ-summable.
Duality with K-theory	
4° Riemannian (Spin^c) structures	
Spin^c manifolds	C^*-algebras with "Dirac-Fredholm modules"
Dirac operator	

2 This Gelfand *"structure theory"* is the culmination of the *spectral theorem asserting that commuting operators on Hilbert space are described by functions on their common spectrum (almost equivalent, knowing that C^*-algebras are always realizable by operators on Hilbert space).*

We now comment this table in detail -as a long preamble to our subject proper .

1. Measure theory. The quantum version of measure theory is nowadays completed through a full classification of the hyperfinite factors [5] [6] [7] [8][3].
In his pioneering work with Murray, von Neumann had showed (reduction theory) that "rings of operators" [4] decompose into "factors"[5], and had classified the latter according to the range of their trace on projections, viz. :
- $\mathbb{N}$ for the "I_s" - the only "factors" deserving this name, since obtained as $\mathcal{B}(\mathcal{H}_1) \otimes \mathbf{1}_{\mathcal{H}_2}$ through tensorial factorization of Hilbert space
- [0,1], resp. [0,∞], for the "II_1"$_s$, resp. the "II_∞"$_s$
- {0,∞} for the "III_s" for which the trace is thus inefficient, a circumstance which caused Murray and von Neumann to stop investigating the "III_s", after the construction of a few examples. The "III_s" were long deemed pathological and uninteresting, until algebraic field theorists found them everywhere present in physics (as local algebras of wedges in the vacuum representation of relativistic field theories ; and as weak closures of the whole algebra in temperature situations -relativistic or not [9], [10] [11] [12]).

The type III-deadlock was overcome after Tomita's breakthrough [13] with the advent of the Tomita - Takesaki theory [14], this occurring in parallel with the recognition of the basic role of KMS states in equilibrium statistical mechanics [15]. Armed with the KMS concept (the proper substitute of the trace for the "III_s") Alain Connes then completely elucidated their classification in the hyperfinite case [5] [6] [7]. The last remaining step -uniqueness of the hyperfinite III_1 (the one in physics !)- was effected by Haagerup [8] (see also [16], and [17] [17a] for the relation to physics).

We define briefly the all-important KMS concept. Let (A,α) be a C^*-dynamical-system consisting of a C^*-algebra A with a one-parameter group α_t of automorphisms[6]. A state φ of A (=norm $\mathbf{1}$ positive linear functional) is a *KMS-state to the inverse temperature* β whenever one has [16]

$$(1) \qquad \varphi(ba) = \varphi(a\alpha_{i\beta}(b)) \quad , \quad \begin{cases} a,b \in A \\ b \text{ analytic for } \alpha \end{cases}$$

This *KMS(Kubo, Martin, Schwinger)-condition* (reflecting itself as the fact that the two point function $\varphi(a\alpha_t(b))$ is analytic in the strip $0 \le \mathrm{Im} z \le 1$) contains in fact the same information as the *Gibbs Ansatz* :

$$(2) \qquad \varphi(a) = \mathrm{Tr}\,(e^{-\beta H}a) \,/\, \mathrm{Tr}e^{-\beta H} ,$$

but in a form freed from the unphysical constraints of (2) (system in a box, artificially discretized energy spectrum !) : indeed (1) makes sense without the need of the (conceptually obscure if computationally necessary) "thermodynamic limit", and has a direct physical

3 in contrast "non-commutative ergodic theory" (the study of actions of groups on algebas) is still largely open.

4 now called von Neumann or W^*-algebras

5 W^*-algebras with a trivial center (and accordingly a unique normal trace)

6 If A is represented on a Hilbert space with implementation of the dynamics by a hamiltonian H, one has $\alpha_t(a) = e^{iHt}\, a\, e^{-iHt}$

interpretation (in fact characterization) in physical terms (stability w.r.t. local perturbations [18] or "passivity" [19] - an abstract version of the second principle of thermodynamics). The one-parameter automorphism group appearing in (1) is time development (dynamics) for the (gauge invariant) observables, and a mixture of dynamics and gauge measured by the chemical potential for the "fields" -the latter fact evolving from the first in a purely algebraic way [20].

