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Elementary Acceleration and Multisummability I

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ELEMENTARY ACCELERATION AND MULTISUMMABILITY I [1]

Jean Martinet, Jean-Pierre Ramis

Lorsqu'il suit le bon *rayon* vers la périphérie, le promeneur peut découvrir...

André HARDELLET,"Périphérie".

This paper[2] is extracted from the contents of a forthcoming book by the same authors [MR 3]. Parts ***1*** to ***3*** joined to chapter ***2*** of [MR 2] form a more or less self-contained set: We recall basic definitions about *Borel-summability* (*Borel* [Bo 1], [Bo2]), and its natural generalization *k-summability* (*Leroy* [Le], *Nevanlinna* [Ne] , *Ramis* [Ra 1]). We describe the *"elementary acceleration"* introduced by *Ecalle* [E 4] and different summability operators related to it. If one compares to [E 4], our description is slightly modified in order to fit with our *"geometric"* interpretations [MR 2], [MR 3]. In part ***4***, as an example of application, we give a *"natural", simple* and *general,* definition of ***Stokes multipliers***[3], using a result[4] of *Ramis* [Ra 3] (Cf. also [Ra 2]), and derive a new proof of a theorem of *Ramis* [Ra 4], [Ra 5], about the computation of the *differential Galois group* of a linear differential equation. As a byproduct we get also the description of the ***meromorphic classification*** of ***meromorphic linear differential equations*** on a Riemann surface by the finite dimensional ***linear representations*** of a ***"wild fundamental group"*** (that is a natural generalization of the *Riemann-Hilbert correspondence*). Part ***6*** is *very sketchy*, we describe *"infinitesimal neighborhoods"* of the *analytic geometry* (following an idea of *Deligne* [De 4]), sheaves of *"analytic functions"* on these neighborhoods (***weakly*** analytic and ***wild*** analytic functions); then we are able to give ***a "geometric interpretation" of the notions of acceleration, summability and Stokes phenomena***[5] and various generalizations (the sum of a formal power series being now a *wild* analytic function).

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[1] Part I of this paper contains paragraphs ***1*** to ***4***; paragraphs ***5*** and ***6*** will appear in *Elementary acceleration and multisummability II*. The second author has exposed a part of this paper at 1989 R.C.P. 25 meeting dedicated to *R. Thom*.

[2] A preliminary manuscript version of parts ***1*** to ***4*** of this paper has been distributed during a Luminy Conference (september 1989).

[3] Compare with the program of [Me]. Relations between our description of *Stokes phenomenon* and the *cohomological approach* [Ma 3], [Ma 4], [Si], [De 3], [J], [BJL], [BV], will be explained in ***4***.

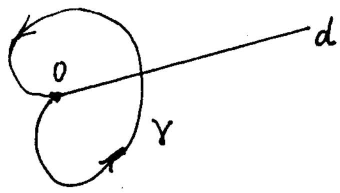
[4] The main steps of the proof of this result, using Gevrey asymptotic expansions technics, are detailed in ***5***.

[5] Partially based upon a ***cohomological version*** of Phragmén-Lindelöf theorem due to Lin [Li] (Cf. also Il'Yashenko's lectures at Luminy Conference).

1. Borel summability, Borel and Laplace transforms.

We denote by B_d the Borel transform in the direction d.

$$B_d f(\xi) = f(\xi) = \frac{1}{2i\pi} \int_{\gamma} f(x)\ (e^{\xi/x} dx/x^2).$$



This formula makes sense with"good" hypothesis on f [MR 2]. We will omit d and write Bf if $B_d f$ is independent of d (up to analytic continuation).

If $\hat{\phi}$ is a convergent power *series* ($\hat{\phi} \in C\{\xi\}$), we will denote by $\phi(\xi) = \mathrm{S}\ \hat{\phi}(\xi)$ its *sum* on a "small disc" centered at zero.

If f is an analytic function in a "small disc" centered at zero, or, more generally, in a "small sector" bisected by the direction d, we will denote by $\bullet_d f$ its *analytic continuation* (if it exists) along d. In the following, when we write $\bullet_d f$, we will *always* suppose that $\bullet_d f$ is defined on a *sector* bisected by d with infinite radius.

Operators S and $\bullet_d$ are clearly *injective homomorphisms of differential algebras* (laws being addition and multiplication, and derivation being $d/d\xi$ or $\xi^2 d/d\xi$) or of *"convolution*[1] *differential algebras"* (laws being addition and convolution, and derivation being multiplication by ξ).

If $\lambda > 0$ and $f(x) = x^\lambda$, we get

$B_d\ f(\xi) = B f(\xi) = \xi^{\lambda-1} / \Gamma(\lambda)$; in particular, for $\lambda = n \in N^*, f(x) = x^n$ $(n \in N)$:

$$B f(\xi) = \xi^{n-1} / \Gamma(n) = \xi^{n-1} / (n-1)!.$$

If we introduce

$\boldsymbol{B}_d\, f = B_d\ f(\xi)\ d\xi$; then for $f(x)=1$, we get as a natural generalization:

$\boldsymbol{B}_d\, f = \delta$ (Dirac distribution).

We can now define a *"formal Borel transform"* $\hat{B}$:

For $\hat{f} \in C[[x]], \hat{f}(x) = \sum_{n \geq 1} a_n x^n$

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[1] The convolution law is defined by $\phi * \psi = \int_0^\xi \phi(t)\psi(\xi - t)dt$ in the analytic case and $\hat{\phi} * \hat{\psi}$ is deduced, in the formal case, from the identities $\frac{\xi^{m-1}}{\Gamma(m)} * \frac{\xi^{n-1}}{\Gamma(n)} = \frac{\xi^{m+n-1}}{\Gamma(m+n)}$.

$\hat{f}(\xi) = \hat{B}\hat{f}(\xi) = \sum_{n\geq 1} a_n \, \xi^{n-1} / (n-1)!$. This definition can be extended, replacing N as a set of indices for the expansion f by a more general semi-group (contained in R^+): $\Lambda^* = \Lambda - \{0\}$,

$$\hat{f}(x) = \sum_{\lambda\in\Lambda^*} a_\lambda x^\lambda \, , \; \hat{B}\hat{f}(\xi) = \sum_{\lambda\in\Lambda^*} a_\lambda \, \xi^{\lambda-1} / \, \Gamma(\lambda)$$

We will also use later formal expansions indexed by $\lambda \in \alpha + N$ $(\alpha \in C)$, and the corresponding *asymptotic expansions* (named *asymptotic expansions at the origin* in the following).

Lemma 1.

We have an isomorphism of differential algebras:

$$\text{Differential algebra } C\{x\} \text{ of convergent power series} \xrightarrow{B} \text{Convolution differential algebra of entire functions of order} \leq 1.$$

Let f be holomorphic with *exponential growth* of order ≤ 1 in a "small" sector bisected by the direction d (or, more generally, infinitely differentiable on d[1], with an *exponential growth* of order ≤ 1) We can define its *Laplace transform along d:*

$$f(x) = L_d \; f(x) \; = \int_d f(\xi) \, (e^{-\xi/x} d\xi)$$

If $f \in C\{x\}$ (resp. f entire of order ≤ 1):

$LBf = f$ and $BLf = f$.

With "good hypothesis":

$L_d B_d = id$ *and* $B_d L_d = id$ [MR 2].

Example: For $f(\xi) = \xi^\mu$ $(\mu > -1)$, *we have* $L \; f(x) = \Gamma(\mu+1) \, \xi^{\mu+1}$.

Let $\hat{f}$ be a formal power series, of *Gevrey order*[2] *1* $(\hat{f} \in C[[x]]_1)$.Then $\hat{B}\hat{f} = \hat{f} \in C\{\xi\}$. If $f = \mathrm{S}\hat{f}$ can be analytically extended along some direction d in a fonction $f = \bullet_d \, \mathrm{S}\hat{f}$ which is analytic with exponential growth of order ≤ 1 on a small sector bisected by d, we can define:

$f_d(x) = L_d \bullet_d \mathrm{S}\hat{f} = L_d \bullet_d \mathrm{S} \, \hat{B}\hat{f}$. By definition f_d is the *"Borel sum"* of $\hat{f}$ in the *direction d* ($\hat{f}$ is *Borel-summable* in the *direction d*).

Clearly if $\hat{f} \in C\{x\}$, $\mathrm{S}\hat{B} = B$ *and* $f_d(x) = \mathrm{S}\hat{f}(x)$. So $\mathrm{S}_d = L_d \bullet_d \mathrm{S} \, \hat{B}$ *extends* the operator S.

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[1] A function "infinitely differentiable on d " is infinitely differentiable on the right at zero, by convention.

[2] For definitions and notations see [MR 1].

Lemma 2.

The operator S_d *is an injective morphism of differential algebras:*

$$\text{Differential algebra of Borel summable series in the direction } d. \xrightarrow{S_d} \text{Differential algebra of germs of holomorphic functions on sectors bisected by } d.$$

So Borel-summability is *"natural"* (i.e. "Galois").

Let $R > 0$ and d a direction.

Let $D_{R;d} = \{t \in C \,/\, |Arg\, t - Arg\, d| < \frac{\pi}{2} \text{ and } Re\,(e^{i\,Arg\,d}t^{-1}) > 1/R\}$.

We denote γ_R the boundary of $D_{R;d}$ oriented in the positive sense.

Let $$B_d f(\xi) = f(\xi) = \frac{1}{2i\pi} \int_{\gamma_R} f(x)\,(e^{\xi/x}dx/x^2),$$

if $f(x) = o(x^2)$, and

$B_d f(\xi) = 1$, if $f(x) = x$. We get $B_d f$ for $f(x) = o(x)$.

Later we will need the "well known"

Lemma 3.

The operator

$$\begin{array}{c} \text{Convolution differential algebra} \\ \text{of functions infinitely differentiable} \\ \text{on } d \text{, with exponential growth} \\ \text{of order } \leq 1 \text{ at infinity.} \end{array} \xrightarrow{L} \begin{array}{c} \text{Differential algebra of functions} \\ \text{analytic on open discs } D_{R;d} \\ (R > 0 \text{ arbitrary}), \text{ with} \\ \text{asymptotic expansion [1] (without} \\ \text{constant term) at zero.} \end{array}$$

is an ***isomorphism*** *of differential algebras.*

Let f be analytic on the open Borel-disc $D_{R;d}$, with an asymptotic expansion (without constant term at zero). Then, using Fubini's theorem, and the formula

$L(e^{-t})(\zeta) = \dfrac{\zeta}{\zeta - 1}$, we get easily $L\,B f = f$ (see [Bo 2]).

Let f be infinitely differentiable on d, with an exponential growth of order ≤ 1. If $Lf = 0$, then $f = 0$ (using inversion of *Fourier transform*).

Now, from $L\,(B\,L f) = L\,B\,(L f) = L f$, we deduce $B\,L f = f$. That ends the proof of *lemma 3*.

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[1] Uniform on closed subdiscs $\bar{D}_{R';d}$ ($R' > R$).

2. *k-summability, k-Borel and k-Laplace transforms.*

Using B_d, L_d, $\bullet_d$, S and *ramification operators* ρ_k $(k>0)$ it is easy to build *new* operators $B_{k;d}$ and $L_{k;d}$ (and the formal operator $\hat{B}_k$ corresponding to $B_{k;d}$):

We will use the notation $(k>0)$: $\rho_k f(x) = f(x^{1/k})$ (x is varying on the Riemann surface of Logarithm); $\rho_{1/k} = \rho_k^{-1}$.

If d^k corresponds to d by the ramification ρ_k, we will set:

$$B_{k;d} = \rho_k^{-1} B_{d^k} \rho_k \text{ and}$$
$$L_{k;d} = \rho_k^{-1} L_{d^k} \rho_k .$$

We have (in general we will simplify our notations: $f_k = f$, $\xi_k = \xi$):

$$B_{k;d} f(\xi_k) = f_k(\xi_k) = \frac{1}{2i\pi} \int_{\gamma_k} f(x)\, (k\, e^{\xi_k{}^k/x^k} dx/x^{k+1})$$

$$L_{k;d} f_k(\xi) = f(\xi) = \int_l f_k(\xi_k)\, (k\, e^{-\xi_k{}^k/x^k}\, \xi_k{}^{k-1} d\xi_k)$$

The operator $L_{k;d}$ can be applied to functions holomorphic with exponential growth of order $\leq k$ on a small sector bisected by d, and an asymptotic expansion at the origin (indexed by $-k+N$) . These functions form a *k-convolution* differential algebra:

the k-convolution is defined by:

$$f_k *_k g_k = \rho_k^{-1} ((\rho_k\ f_k) * (\rho_k\ g_k)) = \rho_k^{-1} (f * g) .$$

Operations are: $+$, $*_k$, and derivation $\partial_k = B_k\, (x^2\, d/dx) L_k$ (∂_k will be explicitely described later; ∂_1 is multiplication by x).

Lemma 4.

We have an isomorphism of differential algebras:

$$\begin{array}{ccc} \textit{Differential algebra } \mathbf{C}\{x\} \textit{ of convergent} & \xrightarrow{B_k} & \textit{k-convolution differential algebra of} \\ \textit{power series .} & & \textit{entire functions of order} \leq k \end{array}$$

We will use the following notations:

$C[[x]]_{1/k}$ is the differential algebra of formal power series of Gevrey *order* $1/k$ (Gevrey *level* k)[1];

$C\{x\}_{1/k;d}$ is the differential algebra of formal power series k-summable in the direction d (definition is given just below);

[1] Notations of [MR 2]. (Be careful, these notations differ from those of [Ra 1], [Ra 2], [Ra 7].)

$C\{x\}_{1/k}$ is the differential algebra of k-summable series (that is of formal power series k-summable in *every direction but* perhaps a *finite* number).

Let $\hat{f} \in C[[x]]_{1/k}$. Then $\hat{f}_k = \hat{B}_k \hat{f} \in C\{\xi_k\}$. If $f_k = S\hat{f}_k$ can be analytically extended along some direction d in a function $\bullet_d f_k = \bullet_d S\hat{f}_k$ analytic with exponential growth of order $\leq k$ on a small sector bisected by d, we can define:

$f_{k;d}(x) = L_{k;d} \bullet_d S\hat{f}_k = L_{k;d} \bullet_d S\hat{B}_k\hat{f}$. By definition $f_{k;d}$ is the "k-sum" of $\hat{f}$ in the direction d ($\hat{f}$ is k-summable in the direction d).

It is clear that $S_{k;d} = L_{k;d} \bullet_d S\hat{B}_k$ extends the operator S (defined for $\hat{f} \in C\{x\}$).

Lemma 5.

The operator $S_{k;d}$ *is an injective morphism of differential algebras:*

$$\begin{array}{ccc} \textit{Differential algebra of} & \xrightarrow{S_{k;d}} & \textit{Differential algebra of germs of} \\ \textit{k-summable series in the direction d.} & & \textit{holomorphic functions on sectors bisected by d.} \end{array}$$

So k-summability is *"natural"* (i.e. "Galois").

We have built a *one parameter family* ($k \in \boldsymbol{R}$, $k > 0$) of *summation processes*. We will now compare these processes for different values of the parameter $k > 0$: if a formal power series is summable by *two processes* then *the two sums are equal*, but this is quite exceptional because k_1-summability and k_2-summability for $k_1 \neq k_2$ requires in some sense *very different* conditions. More precisely:

Proposition 1.

Let $k, k' > 0$ with $k < k'$ and $\hat{f} \in C[[x]]$ k-summable and k'-summable in the direction d. Then :

(i) $S_{k;d}\hat{f} = S_{k';d}\hat{f}$;

(ii) The power series $\hat{f}$ is k-summable in every direction d' with $\arg d' \in \,]\arg d - \pi/k + \pi/k' , \arg d + \pi/k - \pi/k'[$ and the sums $S_{k;d'}\hat{f}$ glue together by analytic continuation;

(iii) The power series $\hat{f}$ is k''-summable in every direction d'' with $\arg d'' \in \,]\arg d - \pi/k + \pi/k'' , \arg d + \pi/k - \pi/k''[$, for $k < k'' < k'$

Moreover $S_{k'';d''}\hat{f} = S_{k;d''}\hat{f}$.

Proposition 2.

Let $k, k' > 0$ with $k < k'$ and $\hat{f} \in C[[x]]_{1/k'}$. If $\hat{f}$ is k-summable, then $\hat{f}$ is a convergent power series (i.e. $C[[x]]_{1/k'} \cap C\{x\}_{1/k} = C\{x\}$).

This result, announced in [Ra 2], is proved in [Ra 5] (for a particular case and exemple, see [RS 1]).