We note that the Gibbs state (2) is of course KMS for α_t as in footnote 6 for the inverse temperature β, as revealed by a two-line verification.

Note that this sketch of "temperature algebraic field theory" (treated at length in [21]) does not touch the "vacuum theory" (Doplicher- Haag-Roberts theory of superselection sectors, cf. quotations in [22]), of recent renewed interest[7]).

To conclude this paragraph, let us mention that the basic appearance of KMS states in von Neumann algebra theory differs technically from that in quantum statistical mechanics. Every von Neumann algebra is automatically equipped by any state φ faithful on its positive cone with a *modular one parameter group* α_t^φ for which the state is KMS (with the conventional choice $\beta = -1$). And the α_t^φ for the different φ differ from each other by cocycles (non commutative Radon-Nicodym derivatives) with values in inner automorphisms. In contrast to this, physical systems are given as pairs -dynamical systems- (A,α) of a C^*-algebra of observables plus a time-development automorphism group α, equilibrium states φ of temperature $(k\beta)^{-1}$ being β-KMS for α (so that $\alpha_{-\beta t}$ is a modular automorphism in the above sense for the weak closure A_φ'' of A in the representation φ generates - A_φ'' is however not a basic entity of the physical system, but rather an attribute of the system plus the state.

2) Topological C^*-K theory We will only mention that the general theory (including Bott periodicity) naturally evolves from phrasing the usual proofs in terms of modules over the algebra rather than in spatial terms. The non commutative generalization arises from the observation (Serre-Swan) that the modules of sections of vector bundles over X are exactly the finite projective modules over $C_0(X)$.

Strikingly enough, the classical proofs [6] when phrased in algebraic language turn to be largely independent of the commutativity axiom. C^*-K theory (including the bivariant KK-theory of Kasparov [26] [29]) is now an ample body of knowledge [24].

3) Quantum smooth structures We now sketch Alain Connes "non commutative differential geometry" [1] [2]. Connes discovered cyclic cohomology whilst recognizing the "Fredholm modules" as the non commutative substitutes for elliptic operators[8]. The classical situation is as follows : an elliptic operator $P : C^\infty(E^\circ) \to C^\infty(E^1)$, E°, E^1 vector bundles over an n-dimensional smooth manifold M, becomes (via extension to appropriate Sobolev-completions) a Fredholm operator $P : H^\circ \to H^1$ with

7 The DHR theory of superselection sectors is presently successfully applied to 2 and 3 dimensional field theories [22] and Schoer's talk at this conference.

8 The starting point was [21] and [22].

quasi inverse $Q = H^1 \to H^\circ$. Introducing the direct sum $H = H^\circ \oplus H'$ (graded with grading involution $\varepsilon = \mathbf{1}_{H^0} \oplus - \mathbf{1}_{H^1}$), P and Q are subsumed by[9]

(3) $$F = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix} \quad ,$$

whilst each $a \in A = C^\infty(M)$ acts on H as

(4) $$(a) = \begin{pmatrix} a^0 & 0 \\ 0 & a^1 \end{pmatrix}$$

(a^i the multiplication by a on $C^\infty(H^i)$). This situation entails the following facts : with $L^p(H)$ the p^{th} Schatten ideal (set of linear operators B on H with $(B^*B)^{p/2}$ trace class), we have

(5) $$(F^2-\mathbf{1}) \;.\; (a) \in L^1(H), \quad a \in A \;,$$

whilst

(6) $$[F,(a)] \in L^p(H) \quad , a \in A \;, \quad p > n \;.$$

Together with the Hölder inequality, this allows Alain Connes to define the *p^{th} character of F* as the (p+1)-linear form :

(7) $$\varphi^{(p)}\, (a_0, a_1, \ldots, a_p) = C_n \operatorname{Tr} \{\varepsilon(a_0)\; [F,(a_1)] \ldots [F,a_p]\},\; a_0, \ldots , a_p \in A$$

(non vanishing only for p even). And Connes makes the key observation that, *if* $F^2 = \mathbf{1}$[10], *$\varphi^{(p)}$ is a Hochschild cocycle* :

(8) $$\varphi \circ b = 0$$

moreover cyclic in the sense

(9) $$\varphi \circ \lambda = 0.$$

Here b is the *Hochschild boundary* [11]

(10) $$b(a_0 \otimes a_1 \otimes \ldots \otimes a_{p+1}) = \sum_{i=0}^{p} (-1)^i\, a_0 \otimes \ldots \otimes a_i \;\; a_{i+1} \otimes \ldots \otimes a_{p+1}$$
$$-(-1)^p\, a_p\, a_0 \otimes a_1 \ldots \otimes a_{p+1} \quad ,$$

whilst λ is the *cyclic permuter*

(11) $$\lambda(a_0 \otimes a_1 \otimes \ldots \otimes a_p) = (-1)^p\, ap \otimes a_0 \otimes \ldots a_{p-1} \;.$$

As it is well known, we have $b^2 = 0$. The corresponding cohomology is the *Hochschild cohomology* of A (with values in the dual of A as an A-bimodule). The fact had however remained unnoticed that *the cyclic Hochshild cochains are a subcomplex of the Hochschild complex,* thereby defining *the cyclic cohomology* of $A = C^\infty(M)$.
So far for the classical situation.

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9 2 x 2 matrices with operators entries corresponding to the decomposition $H = H^\circ \oplus H^1$.

10 relatively mild restriction in view cf (5)

11 recall that (p+1) - linear forms are the same as linear forms on $A^{\otimes(p+1)}$

Alain Connes' capital observation is now that (next occurrence of the "non commutative miracle" !), the results (8) and (9) in fact follow from (3), (4), (6) and $F^2 = \mathbf{1}$ **without the need that A be commutative** This suggests to postulate (3) through (5), defining p-*summable Fredholm modules* of arbitrary complex algebras A as graded Hilbert spaces $H = H_0 \oplus H_1$ carrying a graded representation $a \to (a)$ by bounded operators plus an odd bounded operator F with the properties (5) and (6). *This warrants the existence of the character* (7), *which, in the case* $F^2 = \mathbf{1}$, *is again a cyclic Hochschild cocycle in the sense* (8), (9).

The fact that cyclic cochains are again a subcomplex of the Hochschild complex now yields the definition of the cyclic cohomology for arbitrary complex algebras.

At this point, we make two remarks :

(i) It is obvious from their definition that the p-summable Fredholm modules are the non-commutative substitutes for elliptic operators.

(ii) We are in fact moving towards "non commutative analysis" as suggested by the following observation : defining

(12) $$\delta a = i[F, (a)] \quad , a \in A ,$$

yields a derivation $\delta : A \to B(X)$, moreover of vanishing square if $F^2 = 1$. Viewing $Tr(\varepsilon \cdot)$ as a kind of a Leibnitz integration symbol[12] $\int$, the character (7) now looks like a "multiple integral" :

(7a) $$\int (a_0)\ \delta a_1 \ldots \delta a_n .$$

However, we should note that what we have here is a quantum version of *finite dimensional* manifolds : indeed, condition $p > n$ in the classical case a priori forbids the existence of p-summable Fredholm modules for classical infinite dimensional manifolds. We shall in fact need a modification of what precedes (viz. θ-summable Fredholm modules and entire cyclic cohomology [2], see below) in order to define the "infinite dimensional quantum smooth structures". Before proceeding to this, we need however to develop a few further aspects of cyclic cohomology.