From such a result it is easy to understand that summation operators $S_{k;d}$ (with d and $k > 0$), if very useful, are *not sufficient* if one wants to deal with quite simple, situations as "*non* generic" linear algebraic differential equations:

A formal power series solution of a *"generic"* linear algebraic equation is *k-summable* for some $k > 0$ [Ra 2], [MR 2], [MR 3]. Let now $\hat{f}_1, \hat{f}_2 \in \mathbf{C}[[x]]$, be *divergent*, with $\hat{f}_1$ *k_1-summable*, $\hat{f}_2$ *k_2-summable* ($k_1 \neq k_2$). Then $\hat{f} = \hat{f}_1 + \hat{f}_2$ is *divergent* (proposition 2) and there exists ***no*** $k > 0$ such that $\hat{f}$ is *k-summable* (proposition 1 and 2). If we suppose that there exists $D_1, D_2 \in \mathbf{C}[x][d/dx]$ *with* $D_1 \hat{f}_1 = 0, D_2 \hat{f}_2 = 0,$ then there exists
$D \in \mathbf{C}[x][d/dx]$ such that $D\hat{f} = 0$ (for an explicit exemple see [RS 1]).

Any formal power series solution of ***any*** analytic linear differential equation ***can be summed*** using a *"blend"* of a *finite set* of processes of k-summability (cf. ***4***, ***6***, infra). The corresponding values for k are *computable* using a Newton polygon [Ra 1], [Ra 7]. We get this way a *process of summability* (replacing each formal power series in the blend by its *k-sum*). This method gives an *injective morphism of differential algebras* but is *purely theoretical* (i.e.*not* explicit). This motivates the introduction of a more general tool, that is *multisummability.* Multisummability (due to *Ecalle*[1]) is *effective* and a "blend" of k-summable power series is multisummable. Here we have slightly modified *Ecalle*'s presentation in order to be as near as possible of our geometric description of multisummability (cf. ***6***, infra).

[1] It is a particular case of his concept of *"accelerosummability"*.

3. Acceleration and multisummability.

We will introduce here only a very elementary acceleration (for a more general theory cf. *Ecalle* [E 4]). It is sufficient for our applications (and easy to generalize along the same lines [MR 3]). Following *Ecalle, Acceleration operators* are *first* defined using *Laplace, Borel and ramification* operators; afterwards we get an equivalent definition using an *integral formula*. The *important fact* is that this integral formula lead to a ***natural extension of the domain*** of the corresponding operator.

Let $\alpha \geq 1$. Formally the operator $\wp_\alpha$ of *α-acceleration* is the *conjugate* of the operator ρ_α of *ramification* by the *Laplace transform:*

$$\wp_\alpha = L^{-1}\rho_\alpha L = B\,\rho_\alpha L$$

The operator ρ_α is an *isomorphism of differential algebras*, so the operator $\wp_\alpha$ is *an isomorphism of convolution differential algebras*. More precisely:

$$\wp_\alpha = L^{-1}_{d^\alpha}\,\rho_\alpha L_d\,,\ \text{and:}$$

Convolution differential algebra of analytic functions on sectors bisected by d with exponential growth of order ≤ 1 at infinity and asymptotic expansion at zero.	$\xrightarrow{\ \wp_\alpha\ }$	Convolution differential algebra of analytic functions on sectors with opening $> \pi(\alpha - 1)$, bisected by d^α with exponential growth of order ≤ 1 at infinity and "asymptotic expansion" at zero.[1]

is an *isomorphism.*

As ρ_α the operator $\wp_\alpha$ moves the direction d. It is useful to introduce operators of *"normalized acceleration"* not moving d:

$$A_\alpha = \rho_{1/\alpha}\wp_\alpha = \rho_\alpha^{-1}L^{-1}\rho_\alpha L = (L\rho_\alpha)^{-1}\rho_\alpha L = B_\alpha L\,.$$

So A_α is the *commutator* of $B = L^{-1}$ and $\rho_{1/\alpha} = \rho_\alpha^{-1}$.

The operator A_α gives an *isomorphism* of "convolution" *differential algebras:*

Convolution differential algebra of analytic functions, on sectors bisected by d, with exponential growth of order ≤ 1 at infinity and asymptotic expansion at zero.	$\xrightarrow{\ A_\alpha\ }$	α-convolution differential algebra of analytic functions, on sectors, with opening $> \pi/\beta = \pi\,\dfrac{\alpha - 1}{\alpha}$, bisected by d, with exponential growth of order $\leq \alpha$ at infinity and asymptotic expansion at zero.

For the proof ot this statement see below the more general case of $A_{k',k}$.

If necessary we will denote more precisely the operator A_α by $A_{\alpha;d}$.

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[1] This asymptotic expansion is in powers of $x^{1/\alpha}$.

The operator A_α is clearly related to *level 1*. We need now to introduce similar operators for arbitrary levels $k > 0$ Let $k' > k, \alpha = k'/k$, we will denote:

$$A_{k',k} = \rho_{1/k} A_\alpha \rho_k = (\rho_k)^{-1} (\rho_{k'/k})^{-1} L^{-1} \rho_{k'/k} L \rho_k$$
$$A_{k',k} = (\rho_{k'})^{-1} L^{-1} \rho_{k'/k} L \rho_k = (\rho_{k'})^{-1} L^{-1} \rho_{k'} (\rho_k)^{-1} L \rho_k$$

The operator $A_{k',k}$ gives an *isomorphism* of "convolution" *differential algebras*

$$\begin{array}{ccc} \begin{array}{l} k\text{-convolution differential algebra of} \\ \text{analytic functions, on sectors bisected} \\ \text{by } d, \text{ with exponential} \\ \text{growth of order } \le k \text{ at infinity and} \\ \text{asymptotic expansion at zero.} \end{array} & \xrightarrow{A_{k',k}} & \begin{array}{l} k'\text{-convolution differential algebra of} \\ \text{analytic functions, on sectors with} \\ \text{opening} > \pi/\kappa = \pi \dfrac{k'-k}{kk'}, \\ \text{bisected by } d, \text{ with exponential} \\ \text{growth of order } \le k' \text{ at infinity and} \\ \text{asymptotic expansion at zero.} \end{array} \end{array}$$

If necessary we will denote more precisely the operator $A_{k',k}$ by $A_{k',k;d}$.

We have:

$$A_{k',k} (f *_k g) = \rho_{k'}^{-1} L^{-1} \rho_{k'/k} L \rho_k \rho_k^{-1} ((\rho_k f) * (\rho_k g))$$
$$A_{k',k} (f *_k g) = \rho_{k'}^{-1} L^{-1} \rho_{k'/k} L ((\rho_k f) * (\rho_k g))$$
$$A_{k',k} (f *_k g) = \rho_{k'}^{-1} L^{-1} \rho_{k'/k} ((L \rho_k f) (L \rho_k g))$$
$$A_{k',k} (f *_k g) = \rho_{k'}^{-1} (L^{-1} (\rho_{k'/k} L \rho_k f) * (L^{-1} (\rho_{k'/k} L \rho_k g))$$
$$A_{k',k} (f *_k g) = A_{k',k} f *_{k'} A_{k',k} g .$$

To prove that $A_{k',k}$ is an *isomorphism* it suffices to remark that L_d is an *isomorphism* between the convolution differential algebra of analytic functions on sectors bissected by d with exponential growth of order ≤ 1 at infinity and asymptotic expansion at zero, and the differential algebra of analytic functions on sectors with opening $> \pi$ bisected by d and with asymptotic expansion (having no constant term) at zero.

It is natural to set:

$$A_{\infty,k} = L_k$$
$$A_{\infty,1} = L .$$

We have $A_{k,1} = A_k$ and $A_{k,k} = id.$

Let $k'' > k' > k > 0$. When the formula makes sense, we get:

$$A_{k'',k'} A_{k',k} = A_{k'',k} .$$

We will later use the above formula to *extend* the operator $A_{k'',k}$:

The first step is to extend the domain of the operator $A_{k',k}$ and the second to replace $A_{k',k}$ in the formula by $\bullet_d A_{k',k;d}$: $A_{k'',k';d} \bullet_d A_{k',k;d} = A_{k'',k',k;d}$ *(definition).*

More generally, let $k_1 > k_2 > ... > k_r > 0$. When the formula makes sense, we get:

$$A_{k_1,k_2} A_{k_2,k_3} \cdots A_{k_{r-1},k_r} = A_{k_1,k_r} .$$

Using this formula, we will later *extend* the operator A_{k_1,k_r}, using extensions of the operators

$$A_{k_i,k_{i+1};d} \ (i=1,...,r-1) \text{ and}$$

$$A_{k_1,k_2;d} \bullet_d A_{k_2,k_3;d} \cdots \bullet_d A_{k_{r-1},k_r d} = A_{k_1,k_2,...,k_r;d} \,.\ (definition).$$

Let $k' > k$, when the formula make sense we get:

$$L_{k'} A_{k',k} = L_k \text{ (or } A_{\infty,k'} A_{k',k} = A_{\infty,k}).$$

So we can *extend* the operator L_k using $L_{k'} \bullet_d A_{k',k}$. Then

$$id = L_k B_k = L_{k'} A_{k',k} B_k$$

$$\mathrm{S} = L_{k'} A_{k',k} \ \mathrm{S} \hat{B}_k \text{, and, more generally, for } k_1 > k_2 > ... > k_r :$$

$$\mathrm{S} = L_{k_1} A_{k_1,k_2} \cdots A_{k_{r-1},k_r} \mathrm{S} \hat{B}_{k_r} \,.$$

Then it is natural to *extend* the domain $C\{x\}$ of the *summation* operator S, using the new summation operator (along the direction d):

$$\mathrm{S}_{k_1,k_2,...,k_r;d} = L_{k_1;d} \bullet_d A_{k_1,k_2;d} \cdots \bullet_d A_{k_{r-1},k_r;d} \bullet_d \mathrm{S} \hat{B}_{k_r}$$

(in this formula we have written $A_{k_i,k_{i+1};d}$ for an *extension* of $A_{k_i,k_{i+1};d}$ that we will define precisely below).

The domain of definition of the operator $A_{k',k;d}$ is

{analytic functions on sectors bisected by d with exponential growth of order $\leq k$ at infinity and asymptotic expansion at zero}.

We will now see that there exists a *natural extension* of this operator to the *larger* domain

{analytic functions on sectors bisected by d with exponential growth of order $\leq \kappa = \dfrac{kk'}{k'-k}$ at infinity and asymptotic expansion at zero};

$$1/k' + 1/\kappa = 1/k \,;\ \kappa = k \frac{k'}{k'-k} > k \,.$$

It is clearly sufficient to understand how to extend the operator $A_{\alpha;d}$ $(\alpha > 1)$ defined on the domain

{analytic functions on sectors bisected by d with exponential growth of order ≤ 1 at infinity and asymptotic expansion at zero}

to the domain

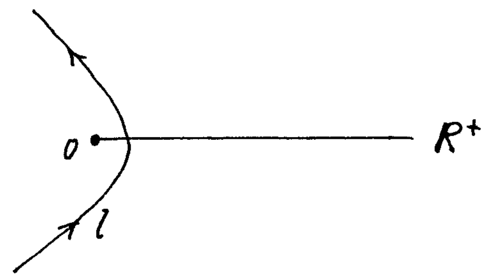
{analytic functions on sectors bisected by d with exponential growth of order $\leq \beta = \dfrac{\alpha}{\alpha - 1}$ at infinity and asymptotic expansion at zero},

$$1/\alpha + 1/\beta = 1.$$

This is done using an *integral formula* for $A_{\alpha;d}$ discovered by *Ecalle* [E 4]:

We introduce a family of *"special functions"* C_α $(0 < \alpha < 1)$, the *"accelerating functions"*:

$C_\alpha(t) = \int_l e^{u-tu^{1/\alpha}} du$; the path l being an Hankel contour:



It is easy to see that C_α is an *entire function* and to compute its analytic expansion at the origin:

$$C_\alpha = 2i \sum_{n\geq 0} \sin \frac{n\pi}{\beta} \frac{\Gamma(1+n/\alpha)}{\Gamma(1+n)} t^n ;$$

with $1/\alpha + 1/\beta = 1$.

Example:

$\alpha = \beta = 2$; then $C_2(t) = i\sqrt{\pi}\, t\, e^{-t^2/4}$.

Functions C_α are *resurgent* at ∞ [E 4], [Ma], [C]. If $\alpha \in Q$ these functions are related to *Mejer G-functions* and *solutions of linear differential equations* (cf. below *"Formulae about accelerating functions"*).

Lemma 6.([E 4], [MR 3][1].)

Let $\beta > 0$, *and* $\alpha = \dfrac{\beta}{\beta - 1}$. *Let* $0 < \theta < \dfrac{\pi}{\beta}$.

Let $V_\theta = \{t \in C \,/\, |Arg\, t| < \frac{\theta}{2}\}$. *Then (on* V_θ*):*

$$|C_\alpha(t)| \leq \frac{K_\alpha}{\sqrt{\cos \beta\theta}} (t^{\beta/2}\, e^{-(t/c_\alpha)^\beta}); \text{ with } K_\alpha > 0 \text{ and}$$

$$c_\alpha = \beta(\alpha-1)^{1/\alpha}.$$

Proposition 3

Let $\alpha > 1$. *Let* $A_{\alpha;d} = (L_{d^\alpha}\rho_\alpha)^{-1} \rho_\alpha L_d$ *and* ϕ *an analytic function on a sector bisected by* d *and with an asymptotic expansion at zero (or, more generally an infinitely differentiable function on* d *with an exponential growth of order* ≤ 1 *at infinity). Then*

$$A_{\alpha;d}\, \phi(x) = x^{-\alpha} \int_d C_\alpha(t/x)\, \phi(t)\, dt .$$

[1] More precisely, using saddlepoint method, it is possible to get an asymptotic expansion of the function C_a on the sector V_θ (and even in $|Arg\, t| < \pi/2$), cf. [HL], page 45, [Bak], page 84, [MR 3].

Definition 1.

Let $\alpha > 1$ and ϕ an infinitely differentiable function[1] *on a direction d. If the integral $\int_d C_\alpha(t/x)\,\phi(t)\,dt$ exists, we will say that ϕ is α-accelerable in the direction d.*

The operator $A_{\alpha;d} = (L_{d^\alpha}\rho_\alpha)^{-1}\rho_\alpha L_d$ is defined on the domain

{analytic functions on sectors bisected by d with exponential growth of order ≤ 1 at infinity and an asymptotic expansion at the origin},

but we have $\beta > 1$ and the operator $\phi \longrightarrow \int_d C_\alpha(t/x)\,\phi(t)\,dt$ is defined on the ***larger*** domain

{analytic functions on sectors bisected by d with exponential growth of order $\leq \beta$ at infinity and an asymptotic expansion at the origin},

(More generally a function infinitely differentiable on d with exponential growth of order $\leq \beta$ at infinity is *α- accelerable.*)

So, *proposition 3* gives the searched extension for the operator $A_{\alpha;d}$ (we will also denote this extension by $A_{\alpha;d}$).

Now using

$$A_{k',k;d}\,\psi(x) = x^{-k'}\int_l \psi(t)\,C_\alpha(t^k/x^k)kt^{k-1}dt,$$

for ψ analytic on a sector bisected by d with an exponential growth of order ≤ 1 at infinity, it is possible to extend the operator $A_{k',k;d}$ to the *larger* domain

{analytic functions on sectors bisected by d with exponential growth of order $\leq \kappa = \dfrac{kk'}{k'-k}$ at infinity and an asymptotic expansion at the origin}.

We can now define the notion of *$(k_1,k_2,\dots,k_r)$-summability* in a direction d and the corresponding summability operator $S_{k_1,k_2,\dots,k_r;d}$ (in the following definition, operators $A_{k_i,k_{i+1};d}$ must be interpreted in the *extended* sense, that is as *integral operators*).

Definition 2.

Let $k_1 > k_2 > \dots > k_r > 0$ and a direction d. A formal power series $\hat{f} \in \mathbf{C}[[x]]$ is $(k_1,k_2,\dots,k_r)$-summable in the direction d if the following conditions are satisfied:

(0) $\hat{f} \in \mathbf{C}[[x]]_{1/k_r}$.

[1] A function infinitely differentiable on d is infinitely differentiable on the right at zero.

(1) $S\hat{B}_{k_r}\hat{f}$ *can be* ***analytically extended*** *along* d *to a function* $\bullet_d S\hat{B}_{k_r}\hat{f}$ *analytic on a sector bisected by* d *with* ***exponential growth of order*** $\leq \dfrac{k_{r-1}k_r}{k_{r-1}-k_r}$.