First, the above heuristic remark (ii) can be given a precise meaning in the following way. To each complex algebra A we associate its *differential envelope* $\Omega(A)$ [1], defined as [13]

(13) $$\Omega(A) = \mathcal{F}/I ,$$

quotient of the free algebra $\mathcal{F}$ generated by the symbols a, da, $a \in A$, through the ideal I corresponding to the relations

(14) $$\begin{cases} \alpha \cdot a \dot{+} \beta \cdot b \dot{-} (\alpha a + \beta b) = 0 \\ a \cdot b \dot{-} (ab) = 0 \\ \alpha \cdot da \dot{+} \beta \cdot db \dot{-} d(\alpha a + \beta b) = 0 \\ da \cdot b \dot{+} a \cdot db \dot{-} d(ab) = 0 \end{cases}$$

12 Think of quantum statistical mechanics, where Tr replaces the classical $\iint dpdq$

13 see also [27] for a $\mathbb{Z}/2$ graded version

(operations marked with a dot are in $\mathcal{F}$; the two first relations aim at having A a subalgebra of $\Omega(A)$; the two others at having d a derivation) $A \to \Omega(A)$) $\Omega(A)$ is properly defined by (13) as a complex algebra generated by elements a and b, a, b $\in$ A, with the rules (14). One wishes however a constructive picture, which one easily obtains as follows : it is clear that "words" in $\Omega(A)$, consisting of arbitrary products of symbols of the type a_i and da_k , may be reordered using the last relation (14), so as to bring all symbols da_k to the right of the symbols a_i : $\Omega(A)$ is thus seen to be linearly generated by elements of the form

$$(15) \qquad \begin{cases} a_0\, da_1 \dots da_n \\ \\ da_1 \dots da_n \end{cases} , a_0, \dots, a_n \in A,\ n \in \mathbb{N} ,$$

this making it intuitive that one has

$$(16) \qquad \begin{cases} \Omega(A) = \sum\limits_{p=0}^{\infty} \Omega(A)^p \\ \Omega(A)^\circ = A \\ \Omega(A)^p = A^{\otimes p+1} \oplus A^{\otimes p} \qquad , n \geq 1 , \end{cases}$$

(or else, adding a formal unit $\tilde{\mathbb{1}}$

$$(16a) \qquad \begin{cases} \tilde{\Omega}(A) = \mathbb{C}\, \tilde{\mathbb{1}} + \Omega(A) = \sum\limits_{p=0}^{\infty} \tilde{\Omega}(A)^p \\ \tilde{\Omega}(A)^\circ = \tilde{A} = \mathbb{C}\, \tilde{\mathbb{1}} \oplus A \\ \Omega(A)^p = \tilde{A} \otimes A^{\otimes p} \qquad , p \geq 1 \end{cases}$$

$\Omega(A)$ is so far specified as a vector space. In order to define the product it clearly suffices (apart from obvious rules) to state the rule

$$(17) \qquad (\tilde{a}_0\, da_1 \dots da_p) a_{p+1} = \tilde{a}_0\, a_1\, da_2 \dots da_n + \sum_{i=1}^{p} \tilde{a}_0\, da_1 \dots (a_i a_{i+p}) \dots da_{p+1}$$

obtained by repeated application of the derivation rule, last line of (14). It is intuitive (and easy to show [23]) that (17) (together with obvious rules) makes $\Omega(A)$ an associative complex algebra fulfilling (14). If we then define d on $\Omega(A)$ as

$$(18) \qquad d(a_0 + \lambda\, \mathbb{1}) da_1 \dots da_p = da_0\, da_1 \dots da_p,\ a_0, a_1, \dots, a_p \in A$$

we make $\Omega(A)$ a differential algebra, universal in the sense that each homomorphism $\varphi : A \to \Omega_1$(as algebras) into a given differential algebra (Ω_1, δ_1) factors through $\Omega(A)$

(19)

$$\begin{tikzcd} \Omega(A) \arrow[dr, "\bar{\varphi}"] & \\ A \arrow[u, hook] \arrow[r, "\varphi"'] & \Omega_1 \end{tikzcd}$$

with $\bar{\varphi} : \Omega(A) \to \Omega$ a homomorphism of differential algebras. The particular case $\Omega_1 = B(H)$, $\delta_1 = \delta$ as given by (12), and (a) = $\varphi(a)$, $a \in A$, proves the above claim that the character (7), pull back by the homomorphism $\bar{\varphi} : (\Omega(A),d) \to (B(H),\delta)$ of the graded trace[14] $Tr \circ \varepsilon$ of $B(H)$ equipped with the obvious $\mathbb{Z}/2$ grading), is itself a graded trace of $\Omega(A)$. This fact is general : the *cyclic cocycles* φ *of* A, *are, via the correspondence*

(20) $$\varphi(a_0,a_1,\ldots,a_p) = \phi(a_0da,\ da_2 \ldots da_p),\quad a_1,\ \ldots\ ap \in A,$$

one-to-one with the graded traces ϕ *of* $\Omega(A)$[15]

The second feature which we need is the fact, discovered by Alain Connes whilst developing homological algebraic aspects of cyclic cohomology [1], that the above cyclic Hochschild cocycles ("geometrically" interpretable as the graded traces of $\Omega(A)$, as we just saw) are in fact part of a more general set of cocycles, neither Hochschild not cyclic, but vanishing under the coboundary $\varphi \to \varphi \circ \Delta$, with

(21) $$\Delta = b + B\quad ,$$

where b is the previously defined Hochschild coboundary (10) whilst B is given by :

(22) $$B = B_o\, A\ ,$$

with A the "cyclicizer"

(23) $$A = \sum_{k=0}^{p-1} \lambda^k \qquad \text{on } A^{\otimes p}\quad ,$$

and B_0 given by

(24) $$B_0(a_0 \otimes \ldots \otimes \cdot a_p) = \mathbb{1} \otimes a_0 \otimes \ldots \otimes a_p + (-1)^p\ \ a_0 \otimes a_1 \otimes \ldots \otimes \mathbb{1}$$

(the algebra A is now assumed unital with unit $\mathbb{1}$ -distinct from the formal unit $\tilde{\mathbb{1}}$ added above to a general A). One checks the relations

(25) $$b^2 = B^2 = bB + Bb = 0\quad ,$$

entailing that Δ is a boundary :

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14 a graded trace ϕ of a $\mathbb{Z}/2$-graded algebra $\mathcal{A}$ is a linear form of $\mathcal{A}$ vanishing on all graded commutators $[a,b] = ab - (1)^{\partial a \partial b}\, ba$, $a,b \in \mathcal{A}$ of respective grades ∂a and ∂b.

15 This is the amount to which the "alternation" of the classical differential forms persists in the non commutative generalisation : $\Omega(A)$ is not anticommutative, as was the case for the De Rham algebra, but has alternation ***under appropriate forms*** (the cyclic cocycles).

(26) $$\Delta^2 = 0.$$

The general cocycles are within the bicomplex $\mathbf{C}$ with entries

(27) $$C^{p,q} = C^{p-q} \qquad ,p\ ,q \in \mathbb{N},\quad p > q,$$

(C^p the set of (p+1)-linear forms on A), a p-cocycle having a finite number of components on the p^{th} antidiagonal of $\mathbf{C}$. The bicomplex $\mathbf{C}$ yields cyclic cohomology as the cohomology of its associated total complex. Each cohomology class contains one cyclic Hochschild cocycle of the previously considered type, this causing the latter to "carry" cyclic cohomology. The shift S along the diagonal of $\mathbf{C}$ is the *"Connes periodicity operator"*, and cyclic cohomology can be "divided" by S", so as to yield *periodic cyclic cohomology* (of period 2) called by Connes *de Rham cohomology* since reducing to the latter in the classical case $A = C^\infty(M)$. The operator S was originally discovered by Connes through the consideration of the *Chern character* arising as follows : with e a projection in the (stabilized) algebra A, ePe is (by virtue of (6)) a Fredholm operator, moreover (with an appropriate choice of the constants C_{2n}), of index given by

(28) $$\text{Index } ePe = C_{2n}\ \varphi^{2m}(e,e,\ \dots\ ,e).$$

Since this holds for all m with $2m \geq p$, the l.h.s. is independent of m, suggesting the existence of a relation between the φ^{2m} : in fact the latter build a "S hierarchy" : $\varphi^{2m+1} = S\ \varphi^{2m}$, and represent the same class of periodic cohomology.