(2) $A_{k_{r-1},k_r;d} \bullet_d S\hat{B}_{k_r}\hat{f}$ *can be* ***analytically extended*** *along* d *to a function* $\bullet_d A_{k_{r-1},k_r;d} \bullet_d S\hat{B}_{k_r}\hat{f}$ *with* ***exponential growth of order*** $\leq \dfrac{k_{r-2}k_{r-1}}{k_{r-2}-k_{r-1}}$.

....

(i) $A_{k_{r-i+1},k_{r-i+2};d} \dots \bullet_d A_{k_{r-1},k_r;d} \bullet_d S\hat{B}_{k_r}\hat{f}$ *can be* ***analytically extended*** *along* d *to a function*
$\bullet_d A_{k_{r-i+1},k_{r-i+2};d} \dots \bullet_d A_{k_{r-1},k_r;d} \bullet_d S\hat{B}_{k_r}\hat{f}$ *with* ***exponential growth of order*** $\leq \dfrac{k_{r-i}k_{r-i+1}}{k_{r-i}-k_{r-i+1}}$.

....

(r) $A_{k_1,k_2;d} \dots \bullet_d A_{k_{r-1},k_r;d} \bullet_d S\hat{B}_{k_r}\hat{f}$ *can be* ***analytically extended*** *along* d *to a function*
$\bullet_d A_{k_1,k_2;d} \dots \bullet_d A_{k_{r-1},k_r;d} \bullet_d S\hat{B}_{k_r}\hat{f}$ *with* ***exponential growth of order*** $\leq k_1$.

If a formal power series $\hat{f} \in \mathbf{C}[[x]]$ is *$(k_1,k_2,\dots,k_r)$-summable* in the direction d, then:
$L_{k_1;d} \bullet_d A_{k_1,k_2;d} \dots \bullet_d A_{k_{r-1},k_r;d} \bullet_d S\hat{B}_{k_r}\hat{f}$ is defined and analytic in a sector bisected by d

We will set

$$S_{k_1,k_2,\dots,k_r;d} = L_{k_1;d} \bullet_d A_{k_1,k_2;d} \dots \bullet_d A_{k_{r-1},k_r;d} \bullet_d S\hat{B}_{k_r};$$

$S_{k_1,k_2,\dots,k_r;d}\hat{f}$ is the *$(k_1,k_2,\dots,k_r)$-sum* of $\hat{f}$ in the direction d.

If $\hat{f} \in \mathbf{C}[[x]]$ is *$(k_1,k_2,\dots,k_r)$-summable* in the direction d, we will write it

$$\hat{f} \in \mathbf{C}\{x\}_{1/k_1,1/k_2,\dots,1/k_r;d}.$$

If $\hat{f} \in \mathbf{C}[[x]]$ is *$(k_1,k_2,\dots,k_r)$-summable* in all directions, but perhaps a finite number,we will denote it by

$\hat{f} \in \mathbf{C}\{x\}_{1/k_1,1/k_2,\dots,1/k_r}$, and say that $\hat{f}$ is *$(k_1,k_2,\dots,k_r)$-summable*.

Lemma 7.

Let $k_1, k_2,\dots, k_r > 0$ and d a given direction Then

(i) $\mathbf{C}\{x\}_{1/k_1,1/k_2,\dots,1/k_r;d}$ *and* $\mathbf{C}\{x\}_{1/k_1,1/k_2,\dots,1/k_r}$ *are differential subalgebras of* $\mathbf{C}[[x]]$;

(ii) The subalgebra of $\mathbf{C}[[x]]$ *generated by the differential algebras* $\mathbf{C}\{x\}_{1/k_1;d}, \mathbf{C}\{x\}_{1/k_2;d},\dots, \mathbf{C}\{x\}_{1/k_r;d}$, *is a differential subalgebra of*

$C\{x\}_{1/k_1,1/k_2,...,1/k_r;d}$. *Moreover if*

$$\hat{f} = \sum_{i\in I} \hat{f}_{i,1}..\hat{f}_{i,r}\text{ , with } I \text{ finite and } \hat{f}_{i,j} \in C[[x]]_{1/k_j;d} \quad (i \in I, \text{ and}$$

$j=1,...,r$), then

$$S_{k_1,k_2,...,k_r;d}\ \hat{f} = \sum_{i\in I} S_{k_1;d}\ \hat{f}_{i,1} \ldots S_{k_r;d}\ \hat{f}_{i,r}\ ;$$ *in particular the analytic function* $\sum_{i\in I} S_{k_1;d}\ \hat{f}_{i,1} \ldots S_{k_r;d}\ \hat{f}_{i,r}$ *is* ***independant*** *of the "decomposition"* $\sum_{i\in I} \hat{f}_{i,1}..\hat{f}_{i,r}$ *of the formal power series* $\hat{f}$.[1]

Proposition 4.

Let $k' > k > 0$. The operator $A_{k',k}$, interpreted in the extended sense (that is as an integral operator) gives an ***injective morphism*** *of "convolution" differential algebras:*

$$\begin{array}{ccc}
\textit{k-convolution differential algebra of} & \xrightarrow{A_{k',k}} & \textit{k'-convolution differential algebra of} \\
\textit{analytic functions, on sectors bisected} & & \textit{analytic functions, on sectors with} \\
\textit{by } d \textit{ with exponential growth of} & & \textit{opening} > \pi/\kappa = \pi\dfrac{k'-k}{kk'} \\
\textit{order } \le \kappa = \dfrac{kk'}{k'-k} \textit{ at infinity,} & & \textit{and arbitrary radius bisected by } d, \\
\textit{and asymptotic expansion at zero.} & & \textit{and asymptotic expansion at zero.}
\end{array}$$

Let f and g be infinitely differentiable (as functions of a real variable) on d, with complex values. If f and g have a growth of order $\le k$ (in particular if f and g have a compact support), we have

$$A_{k',k}\ (f *_k g) = \rho_{k'}^{-1}L^{-1}\ \rho_{k'/k} L\ ((\rho_k\, f) * (\rho_k\, g))$$
$$A_{k',k}\ (f *_k g) = \rho_{k'}^{-1}\ L^{-1}\ \rho_{k'/k}\ ((L\ \rho_k\ f)\ (L\ \rho_k\ g))$$
$$A_{k',k}\ (f *_k g) = A_{k',k} f\ *_{k'} A_{k',k}\ g\ .$$

We get the *same* formula when f and g *only* have growth $\le \kappa$ by a *density argument.* So $A_{k',k}$ is a morphism of "convolution differential algebras".

The proof of *injectivity* is a little more subtle. We will need a little bit of *Ecalle's "deceleration theory"* [E 4]:

We have (definition) $A_\alpha^{-1} = D_\alpha = (\rho_\alpha L)^{-1} L\, \rho_\alpha = L^{-1}\, \rho_\alpha^{-1} L\, \rho_\alpha$ *and* $A_{k',k}^{-1} = D_{k',k} = \rho_k^{-1} L^{-1}\ \rho_{k/k'} L\ \rho_{k'}$ (formally $D_{k',k} = A_{k,k'}$).

[1] This was proved in [Ra 5] using a different method, answering a question of [Ra 2].

There exist *integral formulae* for the operators of *"normalized deceleration"* $\boldsymbol{D}_{\alpha}, \boldsymbol{D}_{k',k}$. To get them we need a new family of "special functions" C^{α} $(\alpha > 1)$, the *"decelerating functions"*:

$$C^{\alpha}(t) = \int_{R^+} e^{-u+tu^{1/\alpha}} du$$

It is easy to see that C^{α} *is an entire function and to compute its analytic expansion at zero:*

$$\sum_{n\geq 0} \frac{\Gamma(1+n/\alpha)}{\Gamma(1+n)} t^n.$$

Example:

$\alpha = \beta = 2$; then $C^2(t) = 1 + \frac{t}{2} e^{t^2/4} \int_{-t}^{+\infty} e^{-u^2/4} du$. *This function is related to "error functions"*[1]: $Erfc(\sigma) = \frac{2}{\sqrt{\pi}} \int_{\sigma}^{+\infty} e^{-v^2} dv = 1 - Erf(\sigma)$.

Functions C^{α} are *resurgent* at ∞ [E 4],[Ma 8], [C]. If $\alpha \in \boldsymbol{Q}$ these functions are related to *Mejer G-functions* and *solutions of linear differential equations (Cf. below "Formulae about decelerating functions")*.

Ecalle's functions C^{α} are particular cases[2] of *Faxén's* integrals:

$$Fi(\mu,\nu;t) = \int_{R^+} e^{-u+tu^{\lambda}} u^{\mu-1} du \quad \text{(see [Ol], [Fa], [BHL])}$$

$$Fi(\alpha^{-1},1;t) = C^{\alpha}(t).$$

There is in fact a very interesting family of functions:

$$F_{P;\pm}(\alpha;\beta;y) = \int_{\gamma_{\pm}} e^{P(v^{\alpha}) \pm vy} v^{\beta} dv; \quad \text{with } \alpha \in \boldsymbol{R}, \beta \in \boldsymbol{C}, P \in \boldsymbol{C}[w],$$

and $\gamma_{\pm}$ a convenient path.

There are many occurences of particular cases of these functions in the litterature; the main sources are *arithmetic* (in connection with exponential sums; cf. the *Hardy - Littlewood's* paper on Waring's problem [HL][3], and more recently works of *N. Katz* [Ka 4], *Deligne*,...), *physics* (*Airy*, *Kelvin*, *Brillouin*[4],...), *analysis* (study of accelerating and decelerating functions, study of Laplace transform: cf.[Ma 5]), and *probabilities* (up to variable and function rescalings, *stable densities* are real parts of

[1] The function C^3 is simply related to Airy function Ai and to Bessel function $K_{1/3}$ (cf. [Bak], page 98).

[2] This was mentionned to us by *A. Duval*. .

[3] Cf. also *Bakhoom* [Bak].

[4] Cf. also [AS], page 1002.

accelerating functions, cf.[Fe], page 548). If $\alpha \in Q$ the function $F_{P;\pm}(\alpha;\beta;y)$ is solution of a differential equation (obtained by a method similar to the derivation of Gauss-Manin connection). These functions[1] would certainly deserve a thoroughful study.

Lemma 8.([E 4], [MR 3][2])

Let $\mathrm{R} > 0$ and $\beta > 0$, $\alpha = \dfrac{\beta}{\beta - 1}$.

Let $\mathrm{D}_{\beta,\mathrm{R}} = \{t \in C \,/\, |Arg\, t| < \dfrac{\pi}{2\beta}$ *and* $Re\, t^{-\beta} > 1/\mathrm{R}^{\beta}\}$. *Then (on* $\mathrm{D}_{\beta,\mathrm{R}}$*):*

$$|C^{\alpha}(t)| \leq K^{\alpha}\mathrm{R}^{\beta/2}\,(t^{\beta-1}\,e^{(t/c_{\alpha})^{\beta}});\ \ with\ \ K^{\alpha} > 0\ \ and$$
$$c_{\alpha} = \beta(\alpha-1)^{1/\alpha}.$$

This Lemma is proved using *saddlepoint method.*

Definition 3.

Let $\alpha > 1$, $\beta = \dfrac{\alpha}{\alpha - 1}$, $R > 0$, *and a direction* d.

Let ψ *be an analytic function on the open* β*-Borel disc*

$\mathrm{D}_{\beta,\mathrm{R};d} = \{t \in C \,/\, |Arg\, t - Arg\, d| < \dfrac{\pi}{2\beta}$ and Re $(t\, e^{-iArg\, d})^{-\beta} > 1/\mathrm{R}^{\beta}\}$, *and continuous on the closure of* $\mathrm{D}_{\beta,\mathrm{R};d}$.

If we denote by γ_{R} *the boundary of* $\mathrm{D}_{\beta,\mathrm{R};d}$ *oriented in the positive sense, we will say that* ψ *is* α*–decelerable in the direction* d *if the integral*

$$\phi(\xi) = \frac{1}{2i\pi}\int_{\gamma_{\mathrm{R}}} \psi(\zeta)\ \zeta^{\alpha}\, C^{\alpha}(\xi/\zeta)\, d\zeta/\zeta^{2}$$ *exists (for* $\xi \in d$, *arbitrary).*

Proposition 5.

Let $\alpha > 1$, $\beta = \dfrac{\alpha}{\alpha - 1}$. *Let* ψ *be an analytic function on a sector, with opening* $> \dfrac{\pi}{\beta}$, *bisected by* d, *with exponential growth of order* $\leq \alpha$ *at infinity and an asymptotic expansion at zero. Then* ψ *is* α*–decelerable in the direction* d *and:*

---

[1] And the similar functions obtained if one replaces the Laplace transform by the Mellin transform in the definition (cf. the functions Γ_P studied in [Du]).

[2] More precisely it is possible, using saddlepoint method, to get an asymptotic expansion of function C^{α} on the disc $\mathrm{D}_{\beta,\mathrm{R}}$ (cf. [MR 3]).

$$D_\alpha \psi(\xi) = L^{-1} \rho_\alpha^{-1} L \rho_\alpha \psi(\xi) = \frac{1}{2i\pi} \int_{\gamma_R} \psi(\zeta)\ \zeta^\alpha C^\alpha(\xi/\zeta)\, d\zeta/\zeta^2 .$$

If the function ψ is analytic on a sector V, with opening $>\frac{\pi}{\beta}$, bisected by d, and if ψ is sufficiently flat at zero, that is if there exists $\lambda > 0$ such that

$\psi = o(\zeta^{1+\beta-\alpha+\lambda})$ on V, then it is *α-decelerable in the direction d* and $D_\alpha \psi$ is analytic on a sector bisected by d, with exponential growth of order $\leq \beta$ at infinity.

If a function ψ is analytic on $D_{\beta,R;d}$ and admits asymptotic expansion at zero, and if there exists a polynomial P such that $\psi = \psi_0 + P$, where ψ_0 is *α–decelerable* in the direction d, we will *also* say that ψ is *α-decelerable* in the direction d and denote it by

$D_\alpha \psi = D_\alpha \psi_0 + D_\alpha P$ ($D_\alpha P$ is computed *"formally"*; see formulae at the end of the paragraph).

The operator $D_{\alpha;d} = L^{-1} \rho_\alpha^{-1} L \rho_\alpha$ is defined on the domain

{analytic functions on sectors, with opening $>\frac{\pi}{\beta}$, bisected by d, with exponential growth of order $\leq \alpha$ at infinity and asymptotic expansion at the origin}.

The operator $\psi \longrightarrow \frac{1}{2i\pi} \int_{\gamma_R} \psi(\zeta)\ \zeta^\alpha C^\alpha(\xi/\zeta)\, d\zeta/\zeta^2$ is defined on the *larger* domain

{analytic functions on sectors, with opening $>\frac{\pi}{\beta}$, with arbitrary radius, bisected by d, with asymptotic expansion at the origin}.

So, *proposition 5* gives an extension for the operator $D_{\alpha;d}$.

Lemma 9.

The function C^α is α-accelerable in the direction $\mathbf{R}^+$ and

$$A_\alpha C^\alpha(\zeta) = \zeta/\zeta^\alpha(1-\zeta).$$

Proposition 6.

Let $\alpha > 1,\ \beta = \dfrac{\alpha}{\alpha - 1}$.

(i) If a function ψ is α-decelerable in the direction d, then $D_\alpha \psi$ is α-accelerable in the direction d and:

$$A_\alpha D_\alpha \psi = \psi.$$

(ii) If a function ϕ is infinitely differentiable on d, with an exponential growth of order $\leq \beta$ at infinity, then $A_\alpha \phi$ is α-decelerable in the direction d and:

$$D_\alpha A_\alpha \phi = \phi.$$

The proof of *(i)* is easy, using Fubini's theorem and *lemma 9*.

To prove *(ii)*, using *lemma 3*, we first prove it when ψ is infinitely differentiable on d, with exponential growth of order ≤ 1 at infinity (in particular for ψ with compact support); then, for ψ with *only* an exponential growth of order $\leq \beta$, we conclude by a *density argument*.

From proposition 5 *(ii)* we deduce the *injectivity* of $A_{\alpha;d}$. The *injectivity* of $A_{k',k;d}$ follows. That ends the proof of proposition 4.

The following result is essential:

Theorem 1.

*Let $k_1 > k_2 > ... > k_r > 0$, and d a given direction. Then the **summation** operator*

$$C\{x\}_{1/k_1,1/k_2,...,1/k_r;d} \xrightarrow{\mathrm{S}_{k_1,k_2,...,k_r;d}} \text{Differential algebra of germs of analytic functions on sectors bisected by } d.$$

*is an **injective morphism of differential algebras.***

Operators S and $\bullet_d$ are *isomorphisms* of differential algebras and of k-convolution differential algebras. Operator $\hat{B}_{k_r}$ is an *isomorphism* of differential algebras between the differential algebra $C[[x]]$ and the k_r-convolution differential algebra $C[[x]]$. Operator L_{k_1} is an *isomorphism* between the convolution differential algebra of analytic functions on sectors bisected by d with exponential growth of order $\leq k_1$ at infinity and asymptotic expansion at zero, and the differential algebra of analytic functions on sectors with opening $> \pi/k_1$, bisected by d, and with asymptotic expansion (having no constant term) at zero. We can now end the proof of theorem 1, using *proposition 4* with $k'=k_{i-1}$, $k=k_i$ $(i = r,...,2)$.

In fact it follows from this proof that the image of the operator $\mathrm{S}_{k_1,k_2,...,k_r;d}$ is *contained* in the differential algebra of analytic functions on sectors with opening $> \pi/k_1$, bisected by d, and with asymptotic expansion (having no constant term) at zero.

It is possible to extend *proposition 2* :

Proposition 7.

Let $k' > k_1 > k_2 > ... > k_r > 0$. Then :

$$C[[x]]_{1/k'} \cap C\{x\}_{1/k_1,1/k_2,...,1/k_r} = C\{x\}.$$

Proposition 8.

Let $k'_1, k'_2,..., k'_{r'} > 0$ and $k''_1, k''_2,..., k''_{r''} > 0$. If

$\{k_1, k_2,..., k_r\} = \{k'_1, k'_2,..., k'_{r'}\} \cap \{k''_1, k''_2,..., k''_{r''}\}$, *with* $k_1 > k_2 > ... > k_r > 0.(r \le r', r'')$, *then* :

$$C\{x\}_{1/k'_1,1/k'_2,...,1/k'_{r'}} \cap C\{x\}_{1/k''_1,1/k''_2,...,1/k''_{r''}} = C\{x\}_{1/k_1,1/k_2,...,1/k_r}.$$

If $\hat{f} \in C[[x]]$ is $(k'_1,k'_2,...,k'_{r'})$-summable, the *smallest set* $\{k_1, k_2,..., k_r\}$ (with $k_1 > k_2 > ... > k_r > 0$), such that $\hat{f}$ is $(k_1,k_2,...,k_r)$-summable, is a subset of $\{k'_1, k'_2,..., k'_{r'}\}$ and depends *only* on $\hat{f}$. The numbers $k_1, k_2,..., k_r$ are ***the singular levels*** of $\hat{f}$:

$$\{k_1, k_2,..., k_r\} = N\Sigma(\hat{f}) \subset]0,+\infty[\ (definition).$$

The situation is very different if $\hat{f} \in C[[x]]$, *is* $(k'_1,k'_2,...,k'_{r'})$*-summable* in a *direction* d. It is easy to prove then that there exists $\varepsilon > 0$, such that $\hat{f}$ is $(k'_1 - \varepsilon',k'_2 - \varepsilon',...,k'_{r'} - \varepsilon')$-summable in the direction d *for every* $\varepsilon' \in [0,\varepsilon]$.

We will identify the real analytic blow-up of the origin in the complex plane[1] with the circle S^1. Then we introduce the ***"analytic halo"*** of the origin in the complex plane:

$$HA_0 =]0,+\infty] \times S^1 = \{(k,d)/k \in]0,+\infty], d \in S^1\}.$$

The complex plane with an analytic halo at zero is:

$$CH_0 = \{0\} \cup HA_0 \cup C^* = ((\{"0"\} \cup "]0,+\infty]") \cup]0,+\infty[) \times S^1)/\mathcal{R};$$

$\mathcal{R}$ being the identification of $"\{0\}" \times S^1$ with a point $"\{0\}"$.

On the set $\{"0"\} \cup "]0,+\infty]") \cup]0,+\infty[)$ we put the ordering relation: Ordinary ordering relation on $]0,+\infty[$ and $"]0,+\infty]"$, $\rho > 0 > k$, if $\rho \in]0,+\infty[$, $k \in "]0,+\infty]"$. ($"+\infty"$ is identified with 0). We endow $\{"0"\} \cup HA_0 \cup C^*$ with the corresponding topology (quotient of the product topology). We will consider $"\{"0"\} \times S^1$ as the *"real blow up"* of 0 in CH_0 (that is the set of directions starting from 0 in CH_0).

The universal covering of $(S^1,1)$ is $(R,0)$. We will interpret $\widetilde{HA}_0 =]0,+\infty] \times (R,0)$ as the *"universal covering* of HA_0 pointed on the direction $"R^+" \in \{"0"\} \times S^1"$.

Let $U \subset S^1$ be an open arc. Let $k_1 > k_2 > ... > k_r > 0$. If $\hat{f} \in C[[x]]$ is $(k_1,k_2,...,k_r)$*-summable* in every direction $d \in U$, then the sums $f_{k_1,k_2,...,k_r;d}$ glue together in a function f analytic on a "sector" with opening equal to

(opening of $U + \pi/k_1$).

If now $U \subset S^1$ is an open arc bisected by d, let

$U^+ = \{d^+ \in U / Arg\, d^+ > Arg\, d\}$, *and*

$U^- = \{d^- \in U / Arg\, d^- < Arg\, d\}$.

If $\hat{f} \in C[[x]]$ *is* $(k_1,k_2,...,k_r)$*-summable* in every direction $d' \in U - \{d\}$, we denote

[1] If we use *polar coordinates* for the points of C^* :
$C^* = \{(\rho,\theta)/ \rho>0, \theta \in S^1\} =]0,+\infty[\times S^1$, *this set corresponds to* $\{0\} \times S^1$.

$$f^{+}_{k_1,k_2,\ldots,k_r;d} = \mathrm{S}^{+}_{k_1,k_2,\ldots,k_r;d}\,\hat{f} \text{ and}$$
$$f^{-}_{k_1,k_2,\ldots,k_r;d} = \mathrm{S}^{-}_{k_1,k_2,\ldots,k_r;d}\,\hat{f}$$ the sums of $\hat{f}$ for $d^{+}\in \mathrm{U}^{+}$ and $d^{-}\in \mathrm{U}^{-}$ respectively. They are in particular defined on a *common "sector"* bisected by d, with opening equal to π/k_1.

If $\hat{f}\in \boldsymbol{C}[[x]]$ is $(k_1,k_2,\ldots,k_r)$-summable, then $\mathrm{S}^{+}_{k_1,k_2,\ldots,k_r;d}\,\hat{f}$ and $\mathrm{S}^{-}_{k_1,k_2,\ldots,k_r;d}\,\hat{f}$ are defined for *every* direction $d\in \mathrm{S}^1$.

We can along the same lines define operators $L^{\varepsilon}_{k_1;d}$ and $A^{\varepsilon}_{k_{j-1},k_j;d}$, for $\varepsilon\in\{1,-1\}$.

Using decelerating operators, we get easily the *very important:*

Lemma 10.

Let $k_1 > k_2 > \ldots > k_r > 0$ and d a given direction Then if $\hat{f}\in \boldsymbol{C}[[x]]$, is $(k_1,k_2,\ldots,k_r)$-summable in every direction of $\mathrm{U}-\{d\}$, *the following conditions are equivalent:*

(i) $\hat{f}$ is $(k_1,k_2,\ldots,k_r)$-summable in the direction d;

(ii) $\mathrm{S}^{+}_{k_1,k_2,\ldots,k_r;d}\,\hat{f} = \mathrm{S}^{-}_{k_1,k_2,\ldots,k_r;d}\,\hat{f}$ on a sector bisected by d.

Moreover if these conditions are satisfied, then

$$\mathrm{S}^{+}_{k_1,k_2,\ldots,k_r;d}\,\hat{f} = \mathrm{S}^{-}_{k_1,k_2,\ldots,k_r;d}\,\hat{f} = \mathrm{S}_{k_1,k_2,\ldots,k_r;d}\,\hat{f}.$$

If the conditions of *lemma 10* are not satisfied, we will say that d is a *singular direction* for the formal power series $\hat{f}$, and we will write $d\in\Sigma(\hat{f})$; the *"singular support"* $\Sigma(\hat{f})$ of $\hat{f}$ is clearly *finite,* and $\Sigma(\hat{f})=\varnothing$ is equivalent to $\hat{f}\in \boldsymbol{C}\{x\}$. We will see below that the *"jump"* from

$\mathrm{S}^{+}_{k_1,k_2,\ldots,k_r;d}\,\hat{f}$ to $\mathrm{S}^{-}_{k_1,k_2,\ldots,k_r;d}\,\hat{f}$ is a natural generalisation of the classical ***"Stokes phenomenon"*** for solutions of linear differential equations.

We will give below *(cf. 5)* a very natural *interpretation of multisummability:*

A formal power series $\hat{f}\in \boldsymbol{C}[[x]]$ is *multisummable in the direction d* (that is there exist $k_1 > k_2 > \ldots > k_r > 0$ such that $\hat{f}$ is *$(k_1,k_2,\ldots,k_r)$-summable in the direction d*) if and only if it is *"analytic"* (*"wild* analytic") in an *"infinitysimal disc"* [1] and can be ***"extended analytically"*** along d, across the *"infinitysimal neighborhood"* [2] in a wild analytic function on a sector bisected by d, with a "non infinitysimal" radius $\mathrm{R} > 0$.

Then, just like one can give a *direct* (that is *without* using Borel and Laplace transforms) definition of *Borel-summability* and *k-summability* using ***Gevrey estimates*** [Ra 2], [MR 1], [MR 2], [MR 3], it is also possible to give a *direct* (that is *without* any use of *Ecalle's* acceleration operators) definition of *multisummability* using the ***wild Cauchy theory*** recently introduced by the authors [MR 3]. This *"geometric"* definition is *easier to check* in the *usual applications* . Converserly the *"analytic"*

1 The corresponding punctured disc has a radius $\geq k>0$ in the analytic halo at zero.

2 This infinitesimal neighborhood is the union of zero and the analytic halo at zero.

definition gives an *"explicit"* way for the computation of the sum (for instance if one has in mind *numerical computations*).

Let $U \subset S^1$ be an open arc bisected by d. Let $k_1 > k_2 > \ldots > k_r > 0$ and let $\hat{f} \in$ *$C[[x]]$, be $(k_1,k_2,\ldots,k_r)$-summable* in every direction $d' \in U - \{d\}$. There is a natural way to generalize the sums $S^+_{k_1,k_2,\ldots,k_r;d}\ \hat{f}$ and $S^-_{k_1,k_2,\ldots,k_r;d}\ \hat{f}$:

Let $\varepsilon = (\varepsilon_1, \varepsilon_2,\ldots,\varepsilon_r)$, *with* $\varepsilon_i \in \{1,-1\}$ $(i = 1,\ldots,r)$. We will say that $(d;\varepsilon)$ defines a "path".[3] We can now introduce the notion of *$(k_1,k_2,\ldots,k_r)$-summablility along the path $(d;\varepsilon)$*:

Definition 3.

*Let $U \subset S^1$ be an open arc bisected by d. Let $k_1 > k_2 > \ldots > k_r > 0$ and let $\hat{f} \in C[[x]]$, be $(k_1,k_2,\ldots,k_r)$-summable in every direction $d' \in U -\{d\}$. Let $\varepsilon = (\varepsilon_1, \varepsilon_2,\ldots,\varepsilon_r)$, with $\varepsilon_i \in \{1,-1\}$ $(i = 1,\ldots,r)$. We will say that $\hat{f}$ is $(k_1,k_2,\ldots,k_r)$-**summable along the path** $(d;\varepsilon)$ if*

$$S^{\varepsilon}_{k_1,k_2,\ldots,k_r;d}\ \hat{f} = L^{\varepsilon_1}_{k_1,k_2;d} \bullet_{d^{\varepsilon_1}} A^{\varepsilon_2}_{k_1,k_2;d} \ldots A^{\varepsilon_r}_{k_{r-1},k_r;d} \bullet_{d^{\varepsilon_r}} S\,\hat{B}_{k_r}\,\hat{f}$$

*exists. Then $S^{\varepsilon}_{k_1,k_2,\ldots,k_r;d}\ \hat{f}$ is the **sum** of $\hat{f}$ **along the path** $(d;e)$.*

Theorem 2.

*Let $k_1 > k_2 > \ldots > k_r > 0$,.a direction d and $\varepsilon = (\varepsilon_1, \varepsilon_2,\ldots,\varepsilon_r)$, with $\varepsilon_i \in \{1,-1\}$ $(i = 1,\ldots,r)$. Then the **summation** operator*

$$\begin{array}{ccc} (k_1,k_2,\ldots,k_r)\text{-summable} & \xrightarrow{\quad S^{\varepsilon}_{k_1,k_2,\ldots,k_r;d}\quad} & \text{Differential algebra of germs of} \\ \text{power series } \hat{f} \in C[[x]] & & \text{analytic functions on sectors} \\ \text{along the path } (d;e). & & \text{bisected by } d\,. \end{array}$$

*is an **injective morphism of differential algebras.***

Comparison between $S^{\varepsilon}_{k_1,k_2,\ldots,k_r;d}\ \hat{f}$ and $S^{\varepsilon'}_{k_1,k_2,\ldots,k_r;d}\ \hat{f}$ for different ε, ε' will give birth to a *"generalized Stokes phenomenon"*.

We will finish this paragraph with a small list of ***useful formulae:***

Let $k, k', \lambda, \mu > 0$. Then:

$$\rho_k(x^{\lambda}) = \xi^{\lambda/k}\,\rho_\alpha(t^{\mu}) = \frac{\Gamma(1+\mu)}{\Gamma(\frac{1+\mu}{\alpha})}\,x^{(1+\mu-\alpha)/\alpha}$$

$$B_k(x^{\lambda}) = t^{\lambda-k}/\Gamma(\lambda/k) \qquad\qquad L_k(t^{\mu}) = \Gamma(1+\mu/k)\,x^{\mu+k}$$

[3] Later we will see that such a $(d;\varepsilon)$ corresponds to a *wild homotopy class of paths* in the analytic halo of the origin, avoiding *"singularities"* of f in this halo.

$$A_\alpha(t^{\mu}) = \frac{\Gamma(1+\mu)}{\Gamma(\frac{1+\mu}{\alpha})} x^{1+\mu-\alpha}$$

$$A_{k',k}(t^{\mu}) = \frac{\Gamma(\frac{k+\mu}{k})}{\Gamma(\frac{k+\mu}{k'})} x^{\mu+k-k'}$$

$$D_\alpha(x^{\nu}) = \frac{\Gamma(1+\nu/\alpha)}{\Gamma(\nu+\alpha)} t^{\nu-1+\alpha}$$

$$D_{k',k}(x^{\nu}) = \frac{\Gamma(\frac{k'+\nu}{k'})}{\Gamma(\frac{k'+\nu}{k})} t^{\nu+k'-k}$$

$$\partial_k(x^{\lambda}) = (\lambda+k)\ \Gamma(1+\lambda/k)\, x^{\lambda+1}/\Gamma(\tfrac{\lambda+1}{k}+1)$$

$$x^{\lambda} *_k x^{\mu} = \frac{\Gamma(1+k\lambda)\,\Gamma(1+k\mu)}{\Gamma(1+k(\lambda+\mu))}\ x^{\lambda+\mu+1/k}\,.$$

When k varies from 1 to $+\infty$, the *k-convolution* $*_k$ varies from the *ordinary convolution* $*$ to the *ordinary product* $\bullet$.

Formulae about accelerating and decelerating functions.