Assuming now that A is a Banach algebra, we are ready to describe Connes "quantum smooth structures" for the infinite dimensional case. The substitutes of elliptic operators are now the *θ-summable Fredholm modules* specified as follows : as previously, one has a graded Hilbert space $H = H^\circ \oplus H'$ carrying a graded representation $a \to (a) \in B(H)$ of the algebra A ; and there is moreover an (unbounded) selfadjoint operator $D = D^*$ of odd grade, fulfilling $[\Delta,(a)] \in B(H)$, $a \in A$, plus the (high temperature) condition :

(29) $$e^{-\beta D^2} \in L^1(H)\ .$$

To each such θ-summable module, Connes associates a character [2], now a cocycle for *the entire cyclic cohomology*[16] , an enlargement of periodic cyclic cohomology which encompasses the "infinite dimensional case". The corresponding complex has period 2 :

(30) $$C^{even} \xrightarrow{\ \Delta\ } C^{odd} \xrightarrow{\ \Delta\ } C^{even}$$

with cochains[17]

(31) $$C^{even} = \{ (\phi_{2p})_{p\in\mathbb{N}}\ ,\ \phi_{2p} \in C^{2p}\ ;\ \sum_{n\in\mathbb{N}} \frac{(2p)!}{p!}\ ||\phi_{2p}||\ z^p \text{ is entire}\}$$

(32) $$C^{odd} = \{ (\phi_{2p+1})_{p\in\mathbb{N}}\ ,\ \phi_{2p+1} \in C^{2p+1}\ ;\ \sum_{n\in\mathbb{N}} \frac{(2p+1)!}{p!}\ ||\phi_{2p}||\ z^p \text{ is entire}\}$$

16 There is an injection of periodic cyclic cohomology into entire cyclic cohomology

17 The growth conditions of the entire function type motivate the name entire cyclic cohomology.

We have no place to describe Connes' characters of the θ-summable modules [2] (see [28] for an introduction). The latter are *"normalized" entire cyclic cocycles* (a generalization of the previous notion of cyclic Hochschild cocycles, now geometrically interpretable as traces of the *Cuntz differential envelope* q A (cf. [29], [2], [32]). Let us just mention that in order to construct his character, Connes had to resort to taking a formal square root so-to-speak *"enforcing supersymmetry"*, which lead him to conjecture a deep relationship between cyclic cohomology, supersymmetry, and the KMS structure [30]. This is in line with the fact, displayed by Jaffe, Lesniewski and Weitsman [31], that the supersymmetric Wess-Zumino model (recognized by Witten as the Dirac operator of loop space) yields a remarkable θ-summable module (here pertaining to loop space as an infinite dimensional classical manifold).

This example has led Jaffe et al. to propose an alternative interesting version of the character of a θ-summable Fredholm module [3] (however not normalized in Connes' sense, hence without a geometrical interpretation in terms of the Cuntz envelope)[18]. We do not describe the Jaffe et al. character at this point, because it will appear as a special (suggestive) case of the entire cyclic cocycles attached in [3] to graded KMS functionals. Before coming to the latter, we need one more remark : there is a natural generalization (spelled out in [27], [32]) of cyclic (or, for that matter, entire cyclic) cohomology to $\mathbb{Z}/2$ *graded (Banach)* algebras ($A = A^{\circ} \oplus A'$ with $A^i A^j \subset A^{i+j \bmod 2}$). The latter is obtained by inserting, in the definition formulae of b and λ, the sign factors characteristic of the $\mathbb{Z}/2$ graded frame. The new formulae are