The following results were obtained recently (january 1990) by *A. Duval:*

$C_3(t) = i\sqrt{3}\ \ G^{2,0}_{0,2}\left((t/3)^3 \,\middle|\, {1,\ \ 2 \atop 1/3,\ 2/3}\right);$

$C^2(t) = \frac{1}{\sqrt{\pi}}\ G^{2,1}_{1,2}\left((t/2)^2 \,\middle|\, {0,\ \ 1 \atop 0,\ 1/2}\right) = \frac{1}{2}\,\psi(1,1/2;\, t^2/4);$

G is a *Mejer G-function* [Lu].

$$C_\alpha(t) = \int_{+\infty}^{(0^-)} \frac{\Gamma(-s)}{\Gamma(-s/\alpha)}\, t^{s}\, ds \quad \text{(Hankel type contour around } \boldsymbol{R}^+\text{)},$$

$$C^{\alpha}(t) = \frac{1}{2i\pi} \int_{+\infty}^{(0^-)} \Gamma(-s)\ \Gamma(1+s/\alpha))\ \ (-t)^{s}\, ds\ \ .$$

If $\alpha = p/q$, with p and q positive integers, $q > p > 0,\ (p,q) = 1$:

$$C_{q/p}(t) = 1/\sqrt{pq(2\pi)^{q-p}} \int_{+\infty}^{(0^-)} \frac{\prod_{j=1,\dots,q-1} \Gamma(-s+j/q)}{\prod_{j=1,\dots,p-1} \Gamma(-s+j/p)} (p^p(t/q)^q)^s\, ds$$

$$C_{q/p}(t) = 2i\pi/\sqrt{pq(2\pi)^{q-p}}\ G^{\,q-1,\,0}_{\,p-1,\,q-1}\left(p^p(t/q)^q \,\middle|\, {1/p, 2/p, \dots, (p-1)/p \atop 1/q, 2/q, \dots, (q-1)/q}\right);$$

$$C^{q/p}(t) = -i\sqrt{pq/(2\pi)^{q+p}} \int_{+\infty}^{(0^-)} \prod_{j=0,\dots,q-1} \Gamma(-s+j/q) \prod_{j=0,\dots,p-1} \Gamma(s+1/p-j/p)\ (p^p(-t/q)^q)^s\, ds$$

$$C^{q/p}(t) = 2\pi \sqrt{pq/(2\pi)^{q+p}}\ G^{\,q-1,\,0}_{\,p-1,\,q-1}\ \left(p^p(t/q)^q \,\middle|\, {0,1/p,2/p,\dots(p-1)/p \atop 0,1/q,\,2/q,\dots,(q-1)/q}\right)$$

Accelerating functions $C_{q/p}$ are *solutions* of the *differential operators* (respectively of order $q-1$ and q):

$$q \prod_{j=1,\dots,q-1} (\delta - j) - (-1)^{q-p}\, p\, t^q \prod_{j=1,\dots,p-1} (\tfrac{p}{q}\delta + j) \ (\delta = td/dt),$$

and

$$D^q - (-1)^{q-p} \prod_{j=1,\dots,p} (\tfrac{p}{q} tD + j) \ (D = d/dt).$$

We get in particular, for $q = n, p = 1$:

$$D^n + (-1)^n (\tfrac{1}{n} tD + 1).$$

Decelerating functions $C^{q/p}$ are *solutions* of *differential operators*

$$D^q - \prod_{j=1,\dots,p} (\tfrac{p}{q} tD + j).$$

If $p-q$ is *even,* we get the *same differential equation* fot the *accelerating* and *decelerating* functions.

We get in particular, for $q = n, p = 1$:

$$D^n - (\tfrac{1}{n} tD + 1).$$

4. *Stokes multipliers.*

Let $\Delta = d/dx - A$, with $A \in End(n;\mathbf{C}\{x\}[x^{-1}])$, be a germ of *meromorphic differential operator* at the origin of the complex plane $\mathbf{C}$.

It is well known [Ma 2] that Δ admits a ***formal** fundamental solution*[1]:

$$\hat{F}(x) = \hat{H}(u)\, u^{\nu L}\, e^{Q(1/u)}, \text{ with:}$$

$u^{\nu} = x$ (for some $\nu \in N^*$), $L \in End(n;\mathbf{C})$, $\hat{H} \in GL(n;\mathbf{C}[[u]][u^{-1}])$, and Q a diagonal matrix with entries in $u^{-1}\mathbf{C}[u^{-1}]$, invariant, up to permutations of the diagonal entries, by the transformation corresponding to $u \longrightarrow e^{2i\pi/\nu}u$ $(x \longrightarrow e^{2i\pi}x)$ and satisfying $[e^{2i\pi\nu L}, Q] = 0$ (and $[L,Q] = 0$, if $\nu = 1$); L can be supposed in *Jordan form.*

If $Q = Diag\{q_1, q_2,\ldots, q_n\}$, then the *set* $\{q_1, q_2,\ldots, q_n\}$ is a subset of $u^{-1}\mathbf{C}[u^{-1}]$ which is *independent* of the choice of the fundamental solution F (ν is choosen *minimal*).

We will set $\{q_1, q_2,\ldots, q_n\} = \mathbf{q}(Q) = \mathbf{q}(\Delta)$; the ***set*** $\mathbf{q}(\Delta)$ is clearly a *formal invariant* of Δ (invariant by the transformation $\mathbf{q}(\Delta)(u) \longrightarrow \mathbf{q}(\Delta)(e^{2i\pi/\nu}u)$).

Proposition 9.

Let $k_1 > k_2 > \ldots > k_r > 0$, and $\nu \in N^*$. Let d be a fixed direction. Let $\alpha_1, \alpha_2,\ldots, \alpha_m \in \mathbf{C}$, and $q_1, q_2,\ldots, q_n \in x^{1/\nu}\mathbf{C}[x^{1/\nu}]$. Then the summation operator

$$\mathbf{C}\{x\}_{1/k_1,1/k_2,\ldots,1/k_r;d} \xrightarrow{\ \mathrm{S}_{k_1,k_2,\ldots,k_r;d}\ } \textit{Differential algebra of germs of analytic functions on sectors bisected by } d.$$

*can be **uniquely extended** to a summation operator (still denoted by $\mathrm{S}_{k_1,k_2,\ldots,k_r;d}$)*

$$\underset{(i=1,\ldots,m;j=1,\ldots,n)}{\mathbf{C}\{x\}_{1/k_1,1/k_2,\ldots,1/k_r;d} <x^{\alpha_i}, e^{q_j}, Log\ x>} \longrightarrow \textit{Differential algebra of germs of analytic functions on sectors bisected by } d.$$

such that (a "branch" of Log x being fixed[2]):

$\mathrm{S}_{k_1,k_2,\ldots,k_r;d}(x^{\alpha_i}) = e^{\alpha_i Log\, x}$, $\mathrm{S}_{k_1,k_2,\ldots,k_r;d}(e^{q_j}) = e^{q_j}$, *and* $\mathrm{S}_{k_1,k_2,\ldots,k_r;d}(Log\ x) = Log\ x$.

*This operator is **an injective morphism of differential algebras.***

It is easy to extend the definition of the operator $\mathrm{S}_d = \mathrm{S}_{k_1,k_2,\ldots,k_r;d}$ to the elements of $\mathbf{C}\{x\}_{1/k_1,1/k_2,\ldots,1/k_r;d} <x^{\alpha_i}, Log\ x>$ $(i=1,\ldots,m)$. Then, using *asymptotic expansions* (the inverse of S_d, restricted to $Im\ \mathrm{S}_d$, is the *asymptotic expansion* operator in the classical sense), we get

$\mathbf{C}\{x\}<e^{q_j}> \cap \mathbf{C}\{x\}_{1/k_1,1/k_2,\ldots,1/k_r;d} <x^{\alpha_i}, Log\ x> = \mathbf{C}\{x\}$ $(i=1,\ldots,m; j=1,\ldots,n)$. The result follows.

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[1] Cf. infra for a more precise description of $\hat{F}$ when $\nu \geq 2$ (*"ramified* case").
[2] *Log x* is "formal" in the "left expression", and an actual function in the "right expression".

Theorem 3.

Let $\Delta = d/dx - A$, with $A \in End(n;\mathbf{C}\{x\}[x^{-1}])$, be a germ of meromorphic differential operator at the origin of the complex plane $\mathbf{C}$.

We denote by $k_1 > k_2 > \ldots > k_r$ the positive (non zero) slopes of the Newton polygon of the (rank n^2) differential operator

$$End\,\Delta = d/dx - [A,.].$$

*Let $\hat{F}$ be a **formal** fundamental solution of Δ. Then there exists a "natural decomposition"* [1]

$$\hat{H} = \hat{H}_1\hat{H}_2 \ldots \hat{H}_r\,,\ \text{where } \hat{H}_i \in GL(n;\mathbf{C}[[u]][u^{-1}]) \text{ is } k_i\text{-}\mathbf{summable} \text{ as a}$$

"function" of x (i. e. νk_i-summable as a "function" of u), for $i = 1,\ldots,r$, and such that

*(i) $F^i(u^\nu) = \hat{H}_i(u)\hat{H}_{i+1}(u)\ldots\hat{H}_r(u)\; u^{\nu L}\, e^{Q(1/u)}$ is a **formal fundamental solution** of a **meromorphic** differential operator $\Delta_\nu{}^i = d/dx - A_\nu{}^i$, with*

$$A_\nu{}^i \in End(n;\, \mathbf{C}\{u\}[u^{-1}]),\ \text{for } i = 1,\ldots,r;\ ^2$$

(ii) If $\Sigma(\hat{F}) = \Sigma(\hat{H}) = \bigcup_{i=1,\ldots,r} \Sigma(\hat{H}_i)$, $H_{i;d} = S_{k_i;d}\,\hat{H}_i$ (for $i = 1,\ldots,r$)

and

$$H_d = H_{1;d}H_{2;d} \ldots H_{r;d}\,,$$

then, for $d \notin \Sigma(\hat{H})$, and every determination of Log x ($u = e^{(Log\,x)/\nu}$ and $u^L = e^{L\,Log\,u}$):

*$F_d(x) = H_d(u)\, u^{\nu L}\, e^{Q(1/u)}$ is an **actual** analytic fundamental solution of the operator Δ on a sector bisected by d.*

From this result (using *proposition 9*) it is easy to deduce the

Theorem 4.

*Let $\Delta = d/dx - A$, with $A \in End(n;\mathbf{C}\{x\}[x^{-1}])$, be a germ of meromorphic differential operator at the origin of the complex plane $\mathbf{C}$. Let $\hat{F}$ be a **formal fundamental solution** of Δ. If we denote by $\mathbf{C}\{x\}[x^{-1}]\langle\hat{F}\rangle$ the differential field generated, on $\mathbf{C}\{x\}[x^{-1}]$, by the entries of $\hat{F}$, then, for $d \notin \Sigma(\hat{F})$, the map*

$\mathbf{C}\{x\}[x^{-1}]\langle\hat{F}\rangle \longrightarrow$ Differential field generated, on $\mathbf{C}\{x\}[x^{-1}]$, by the analytic solutions of the operator Δ in a germ of sector bisected by d.

defined by "identity" on $\mathbf{C}\{x\}[x^{-1}]$ and $\hat{F} \longrightarrow F_d$,

---

[1] Unique up to "natural" analytic transformations (see [Ra 4]); in particular, the matrices H_i are well defined up to analytic (in u) conjugation.

[2] Moreover the matrices $A_\nu{}^i$ and H_{i+1} have a common "blockstructure" and $\Delta_\nu{}^i$ can be reduced by a transform *"$Y = Exp(Q_i)\,Z$"* to a differential operator whose *Katz's invariant* [De 1] is k_{i+1} ; Q_i being a diagonal matrix whose entries are monomials in u (fixed for each block)of degree νk_i [J] , [Ra 6] .

is an ***isomorphism*** *of* ***differential fields.***

We will first admit *theorem 3,* and will go back in *5* to some indications about its proof, after some applications. It is very easy to deduce *theorem 4* from *theorem 3,* using ***multisummability*** (other ways to do that are explained in [Ra 5], [Ra 6], and [De 4][1]):

From *theorem 3* and *lemma 7* we get

Theorem 5.

Let $\Delta = d/dx - A$, *with* $A \in End(n;\mathbf{C}\{x\}[x^{-1}])$, *be a germ of meromorphic differential operator at the origin of the complex plane* $\mathbf{C}$. *Let* $\hat{F}$ *be a* ***formal fundamental solution*** *of* Δ. *We denote by* $k_1 > k_2 > ... > k_r$ *the positive (non zero) slopes of the Newton polygon of the operator*

$$End\,\Delta = d/dx - [A,.].$$

Then $\hat{F}$ *is* $(k_1,k_2,...,k_r)$*-summable in every direction, but perhaps a finite number belonging to* $\Sigma(\hat{F}) \subset S^1$.

Clearly (using *lemma 7*) the sums (in a common non singular direction) given by *theorems* 2 and 4 are the same.

If $d \notin \Sigma(F)$, the operator $S_{k_1,k_2,...,k_r;d}$ is injective and *Galois-differential.* So *theorem 4* follows from *theorem 5.* Moreover we have got an ***"explicit"*** method of ***summation of formal solutions of linear differential equations.***[2] It is interesting to remark that $k_1, k_2,..., k_r$ are ***rational numbers***, so $k_i/k_{i-1} = \alpha_i \in \mathbf{Q}$ and $\mathbf{C}_{\alpha_i}$ $(i=1,...,r)$ is *a solution of a linear differential equation*; moreover all the functions written under $\int$ when we apply the successive computations of the resummation algorithm are *solutions of linear differential equations.* A consequence is that, for numerical computations, we can apply efficient algorithms in order to compute the successive *analytic continuations* $\bullet_d$ (Runge-Kutta algorithm, Chudnovskys algorithm [Chu],...).

Let now $d \in \Sigma(\hat{F})$ be a *singular* direction:

Then (a *"branch"* of Logarithm being choosen)

$S^+_{k_1,k_2,...,k_r;d}\,\hat{F}$ and $S^-_{k_1,k_2,...,k_r;d}\,\hat{F}$ are *(different)* ***actual*** *fundamental solutions* of Δ, *analytic on a* ***common*** *sector bisected by* d, *with opening* π/k_1, on the Riemann surface of Logarithm. So we get

[1] The methods differs by the respective proportions of analysis and algebra used.

[2] There exists an algorithm for the explicit computation of the levels $k_1, k_2,..., k_r$ [Ma 2]. An effective computation is possible on a computer using the systems "Reduce", "Desir" and "D5" [Tou]. For the ("generic") one-level case there are efficient numerical algorithms of summation [Th]; for the multilevelled case, algorithms are studied by *Thomann.*

$F_d^+ = F_d^- \ \mathrm{St}_d$, with $\mathrm{St}_d \in GL(n;\mathbf{C})$. By *definition* St_d is ***the Stokes matrix*** associated to the *formal fundamental solution* F of Δ, to the *direction* d, and to the *choice of branch of Logarithm.*

The operator $(\mathrm{S}^+_{k_1,k_2,\dots,k_r;d})^{-1}(\mathrm{S}^-_{k_1,k_2,\dots,k_r;d}) = St_d$ is clearly a K-automorphism of the differential extension $\mathbf{C}\{x\}[x^{-1}]<\hat{F}>$ (which is a *Picard-Vessiot extension* of $\mathbf{C}\{x\}[x^{-1}]$ associated to Δ [Kap], [Kol]), that is an element of the *Galois differential group,* clearly *independent* of the choice of $\hat{F}$). Later we will *systematically* write the operation of elements of St_d, and, more generally, of differential automorphisms, ***on the right*** (and ask the reader to be careful with the *ordering* of compositions...). We will also denote by St_d the induced automorphism (this automorphism depends on $d \in \mathrm{S}^1$ and on the choice of branch of Logarithm[1], that is on $d \in (\mathbf{R},0)$ (*universal covering* of $(\mathrm{S}^1,0)$) *"above"* d) of the $\mathbf{C}$*-vector space of formal solutions* of Δ (the matrix of this automorphism in the basis formed by the columns of $\hat{F}$ is St_d). So the ***Stokes matrix*** St_d ***is an element of the representation of "the" differential Galois group*** $Gal_K(\Delta) = Aut_K\, K<\hat{F}>$ $(K = \mathbf{C}\{x\}[x^{-1}])$ in $GL(n;\mathbf{C})$ given by *the formal fundamental solution* $\hat{F}$.

Here one must be very careful: *Stokes matrices defined by our method* (very near of *Stokes original method* [Sto](cf. references and comments in [MR 2], chapter 3)) are "in" the Galois differential group, but this is in general *completely false for "classical" Stokes matrices.* Classical definition, starting from asymptotic expansions in Poincaré's sense[2], is "unnatural" and corresponds to a misunderstanding of the original Stokes ideas (Stokes was working by numerical computations with in mind something like an idea of *"exact asymptotic expansions"*).

Remark.

Stokes operators St_d and Stokes matrices St_d are ***unipotent*** (see infra), so we can define their *logarithms* st_d and st_d respectively (the idea of a systematical use of these logarithms seems essentially due to *Ecalle* in a more general context):

$St_d = Exp\ st_d$ and $\mathrm{St}_d = Exp\ \mathrm{st}_d$. Then

$F_d = F_d^+ Exp\left(-\frac{1}{2}\mathrm{st}_d\right) = F_d^- Exp\left(\frac{1}{2}\mathrm{st}_d\right)$, and we can choose F_d as *sum* of $\hat{F}$ in the *singular direction* $\boldsymbol{d}$ (this idea is already in *Dingle*'s book [Din]; this has been recently extended to *extremely general* situations by *Ecalle: "sommation médiane"*). If the differential operator Δ is ***real***, if $\hat{G}$ is a ***real formal fundamental solution***, and if $\boldsymbol{d} = \mathbf{R}^+$, then we can choose the *fundamental determination* of the

[1] Up to conjugation by the "formal monodromy" (Cf. infra).

[2] Asymptotic expansions in Poincaré's sense must be replaced by *"transasymptotic expansions"* (*Ecalle's* terminology): the transasymptotic expansion map is the *inverse* of the summation map). Transasymptotic expansions can only make *"exponentially small jumps"* on *singular lines* ("antiStokes lines"), but Poincaré asymptotic expansions can only make *"jumps"* on *"Stokes lines"* (consequence of transasymptotic expansion "jumps", in *"quadrature of phasis"*).

Logarithm , and the *"median sum"* G_d is ***real*** (this can be applied to *Airy* equation at infinity, cf. [MR 2], chapter 3). Moreover st_d is a ***Galois derivation*** (i.e. *commuting* with the *derivation* of the differential field) of the differential field $K<\hat{G}>$, and $Exp\ (\frac{1}{2} st_d) \in Aut_K\, K<\hat{G}>$, then, when the reality conditions given above are satisfied, the map $\boldsymbol{R}\{x\}[x^{-1}]<\hat{G}> \longrightarrow$ *germs of real meromorphic functions at* $0 \in\,]0,+\infty[$, defined by

$\hat{G} \longrightarrow G_d$ on $\hat{G}$, and equal to S on $\boldsymbol{R}\{x\}[x^{-1}]$ is ***an injective morphism of differential fields.***

The following generalization of a *Schlesinger's* theorem[1] [Sch] was first proved in [Ra 4], [Ra 5], using a different method[2]:

Theorem 6.

Let $K = \boldsymbol{C}\{x\}[x^{-1}]$. *Let* $\Delta = d/dx - A$, *with* $A \in End(n;K)$, *be a germ of meromorphic differential operator at the origin of the complex plane* $\boldsymbol{C}$. *Let* $\hat{F}$ *be a* ***formal fundamental solution*** *of* Δ. *Let* H *be the subgroup of* $GL(n;C)$ *generated by the* ***formal monodromy*** *matrix* $\hat{\mathrm{M}}$, ***the exponential torus*** T, *and the* ***Stokes matrices*** *of* Δ *associated to the given formal fundamental solution* $\hat{F}$. *Then the representation of the* ***Galois differential group*** $Gal_K(\Delta)$ *of* Δ *in* $GL(n;C)$, *given by* $\hat{F}$, *is the* ***Zariski closure*** *of* H *in* $GL(n;C)$.

Using *"Galois correspondence"* [Kap], it suffices to prove that the *invariant field* of H (that is the subfield of $K<\hat{F}>$ consisting of the *invariant* elements by H) is K.

First we must define the *"formal monodromy"* and the *"exponential torus"* of Δ.

Replacing u by $u\, e^{2i\pi}$ in $F(u)$, we get a (in general *new*) *fundamental solution* of the differential operator Δ:

$F(u\, e^{2i\pi}) = F(u)\, \hat{\mathrm{M}}$, with $\hat{\mathrm{M}} \in GL(n;C)$. By definition $\hat{\mathrm{M}}$ is the *formal monodromy matrix* associated to Δ and to the fondamental solution $\hat{F}$. The corresponding element $\hat{M}$ of $Aut_K K<\hat{F}>$ is clearly *independant* of the choice of $\hat{F}$ and is a *formal invariant* of Δ; it is the *formal monodromy* of Δ. (We will later systematically write the operation of $\hat{M}$ on the ***right.***)

We will now define the *"exponential torus"*.

Let $\hat{\mathbb{K}} = \hat{K}_v < u^L, e^Q>$ the differential field generated by $\hat{K}_v = \boldsymbol{C}[[u]][u^{-1}]$ *and the entries of the matrices* u^L and e^Q.

Let $\hat{\mathbb{L}}_v = \hat{K}_v < e^Q > = \hat{K}_v < e^{q_1}, e^{q_2},\ldots, e^{q_n}> \subset \hat{\mathbb{K}}$.

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[1] Schlesinger's theorem is for the case of Fuchsian equations.

[2] A second proof has been given by *Deligne* using "Tannakian" ideas [De 4], and, during Luminy conference (september 1989), I have learned from *Y. Il'Yashenko* that he has also recently got another proof...

If μ is the dimension of the (free) abelian *Z-module* $E(\Delta) \subset u^{-1}C[u^{-1}]$ generated by $q_1, q_2, ..., q_n$, the *Galois differential group* $Aut_{\hat{K}_\nu}\hat{\mathbb{L}}_\nu = Aut_{K_\nu}\mathbb{L}_\nu$ *is a torus* $\mathcal{T}(Q) = \mathcal{T}_\nu(Q) = \mathcal{T}(q(\Delta))$ *isomorphic to* $(C^*)^\mu$ (clearly $\mu \leq n$). (We have set $K_\nu = C\{u\}[u^{-1}]$ and $\mathbb{L}_\nu = K_\nu < e^Q >$.)

We have $\hat{\mathbb{L}}_\nu \cap \hat{K}_\nu < u^L > = \hat{K}_\nu$. Then $\mathcal{T}(Q)$ can be identified with a subgroup of $Aut_{\hat{K}_\nu}\hat{\mathbb{K}}$ leaving $K_\nu < u^L >$ *fixed* (still denoted by $\mathcal{T}(Q)$).

We have $K<\hat{F}> \subset \hat{\mathbb{K}}$, and K and $K<\hat{F}>$ are invariant by $\mathcal{T}(Q)$; so $\mathcal{T}(Q)$ can be identified with a subgroup of $Aut_K K<\hat{F}> = Gal_K(\Delta)$. This group is clearly independent of the choice of $\hat{F}$. By *definition* we call this group *"the exponential torus" of* Δ. It will be denoted by $T(\Delta)$ (it depends only on $q(\Delta)$ and is a *formal invariant* of Δ). Its representation in $GL(n;C)$ given by the fundamental solution $\hat{F}$ will be denoted by $\mathrm{T} = \mathrm{T}(\Delta) = \mathrm{T}(Q(\Delta))$ (and still named *"exponential torus"*).

Let $K'_\nu = C\{u\}_{1/\nu k_1, 1/\nu k_2, ..., 1/\nu k_r}$. We have $K < \hat{F} > \subset K'_\nu < u^L, e^Q > = \mathbb{K}'$.

Let now $\xi \in K<\hat{F}>$ be an ***invariant*** element by H (more precisely by the subgroup of $Aut_K K<\hat{F}>$ corresponding to H). If $x = u^\nu$, then ξ is invariant by $\hat{\mathrm{M}}^\nu$, that is by the *formal monodromy* "in u", so $\xi \in K'_\nu < e^Q >$. But ξ is also invariant by the *exponential torus* and $\xi \in K'_\nu$. From the *invariance* of ξ by the *Stokes matrices* we deduce that the $(k_1,k_2,...,k_r)$*-summable* power series ξ admits ***no singular direction*** *(Lemma 10),* so ξ is ***convergent*** and $\xi \in K_\nu$. The action of the monodromy matrix $\hat{\mathrm{M}}$ on $\xi \in K_\nu$ is the same as the action of the *(ordinary)* Galois group $Aut_K K_\nu$ (isomorphic to $Z/\nu Z$), so ξ is invariant by $Aut_K K_\nu$ and $\xi \in K$ (by the *ordinary* Galois correspondence). That ends the proof of *Theorem 5.*

Examples.

From fundamental systems of solutions at infinity $(z = x^{-1}; x = 0)$ for *Airy* and *Kummer differential equations it is possible* to compute *formal monodromies, exponential torus* and *Stokes multipliers*. From these results it is possible to compute the Galois differential groups of our differential equations[1]. See [MR 3]).

For a deeper study of germs of analytic linear differential equations we need now a little *"toolbox"*[2] (built with elementary linear algebra).

Let $E_\nu = x^{-1/\nu}C\{x^{-1/\nu}\}$ $(n \in N^*)$ *and* $E = \bigcup_{\nu \in N^*} E_\nu$. If $q = \{q_1, q_2, ..., q_n\} \subset E$, we denote $\quad E(q) = Z\, q_1 + Z\, q_2 + ... + Z\, q_n \subset E$,

[1] "Classical computation" of the Galois differential group of Airy equation is in [Kap]; the computation of the Galois differential group of Kummer equations is , as far as we know, new (it is possible to do the computations "classically", using improvements of Kovacic's algorithm [Kov], [DLR], [MR 3]).

[2] A first version ot these tools was first introduced by *Balser, Jurkat, Lutz* [BJL 1], [J]. In our presentation we have also used ideas of *Deligne, Malgrange* [De 3], [Ma 3], [Ma 4], *Babbitt, Varadarajan* [BV], and the systematic treatment of *M. Loday-Richaud* [LR1] .

the sublattice of E *generated by* $q_1, q_2,..., q_n$. The *smallest* integer ν such that $E(q) \subset x^{-1/\nu}C\{x^{-1/\nu}\}$ is, by *definition*, the *ramification* of q, *or* $E(q)$. We have:

$$E = \bigcup_q E(q) = \varinjlim_{q} E(q).$$

We define an action of the (classical) Galois group $Aut_K K_\nu \approx Z/\nu Z$ on a sublattice E' of E_ν, *by*

$q(x^{-1/\nu}) \longrightarrow q(e^{-2i\pi/\nu}x^{-1/\nu})$ (corresponding to $x \longrightarrow e^{-2i\pi}x$). If E' is *invariant* by this action we will say that E' is *Galois invariant.* The lattice $E(q)$ is Galois invariant if and only if the *set* q is invariant by the corresponding action (Galois invariant).

If $q \in E(q)$, its *"degree"* $\delta(q)$ is the rational number $m/\nu \in \frac{1}{\nu}Q$, where m is the degree of q as a polynomial in $x^{1/\nu}$. There is a natural filtration of E by the degree, that is by the sublattices

$E^m = \{q \in E \ / \ \delta(q) \leq m\}$.

We identify the *universal covering* of $(S^1,1)$ to $(R,0)$. By definition the *"front"* $Fr(q)$ of $q \in E$ is the subset of $(R,0)$ whose elements are the *"lines of maximal decrease"* of e^q (we will also call "front" the natural projection of this set on the ν-covering of $(S^1,1)$, identified with another copy of $(S^1,1)$); the front of q depends clearly *only* on the monomial of maximal degree $\delta(q)$ of q. If d is a direction of the *front* of q (or of its projection on S^1), we will say that q is *"carried"* by d.

Let $x = u^\nu$, $K_\nu = C\{u\}[u^{-1}]$, *and* $\hat{K}_\nu = C[[u]][u^{-1}]$.

Let $\hat{\mathbb{L}}_\nu = \hat{K}_\nu < e^{q_1}, e^{q_2},..., e^{q_n} >$, and $\mathbb{L}_\nu = K_\nu < e^{q_1}, e^{q_2},..., e^{q_n} >$. As above we set $Aut_{\hat{K}_\nu}\hat{\mathbb{L}}_\nu = Aut_{K_\nu}\mathbb{L}_\nu = \mathcal{T}(q)$.

To each $q \in E(q)$, we can associate a *character* of the exponentiel torus $\mathcal{T}(q)$, that is a (continuous) homomorphism of groups (denoted still by q):

$$q: \ \mathcal{T}(q) \longrightarrow C^*$$

$$q: \ \ \theta \ \longrightarrow q(\theta), \text{ with}$$

$$(e^q)\,\theta = q(\theta)\, e^q \ \ (e^q \in \mathbb{L}_\nu \text{ and } \theta \text{ acts on } \mathbb{L}_\nu).$$

Let $(p_1, p_2,..., p_\nu)$ be a Z-*basis* of the lattice $E(q)$

We get an *isomorphism*

$$(p_1, p_2,..., p_\nu): \ \mathcal{T}(q) \longrightarrow (C^*)^\nu$$

$$(p_1, p_2,..., p_\nu): \quad \theta \ \longrightarrow (p_1(\theta), p_2(\theta),..., p_\nu(\theta)).$$

In the following the *exponential lattice* $E(q)$ will be identified with the *lattice of* **characters** *on the exponential torus* $\mathcal{T}(q)$.

Let $d \in (R,0)$ (the universal covering of $(S^1,1)$), we set

$E_d(q) = \{q \in E(q) \ / \ q$ is carried by $d\}$; $E_d(q)$ is a *semi-lattice* of $E(q)$, and depends clearly only on the projection d of d on the ν-*covering* of S^1:

$E_d(q) = E_d(q)$.

To the set $q = \{q_1, q_2,..., q_n\} \subset E$, after the choice of an ***ordering***, we associate the diagonal matrix e^Q, with $Q = Diag\{q_1, q_2,..., q_n\}$.

We will use *ordering relations* associated to a direction $d \in (\mathbf{R},0)$:

$q >>_d q'$, if and only if $q' - q \in E_d(q)$ (*i.e.* $q' - q$ is carried by d):

$q >_d q'$, if and only if $e^{q'-q}$ is *infinitely flat on* d;

$q \geq_d q'$, if and only if $e^{q'-q}$ is bounded on d.

Clearly, if $q >>_d q'$, then $q >_d q'$; and, if $q >_d q'$, then $q \geq_d q'$.

We will also use an *equivalence relation* on the space E associated to a rational number $k > 0$, $k \in \mathbf{Q}$:

$q =_k q'$ *if and only if* $\delta(q - q') < k$ (if $\delta(q - q') \geq k$, we will write $q \neq_k q'$).

To a rational number $k > 0$, we associate the *partition* of the set $q = \{q_1, q_2,..., q_n\}$, defined by the relation $=_k$. This partition is named the *"k-partition"*. The only "significative" values for k are in the set $\{k_1, k_2,..., k_r\} = N\Sigma(q)$ of values taken by $\delta(q_i - q_j)$ $(q_i \neq q_j)$. We will ***always*** suppose in the following that we have chosen an ordering on $q_1, q_2,..., q_n$ such that, for every $k>0$, $k \in \mathbf{Q}$, the elements of each subset of the k-partition are *consecutive*. Then, there exists a *unique block-decomposition* (by definition the *k-block-decomposition*) of the matrix Q, which is invariant by transposition, and inducing the *k-partition* on the diagonal. For $k = k_1, k_2,..., k_r$ we get, by definition, the "iterated block-decomposition" (cf. [BJL 1], [J]). If a matrix A admits the same k-block-decomposition than Q, we will say that A admits a *(Q,k)-block-structure*. Moreover, a direction d being fixed, it is possible to choose an *indexation* (called by definition a ***d**-indexation*) of the elements q_i of q such that:

$q_1 \leq_d q_2 \leq_d ... \leq_d q_n$. The corresponding ordering on q satisfies the above conditions; the corresponding iterated block-decomposition is named a *d-iterated block-decomposition.*

The set q and the direction d, being fixed, and an *order* (perhaps depending on d) being chosen on q, the diagonal matrix Q is defined. To this matrix and a ***fixed direction*** $d \in (\mathbf{R},0)$, we will associate families of subgroups of $GL(n;\mathbf{C})$, indexed by $k_m \in \{k_1, k_2,..., k_r\} = N\Sigma(q)$ (isotropy groups, and Stokes groups).

All these groups are ***unipotent***. More precisely, if P is a matrix in one of this group, all the diagonal terms of P are 1, and $I - P$ is ***nilpotent*** (if the order on q corresponds to a d-indexation, P is ***upper-triangular***).

Let $\Lambda(Q;d) = \{C = (c_{ij})$ / *if* $i=j$, $c_{ij} = 1$,*and, if* $i \neq j$, *and* $c_{ij} \neq 0$, *then* $q_i <_d q_j\}$;

$\Lambda(Q;d)$ is a subgroup of $GL(n;\mathbf{C})$, named the *isotropy subgroup* in the direction d.

Let $\mathrm{Sto}(Q;d) = \{C = (c_{ij})$ / *if* $i=j$, $c_{ij} = 1$,*and, if* $i \neq j$, *and* $c_{ij} \neq 0$, *then* $q_i <<_d q_j\}$;

$\mathrm{Sto}(Q;d)$ is a subgroup of $\Lambda(Q;d)$, named the *Stokes subgroup* in the direction d.