(33) $$b\,(a_0 \otimes a_1 \otimes \dots \otimes a_{p+1}) = \sum_{i=1}^{p} (-1)^i\, a_0 \otimes \dots \otimes a_i\, a_{i+1} \otimes \dots \otimes a_{p+1}$$

$$- (-1)^{p+\partial a_{p+1} \sum_{k=0}^{p} \partial a_k}\, a_{p+1} a_0 \otimes a_1 \otimes \dots \otimes ap$$

(34) $$\lambda(a_0 \otimes a_1 \otimes \dots \otimes ap) = (-1)^{p+\partial a_{p+1} \sum_{k=0}^{p} \partial a_k}\, a_p \otimes a_0 \otimes \dots \otimes a_{p-1}$$

and we have as above (21), (22) and (23).

We are now ready for

A. *Definition*. Let $A = A^{\circ} \oplus A'$ be a $\mathbb{Z}/2$ graded C^* algebra, with α_t a continuous one parameter-group of automorphisms. The dynamical system (A, α) is called *supersymmetric* whenever

(i) α preserves the $\mathbb{Z}/2$-grading :

(35) $\alpha_t(A^i) \subset A^i$, $i = 1, 2,$ $t \in R$

(ii) the infinitesimal generator[19] of α :

(36) $$D = \frac{d}{dt}\Big|_{t=0} \alpha_t$$

18 This poses a problem of interpretation. One would wish to understand better the relationship between the Connes and the Jaffe et al character.

19 D is an even derivation of A as a result of (35)

is the square of an odd derivation δ of A :

(37) $\qquad D = \delta^2$

This definition aims at capturing the essence of supersymmetry.

B. *Definition*. With (A,α) a supersymmetric C^*-dynamical system and D,δ as in A, a bounded linear form ϕ of A is *graded β-KMS* , $\beta \in \mathbb{R}$, whenever one has $\varphi \circ \delta = 0$ and

(39) $\qquad \varphi(ba) = (-1)^{\partial a \partial b} \varphi(a\ \alpha_{i\beta}\ (b))$, a, b$\in$A, b analytic for α

Note that the restriction of φ to the bosonic part is β-KMS in the usual sense (1), hence may be a state of A°.

C. *Theorem Let φ be a graded β-KMS linear form of a supersymmetric C^*-dynamical system (A,α), with δ as in* A. *Then, defining, for homogeneous and α-analytic* $a_0, a_1, \dots, a_n \in A$

(40) $\qquad \varphi^{\beta}(a_0, a_1, \dots, a_n) =$

$$\beta^{-\frac{n}{2}} i^n (-1)^{n-1+\partial a_n \sum_{k=0}^{n-1} \partial a_k} \varphi\left(a_0 \int_{I_\beta^n} \alpha_{it_1}(\delta a_1) \dots \alpha_{it_1}(\delta a_1) \dots \alpha_{it_n}(\delta a_n)\, dt \right),$$

where $I_\beta^n = \{ t \in (t_1, \dots t_n)\ ;\ 0 \le t_1 \le \dots \le t_n \le \beta \}$, *yields an entire cyclic cocycle* of A.

The Jaffe et al. character of the θ-summable module (H,D) [3] is the special case obtained from $\varphi^{\beta} = \mathrm{Tr}(\varepsilon e^{-\beta D^2}\ \bullet\ )$ (special case of a type I or II flavour).

We conclude with a few words about our claim that KMS states generalize elliptic operators in the sense of generalizations (i) and (ii) considered in the outset. This is because KMS states appear here as generalizations of (the characters of) θ-summable Fredholm modules, themselves substitutes of elliptic operators. Note that this type of generalization is already expected to occur in the infinite dimensional classical case. Indeed, most elliptic operators on an infinite-dimensional manifold will, unlike the Dirac operator on loop space, be obtained through a "thermodynamic limit" spoiling formal properties.

We conclude in stressing the challenging apparent need of a supersymmetric frame for establishing the basic relationship we find between KMS states and entire cyclic cohomology.

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