Let be now $k_m \in \{k_1, k_2,..., k_r\} = N\Sigma(q)$. We set:

$\Lambda^{\geq k_m}(Q;d) = \{C = (c_{ij}) \,/\, \text{if } i=j,\ c_{ij} = 1$, and, if $i \neq j$, and $c_{ij} \neq 0$, then $q_i <_d q_j$ and $q_i \neq_{k_m} q_j\}$;

$\Lambda^{k_m}(Q;d) = \{C = (c_{ij}) \,/\, \text{if } i=j,\ c_{ij} = 1$, and, if $i \neq j$, and $c_{ij} \neq 0$, then $q_i <_d q_j$, $q_i \neq_{k_m} q_j$ and $q_i =_{k_{m-1}} q_j\}$;

$\Lambda^{<k_m}(Q;d) = \{C = (c_{ij}) \,/\, \text{if } i=j,\ c_{ij} = 1$, *and, if* $i \neq j$, *and* $c_{ij} \neq 0$, *then* $q_i <_d q_j$ and $q_i =_{k_m} q_j\}$;

and

$\mathrm{Sto}^{\geq k_m}(Q;d) = \{C = (c_{ij}) \,/\, \text{if } i=j,\ c_{ij} = 1$, *and, if* $i \neq j$, *and* $c_{ij} \neq 0$, *then* $q_i <<_d q_j$ and $q_i \neq_{k_m} q_j\}$;

$\mathrm{Sto}^{k_m}(Q;d) = \{C = (c_{ij}) \,/\, \text{if } i=j,\ c_{ij} = 1$, *and, if* $i \neq j$, *and* $c_{ij} \neq 0$, *then* $q_i <<_d q_j$, $q_i \neq_{k_m} q_j$ and $q_i =_{k_{m-1}} q_j\}$;

$\mathrm{Sto}^{<k_m}(Q;d) = \{C = (c_{ij}) \,/\, \text{if } i=j,\ c_{ij} = 1$, *and, if* $i \neq j$, *and* $c_{ij} \neq 0$, *then* $q_i <<_d q_j$ and $q_i =_{k_m} q_j\}$.

Proposition 10.

Let Q be a diagonal matrix with entries in $\mathbf{E}$, and $d \in (\mathbf{R},0)$ be a fixed direction. Then, for every $k > 0$, $k \in \mathbf{Q}$, the four sequences

$$\{id\} \longrightarrow \Lambda^{\geq k_m}(Q;d) \longrightarrow \Lambda(Q;d) \longrightarrow \Lambda^{<k_m}(Q;d) \longrightarrow \{id\},$$
$$\{id\} \longrightarrow \Lambda^{k_m}(Q;d) \longrightarrow \Lambda^{\leq k_m}(Q;d) \longrightarrow \Lambda^{<k_m}(Q;d) \longrightarrow \{id\},$$
$$\{id\} \longrightarrow \mathrm{Sto}^{\geq k_m}(Q;d) \longrightarrow \mathrm{Sto}(Q;d) \longrightarrow \mathrm{Sto}^{<k_m}(Q;d) \longrightarrow \{id\},$$
$$\{id\} \longrightarrow \mathrm{Sto}^{k_m}(Q;d) \longrightarrow \mathrm{Sto}^{\leq k_m}(Q;d) \longrightarrow \mathrm{Sto}^{<k_m}(Q;d) \longrightarrow \{id\}$$

*are **exact sequences** of (algebraic) groups which are **split**.*

Maps are evident inclusions and evident "projections" (by "suppression" of some entries). The sequences are split by the inclusion maps $\Lambda^{<k_m}(Q;d) \longrightarrow \Lambda(Q;d)$,...

Proposition 9 is a set of *"block variations"* on the

Lemma 11.

Let D_n *be the subgroup of* $GL(n;\mathbf{C})$ *of diagonal invertible matrices. Let* T_n *be the subgroup of* $GL(n;\mathbf{C})$ *of upper triangular invertible matrices. Let* B_n *be the subgroup of* $GL(n;\mathbf{C})$ *of upper triangular unipotent matrices. Then we have a split exact sequence of groups:*

$$\{id\} \longrightarrow \mathrm{B}_n \longrightarrow \mathrm{T}_n \longrightarrow \mathrm{D}_n \longrightarrow \{id\}.$$

The map $\mathrm{T}_n \longrightarrow \mathrm{D}_n$ is the evident "projection" (we replace by zero the off diagonal entries), and the map $\mathrm{B}_n \longrightarrow \mathrm{T}_n$ is the natural injection; the natural inclusion $\mathrm{D}_n \longrightarrow \mathrm{T}_n$ gives the splitting.

Then T_n is the semi-direct product of B_n and D_n. We will write

$$\mathrm{T}_n = \mathrm{D}_n \ltimes \mathrm{B}_n \ ;$$

$\Lambda(Q;d)$ is the semi-direct product of $\Lambda^{\geq k_m}(Q;d)$ and $\Lambda^{<k_m}(Q;d)$, we will write

$$\Lambda(Q;d) = \Lambda^{<k_m}(Q;d) \ltimes \Lambda^{\geq k_m}(Q;d), \ldots$$

Lemma 12.

If $\{k_1, k_2, \ldots, k_r\} = \{\delta(q_i - q_j) / \ i, j = 1, \ldots, n$ *and* $q_i - q_j \neq 0\}$ $(k_1 > k_2 > \ldots > k_r > 0)$, *we have:*

$$\Lambda(Q;d) = \Lambda^{k_r}(Q;d) \ltimes \Lambda^{k_{r-1}}(Q;d) \ltimes \ldots \ltimes \Lambda^{k_1}(Q;d).$$

If $C \in \Lambda(Q;d)$, *there exists a* ***unique decomposition:***

$$C = C_r \ C_{r-1} \ldots C_1, \text{ with } C_i \in \Lambda^{k_i}(Q;d).$$

We can now go back to *linear differential equations*. We need a more precise version of *theorem 3*.

Let $\Delta = d/dx - A$, with $A \in End(n;\mathbf{C}\{x\}[x^{-1}])$, be a germ of *meromorphic differential operator* at the origin of the complex plane $\mathbf{C}$.

The operator Δ admits a ***formal** fundamental solution:*

$$\hat{F}(x) = \hat{H}(x)\, x^L U e^{Q(1/u)}, \text{ with:}$$

$u^\nu = x$ (for some $\nu \in \mathbf{N}^*$), $L \in End(n;\mathbf{C})$, *in Jordan form*, $\hat{H} \in GL(n;\mathbf{C}[[x]][x^{-1}])$, Q a diagonal matrix with entries in $u^{-1}\mathbf{C}[u^{-1}]$, Galois invariant, unique up to permutations of the diagonal entries, *and* $U \in End(n;\mathbf{C})$ a *"universal"* matrix (depending only on Q) [BJL 1], [J] (ν is choosen *minimal*).

Let $\hat{M} = U^{-1} e^{2i\pi L} U$. We have:

$$\hat{F}(e^{2i\pi} x) = \hat{H}(x)\, x^L U\, \hat{M}\, e^{Q(\exp(-2i\pi/\nu)/u)} = \hat{F}(x)\, \hat{M}, \text{ and}$$

$$e^{Q(\exp(-2i\pi/\nu)/u)} = \hat{M}^{-1} e^{Q(1/u)} \hat{M}. \text{ And } [\hat{M}^\nu, Q] = 0.$$

Theorem 7.

Let $\Delta = d/dx - A$, *with* $A \in End(n;\mathbf{C}\{x\}[x^{-1}])$, *be a germ of meromorphic differential operator at the origin of the complex plane* $\mathbf{C}$.

We denote by $k_1 > k_2 > \ldots > k_r$ *the positive (non zero) slopes of the Newton polygon of the (rank* n^2*) differential operator*

$$End\ \Delta = d/dx - [A,.].$$

Let $\hat{F}$ *be a* ***formal*** *fundamental solution of* Δ *as above. Then there exists a "natural decomposition" (unique up to "meromorphic transforms"* [Ra 4]*)*

$$\hat{H} = \hat{H}_1 \hat{H}_2 \ldots \hat{H}_r, \text{ where } \hat{H}_i \in GL(n;\mathbf{C}[[x]][x^{-1}]), \text{ is } k_i\text{-}\textbf{summable} \text{ for}$$

$i = 1, \ldots, r$, *and such that*

(i) $\hat{F}^i(x) = \hat{H}_i(x) \hat{H}_{i+1}(x) \ldots \hat{H}_r(x)\ x^L U\ e^{Q(1/u)}$ *is a* ***formal fundamental solution*** *of a* ***meromorphic*** *differential operator* $\Delta^i = d/dx - A^i$, *with* $A^i \in End(n;\ \mathbf{C}\{x\}[x^{-1}])$, *for* $i = 1, \ldots, r$;

(ii) *If* $\Sigma(\hat{F}) = \Sigma(\hat{H}) = \bigcup_{i=1,\ldots,r} \Sigma(\hat{H}_i)$, $H_{i;d} = S_{k_i;d}\ \hat{H}_i$ *(for* $i = 1, \ldots, r$*)*

and

$$H_d = H_{1;d}\, H_{2;d} \cdots H_{r;d}\,,$$

then, for $d \notin \Sigma(\hat{H})$, *and every determination of* $\mathrm{Log}\, x$ ($u = e^{(\mathrm{Log}\, x)/\nu}$ *and* $x^L = e^{L\, \mathrm{Log}\, x}$):

$F_d(x) = H_d(x)\, x^L U\, e^{Q(1/u)}$ *is an* ***actual*** *analytic fundamental solution of the operator* Δ *in a sector bisected by* d ($\mathbf{d} \in (\mathbf{R},0)$ *"above"* d corresponds to the given branch of Logarithm).

Moreover $\hat{H}^i$ *admits a* (Q,k_{i-1})*-block-structure* $(i=2,\dots,r)$ *and* A^i *admits a* (Q,k_i)*-block-structure* $(i=1,\dots,r)$.

We define $F^i_d(x) = H_{i;d}\, H_{i+1;d} \cdots H_{r;d}\, x^L\, U\, e^{Q(1/u)}$; $F^i_d(x)$ is an *actual analytic fundamental solution* of the operator Δ^i in a sector bisected by d $(i=1,\dots,r)$, and admits a (Q,k_{i-1})*-block-structure* $(i=2,\dots,r)$.

We have:

$$F^i_{\mathbf{d}} = H_{i;d}\, F^{i+1}_{\mathbf{d}} \quad (i=1,\dots,r-1), \text{ and we set } (i=1,\dots,r):$$

$$H_{i;d}^{+}\, F^{i+1}{}_{\mathbf{d}}^{+} = H_{i;d}^{-}\, F^{i+1}{}_{\mathbf{d}}^{+}\ \mathrm{S}_{i;\mathbf{d}}\,.$$

We have $\mathrm{S}_{i;\mathbf{d}} \in GL(n;\mathbf{C})$ $(i=1,\dots,r)$ and $\mathrm{St}_{\mathbf{d}} = \mathrm{S}_{r;\mathbf{d}}\, \mathrm{S}_{r-1;\mathbf{d}} \cdots \mathrm{S}_{1;\mathbf{d}}\,.$

Lemma 13.

Let $\mathbf{q} = \{q_1, q_2,\dots, q_n\} \subset \mathbf{E}$, *and, after an ordering, let* Q *be the diagonal matrix* $Q = \mathrm{Diag}\{q_1, q_2,\dots, q_n\}$. *Let* $\mathrm{C} \in End(n;\mathbf{C})$, *and* $\mathbf{d}$ *a fixed direction* ($\mathbf{d} \in (\mathbf{R},0)$):

(i) The following conditions are equivalent:

(a) $e^Q\, \mathrm{C}\, e^{-Q} = I + \Phi$, *with* Φ ***infinitely flat*** *on* $\mathbf{d}$.

(b) $\mathrm{C} \in \Lambda(Q;\mathbf{d})$.

(ii) The following conditions are equivalent:

(a) $e^Q\, \mathrm{C}\, e^{-Q} = I + \Phi$, *with* Φ ***exponentially flat of order*** $\geq k$ *on* $\mathbf{d}$.

(b) $\mathrm{C} \in \Lambda^{\geq k}(Q;\mathbf{d})$.

(iii) The following conditions are equivalent:

(a) $e^Q\, \mathrm{C}\, e^{-Q} = I + \Phi$, *with* Φ ***exponentially flat of order exactly*** k *on* $\mathbf{d}$.

(b) $\mathrm{C} \in \Lambda^{k}(Q;\mathbf{d})$.

(iv) The following conditions are equivalent:

(a) $e^Q\, \mathrm{C}\, e^{-Q} = I + \Phi$, *with* Φ ***exponentially flat of order*** $\geq k$ *on an open sector* ***with opening*** π/k, ***bisected*** *by* $\mathbf{d}$.

(b) $e^Q\, \mathrm{C}\, e^{-Q} = I + \Phi$, *with* Φ ***exponentially flat of order exactly*** k *on an open sector* ***with opening*** π/k, ***bisected*** *by* $\mathbf{d}$.

(c) $\mathrm{C} \in \mathrm{Sto}^k(Q;\mathbf{d})$.

Theorem 8.

Let $\Delta = d/dx - A$, *with* $A \in End(n;\mathbf{C}\{x\}[x^{-1}])$, *be a germ of meromorphic differential operator at the origin of the complex plane* $\mathbf{C}$.

We denote by $k_1 > k_2 > \ldots > k_r$ *the positive (non zero) slopes of the Newton polygon of the differential operator*

$$End\ \Delta = d/dx - [A,.].$$

Let $\hat{F}(x) = \hat{H}(x)\, x^L\, U\, e^{Q(1/u)}$, *be a* ***formal*** *fundamental solution of* Δ *as above, and* $\hat{H} = \hat{H}_1\hat{H}_2 \ldots \hat{H}_r$, *a decomposition like in theorem 7.*

Let $S_{i;d} \in GL(n;\mathbf{C})$ $(i=1,\ldots,r)$ *defined as above. Then:*

(i) $S_{i;d} \in Sto^{k_i}(Q;\mathbf{d})$ $(i=1,\ldots,r)$.

(ii) $S_d \in Sto(Q;\mathbf{d})$ *and* $St_d = S_{r;d}\, S_{r-1;d} \ldots S_{1;d}$ *is the unique decomposition of* S_d *corresponding to* $\Lambda(Q;\mathbf{d}) = \Lambda^{k_r}(Q;\mathbf{d}) \ltimes \Lambda^{k_{r-1}}(Q;\mathrm{d}) \ltimes \ldots \ltimes \Lambda^{k_1}(Q;\mathbf{d})$.

Assertion *(i)* is a consequence of *lemma 13 (iv):*

We have $(H_{i;d}^{-})^{-1} H_{i;d}^{+} = I + \Psi$, with Ψ *exponentially flat of order* $\geq k_i$ *on an open sector, with opening* π/k, *bisected by* d *(*H_i is k_i*-summable).* We set

$G_i = H_{i+1;d} \ldots H_{r;d}\, x^L U$; it is clear that G_i and G_i^{-1} are analytic on an open sector, with opening π/k_{i+1} ($\pi/k_{i+1} > \pi/k_i$), bisected by d, and admit a moderate growth at the origin on this sector. Then $e^Q\, S_{i;d}\, e^{-Q} = G_i (I + \Psi) G_i^{-1} = I + \Phi$, where Φ *is exponentially flat of order* $\geq k_i$ on an open sector, with *opening* π/k, *bisected by* d. Assertion *(ii)* follows from *(i)* and *lemma 12.*

The Stokes matrices $S_{i;d}$ are *a priori* defined in a *transcendental* way. *Theorem 8* says that we can get them by an *algebraic algorithm,* from the *knowledge of* S_d ***and*** *Q.* We will give later an *"infinitysimal version"* of this computation.

Lemma 14.

Let $k'_1 > k'_2 > \ldots > k'_{r'} > k' > 0$. *Let* $d = \mathbf{R}^+$. *Then:*

$$e^{-1/x^{k'}} = L_{k'_1;d}\, A_{k'_1,k'_2;d} \ldots A_{k'_{r-1},k'_r;d}\, B_{k'_{r'}}\, (e^{-1/x^{k'}}).$$

From this *lemma* and *theorem 8,* we get

Theorem 9.

Let $\Delta = d/dx - A$, *with* $A \in End(n;\mathbf{C}\{x\}[x^{-1}])$, *be a germ of meromorphic differential operator at the origin of the complex plane* $\mathbf{C}$.

We denote by $k_1 > k_2 > \ldots > k_r$ *the positive (non zero) slopes of the Newton polygon of the differential operator*

$$End\ \Delta = d/dx - [A,.].$$

Let $\hat{F}(x) = \hat{H}(x)\, x^L U\, e^{Q(1/u)}$, *be a* ***formal*** *fundamental solution of* Δ *as above, and* $\hat{H} = \hat{H}_1\hat{H}_2 \ldots \hat{H}_r$, *a decomposition like in theorem 7.*

Let $S_{i;d} \in GL(n;\mathbf{C})$ $(i=1,\ldots,r)$ *defined as above.*

Let $\varepsilon = (\varepsilon_1, \varepsilon_2,\ldots,\varepsilon_r)$, *and* $\varepsilon' = (\varepsilon'_1, \varepsilon'_2,\ldots,\varepsilon'_r)$, *with* ε_i, $\varepsilon'_i \in \{1,-1\}$ $(i = 1,\ldots,r)$.

Then, for every direction $d \in (\mathbf{R},0)$*:*

(i) $\hat{H}$ *is* $(k_1,k_2,\ldots,k_r)$***-summable along the paths*** $(d;\varepsilon)$ *and* $(d;\varepsilon')$.

(ii) If $S^{\varepsilon}_{k_1,k_2,\ldots,k_r;d}\,\hat{F} = S^{\varepsilon'}_{k_1,k_2,\ldots,k_r;d}\,\hat{F}\ \mathrm{St}_d^{\varepsilon,\varepsilon'}$,

$\mathrm{St}_d^{\varepsilon,\varepsilon'} \in GL(n;\mathbf{C})$, *and, if* $\varepsilon = (-,-,\ldots,-) = -$*: then*

$$\mathrm{St}_d^{\varepsilon,\varepsilon'} = \varepsilon'(S_{r;d})\,\varepsilon'(S_{r-1;d})\ldots\varepsilon'(S_{1;d}),\ \text{with}$$

$$\varepsilon'(S_{i;d}) = S_{i;d}\ \text{if}\ \varepsilon'_i = +,\ \text{and}\ \varepsilon'(S_{i;d}) = I\ \text{if}\ \varepsilon'_i = -.$$

(iii) If $\varepsilon = (-,-,\ldots,-)$ *and* $\varepsilon' = (-,-,\ldots,+,\ldots,-)$, *with a* $+$ *only at the index i, then*

$\mathrm{St}_d^{\varepsilon,\varepsilon'} = S_{i;d}$, *and* $S_{i;d}$ ***is in the representation in*** $GL(n;C)$ ***of the differential Galois group*** $Gal_K(\Delta)$ *given by the fundamental formal solution* $\hat{F}$ $(i=1,\ldots,r)$.

We will write $S_{i;d} = \mathrm{St}_{d;k_i}$.

Our aim now is to use the preceding results and considerations to give a ***"purely combinatorial" description*** of the category of ***germs of meromorphic connections*** at the origin of the complex plane, as simple as possible. In "down to earth terms" a germ of meromorphic connection is a germ of differential system *up to meromorphic equivalence* [De 1], [Ma 4], [MR 2]; so the searched combinatorial description is equivalent to a *meromorphic classification of germs of differential systems.*

Such a result is well known for the *regular singular* case; it is given by the ***Riemann-Hilbert correspondence*** [De 1], [Ka 2],[MR 2]:

Germs of ***Fuchsian*** *connections at the origin of* **C**.	$\longrightarrow$	*Finite dimensional linear representations of the local fundamental group*[1].
Germ of meromorphic fuchsian differential operator Δ, *up to meromorphic equivalence.*	$\longrightarrow$	*Monodromy* $M(\Delta)$ *"around 0" up to conjugation*

This map is ***bijective***, moreover it is an *equivalence of Tannakian categories* [Saa], [De Mi], [De 2]. The result is *false* if we *suppress* the *fuchsian* hypothesis.

The now "classical" meromorphic classification of germs of meromorphic differential operators is given in terms of *cohomology of sheaves of groups* (isotropy groups of a "normal form") on S^1 [Si], [Ma 3], [Ma 4], [De 3], [MR 1][2]. We have in mind a *"better"* description (adapted in particular to the computation of the Galois differential groups), *extending the Riemann-Hilbert correspondence to the irregular case*, that is a description of connections in terms of ***representations of groups:***

Germs of connections $\longrightarrow$ *Finite dimensional linear representations*

[1] Generated by a *loop* turning "one time" around the origin and isomorphic to **Z**.
[2] We will recall this description in part *5*.

at the origin of $\mathbf{C}$.		*of the local "**wild** fundamental group".*
Germ of meromorphic differential operator Δ, *up to meromorphic equivalence.*	$\longrightarrow$	*????*

We will call *"Gevrey front"* of $q \in \boldsymbol{E}$ the set

$Gfr\ q = \{(d,k)/d \in Fr\ q,\ k = \delta(q)\} \subset \widetilde{HA}_0$, universal covering of the analytic halo HA_0.

Let

$Fr\,(\boldsymbol{q}) = \bigcup_{i,j} Fr\ q_{i,j}\ \ (q_{i,j} = q_i - q_j),$

$Gfr\,(\boldsymbol{q}) = \bigcup_{i,j} Gfr\ q_{i,j}\,,\ \ \Sigma(\boldsymbol{q})$ the *projection* on S^1 of $Fr\,(\boldsymbol{q})$.

We define an action of the free group (γ_0) generated[1] by γ_0 on the (non abelian) free group generated by the γ_d $(d \in Fr\,(\boldsymbol{q}))$ by

$\gamma_0 : \gamma_d \longrightarrow \gamma_{exp(-2i\pi)d}$ $(exp(-2i\pi):$ it is a translation of -2π in $(\boldsymbol{R},0))$.

We denote by $\Pi(\boldsymbol{q})$ the corresponding *semi-direct* product $\Pi = (\gamma_0) \times (\underset{d\in Fr(\boldsymbol{q})}{*} (\gamma_d))$

In $\Pi(\boldsymbol{q})$ we have $\gamma_0\ \gamma_d\ \gamma_0^{-1} = \gamma_{exp(-2i\pi)d}$.

We define an action of the free group (γ_0) generated by γ_0 on the (non abelian) free group generated by the γ_a's $(a \in Gfr\,(\boldsymbol{q}))$ by

$$\gamma_0 : \gamma_a \longrightarrow \gamma_{exp(-2i\pi)a}\ \ (a=(d,k),\ exp(-2i\pi)a=(exp(-2i\pi)d,k)).$$

We denote by $G\Pi(\boldsymbol{q})$ the corresponding *semi-direct* product

$G\Pi(\boldsymbol{q}) = (\gamma_0) \times (\underset{a\in Gfr(\boldsymbol{q})}{*} (\gamma_a))$

In $G\Pi(\boldsymbol{q})$ we have $\gamma_0\ \gamma_a\ \gamma_0^{-1} = \gamma_{exp(-2i\pi)a}$.

The groups $\Pi(\boldsymbol{q})$, and $G\Pi(\boldsymbol{q})$ are "first approximations" of the "wild local fundamental group".[2] We can identify $\Pi(\boldsymbol{q})$ to a subgroup of $G\Pi(\boldsymbol{q})$ by

$$\gamma_d = \gamma_{a_r} * \gamma_{a_{r-1}} * \dots * \gamma_{a_1}\ \ (a_\iota = (d,k_\iota);\iota = 1,\dots,r).$$

We will obtain below a classification in terms of *linear representations* of these groups[3]. Unfortunately there are ***conditions** ("Stokes conditions")* on the representations in order that *they come from a connection.* That is *unsatisfying:* we want a *"wild fundamental group"* whose ***all*** finite dimensional linear representations come from a connection, like in the Riemann-Hilbert correspondence. We will be led to the "good" group $\pi_{1,s}(\mathbf{C}^*,0)$ by a ***"Fourier analysis"*** of the (Galois differential)*"unfolding"* of the

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[1] Here γ_0 and the γ_d, γ_a are "labels"; later γ_0 and γ_a will be interpreted as *loops* turning around respectively 0 and a.

[2] The terminology "wild π_1" (in french "π_1–sauvage") was suggested to the second author by *Malgrange* for the group $G\Pi$ [Ma 7].

[3] If we consider *"isoformal"* families, that is if we fix the *"formal form"*. If we leave it free, we need to "add" a representation of the "formal fundamental group".

Stokes phenomena under the ***adjoint action of the exponential torus***. Moreover we will see that this approach gives[1] a *very natural* interpretation of *Ecalle*'s ***resurgence*** [E 4].

Let $\Delta = d/dx - A$, *with* $A \in End(n;\mathbf{C}\{x\}[x^{-1}])$, *be a germ of meromorphic differential operator at the origin of the complex plane* $\mathbf{C}$.

Let $\hat{F}(x) = \hat{H}(x)\, x^L\, U\, e^{Q(1/u)}$, *be a* ***formal*** *fundamental solution of* Δ *as above. We set*

$F_0(x) = x^L\, U\, e^{Q(1/u)}$.

For $\hat{P} \in GL(n;\mathbf{C}[[x]][x^{-1}])$, we set $A^{\hat{P}} = \hat{P}A\hat{P}^{-1} + \frac{d\hat{P}}{dx}\hat{P}^{-1}$ and $\Delta^{\hat{P}} = d/dx - A^{\hat{P}}$, and we say that *the differential operators* Δ and $\Delta^{\hat{P}}$ are ***formally*** *equivalent*. If $P \in GL(n;\mathbf{C}\{x\}[x^{-1}])$, we will say that *the differential operators* Δ and Δ^{P} are ***analytically*** *equivalent*. We have $(\Delta^{\hat{P}_2})^{\hat{P}_1} = \Delta^{\hat{P}_1\hat{P}_2}$.

It is easy to check [BJL 1] that F_0 is a fundamental solution of a *rational differential operator* $\Delta_0 = d/dx - A_0$, *with* $A_0 \in End(n;\mathbf{C}(x)[x^{-1}]$, *which is formally equivalent* to Δ $(\Delta = \Delta_0^{\hat{H}})$.

We will define:

$$\mathcal{I}_0(\hat{F}) = \{C \in GL(n;\mathbf{C}) /\ C\hat{F} = \hat{F}C\}, \text{ and}$$

$\mathcal{I}(\hat{F}) = \{C \in GL(n;\mathbf{C}) /$ *there exists* $\hat{G} \in GL(n;\mathbf{C}[[x]][x^{-1}])$ *such that* $\hat{G}\hat{F} = \hat{F}C\}$;

$\mathcal{I}_0(\hat{F})$ and $\mathcal{I}(\hat{F})$ are *algebraic subgroups* of $GL(n;\mathbf{C})$ [BV], and $\mathcal{I}_0(\hat{F}) \subset \mathcal{I}(\hat{F})$.

We set:

$\mathcal{I}(\Delta) = \{G \in GL(n;\mathbf{C}[[x]][x^{-1}]) /$ *there exists* $C \in GL(n;\mathbf{C})$ *such that* $\hat{G}\hat{F} = \hat{F}C\}$;

$\mathcal{I}(\Delta)$ is a subgroup of $GL(n;\mathbf{C}[[x]][x^{-1}])$. It is easy to check that $\mathcal{I}(\Delta_0)$ is a subgroup of $GL(n;\mathbf{C}(x)[x^{-1}])$ *containing* $\mathcal{I}_0(F_0)$. It is clear that $\Delta^{\hat{G}} = \Delta$ is equivalent to $\hat{G} \in \mathcal{I}(\Delta)$ ($\mathcal{I}(\Delta)$ is *independant* of the choice of $\hat{F}$).

We leave now Δ_0 *fixed*, and we want to ***classify***, ***up to meromorphic equivalence***, all the ***meromorphic differential operators*** Δ ***formally equivalent to*** Δ_0. Moreover we are also interested in the classification of the *"**marked pairs**"* $(\Delta,\hat{H})$ such that $\Delta^{\hat{H}} = \Delta_0$.

To a differential operator Δ *formally equivalent to* Δ_0 (a *fundamental solution* F_0 of Δ_0 being *fixed*) we can associate representations $\rho_{irr}(\Delta)$ of the groups $\Pi(\mathbf{q})$ and $G\Pi(\mathbf{q})$ in $GL(n;\mathbf{C})$ defined by:

$$\rho_{irr}(\Delta)(\gamma_0) = \hat{M},\ \rho_{irr}(\Delta)(\gamma_d) = St_d(\Delta),\ \rho_{irr}(\Delta)(\gamma_a) = St_{d,k}(\Delta)$$

$(a=(d,k))$. (We use the formulae:

[1] With the tools of part *6*, this approach will lead to an essentially "geometric" description of the resurgence where Laplace transform and convolution no longer play the central characters...The second author was led to this description in particular by *Malgange's* description of a part of *Ecalle's* work [Ma 8].

$$\hat{M}\mathrm{St}_d(\Delta)\hat{M}^{-1} = \mathrm{St}_{exp(-2i\pi)d}(\Delta), \text{ and } \hat{M}\mathrm{St}_a(\Delta)\hat{M}^{-1} = \mathrm{St}_{exp(-2i\pi)a}(\Delta).)$$

These representations are clearly submitted to the ***constraints:***

$\rho_{irr}(\Delta)(\gamma_d) \in \mathrm{Sto}(Q;d)$, and $\rho_{irr}(\Delta)(\gamma_a) \in \mathrm{Sto}^k(Q;d)$ $(a=(d,k))$. We will name these conditions *"Stokes conditions"*. These representations are defined up the action (by conjugation) of $\mathcal{U}(F_0)$: if $\hat{F} = \hat{H} F_0$ is a formal fundamental solution of Δ, and C an element of $\mathcal{U}(F_0)$, $\hat{G}$ the corresponding element of $\mathcal{U}(\Delta_0)$, then $\hat{F}$ C= $\hat{H}F_0$ C = $\hat{H}\hat{G}F_0$, is also a formal fundamental solution of Δ. They do *not* change if we replace Δ by a *meromorphically* equivalent operator ($\hat{H}$ is then changed in $P\hat{H}$, with $P \in GL(n;C\{x\})$, and $\rho_{irr}(\Delta)$ depends *only* on the *connection* ∇ associated to Δ; we can set $\rho_{irr}(\nabla) = \rho_{irr}(\Delta)$.

Theorem 10.

*Let Δ_0 be a fixed differential operator with a fundamental solution $F_0 = x^L U e^{Q(1/u)}$. We denote by ∇_0 the meromorphic connection defined by Δ_0. We set **$q = q(Q)$**, and denote by n the rank of Δ_0.*

(i) The natural map

$$\begin{array}{ccc} \textit{Meromorphic connections } \nabla \textit{ formally equivalent to } \nabla_0. & \xrightarrow{\rho_{irr}} & \textit{Representations of the group } G\Pi(q) \textit{ in } GL(n;C) \textit{, satisfying the Stokes conditions, up to the action of } \mathcal{U}(F_0). \\ \nabla & \longrightarrow & \rho_{irr}(\nabla), \end{array}$$

*is a **bijection**.*

(ii) The natural map

$$\begin{array}{ccc} \textit{Meromorphic connections } \nabla \textit{ formally equivalent to } \nabla_0. & \xrightarrow{\rho_{irr}} & \textit{Representations of the group } \Pi(q) \textit{ in } GL(n;C) \textit{, satisfying the Stokes conditions, up to the action of } \mathcal{U}(F_0). \end{array}$$

*is a **bijection**.*

This result is ***non trivial***. We will deduce its proof from the (non trivial...) *classification of isoformal meromorphic connections* in the form given by *Malgrange* and *Sibuya*, [Ma 3], [Si] [1]. We need before to recall some definitions and results (we will return to this topic in more details in *5*). In the following we will systematically consider a function f (with values in a C-vector space), holomorphic on an *open sector* V as an "object" on the *open arc* U corresponding to V in S^1 (the real analytic blow-up of the

[1] The first general classification (after the work of *Birkhoff* for the "generic case") is in [BJL 2].

origin in $\mathbf{C}$) as in [Ma 3]. We define this way on S^1 the sheaf $\mathcal{A}$ of holomorphic functions (with values in $\mathbf{C}$) on sectors, admitting an asymptotic expansion at the origin (with Taylor expansion in $\mathbf{C}[[x]][x^{-1}]$). We denote by Λ_I the subsheaf of. $End(n;\mathcal{A})$ of germs of analytic matrices ***asymptotic to identity;*** Λ_I is a sheaf of (non abelian) *groups*. If $\mathcal{F}$ is a sheaf on S^1, we will denote by $\mathcal{F}_d$ its fiber at $d \in S^1$.

Theorem 11.*(Malgrange, Sibuya* [Ma 3], [Si].)

There exists a natural isomorphism

$$GL(n;\mathbf{C}\{x\}[x^{-1}])\setminus GL(n;\mathbf{C}[[x]][x^{-1}]) \xrightarrow{\mu} H^1(S^1;\Lambda_I).$$

We recall the definition of the *Malgrange-Sibuya map* μ:

Let $\mathbf{U} = \{U_i\}_{i\in \mathbf{I}}$ be a finite open covering of S^1 by open arcs. We suppose that $U_i \cap U_j \cap U_k = \varnothing$, if $i,j,k \in \mathbf{I}$ are distinct.[1]

Let $\hat{A} \in GL(n;\mathbf{C}[[x]][x^{-1}])$. By *Borel-Ritt theorem* [Wa], we can *"represent"* $\hat{A}$ by a collection $\{A_i\}_{i\in \mathbf{I}}$ (A_i being a *holomorphic* matrix on an open sector V_i corresponding to U_i, with $\hat{A}$ as *asymptotic expansion* at the origin).

We consider $\{A_i\}_{i\in \mathbf{I}}$ as a *0-cochain* (with values in $GL(n;\mathcal{A})$) and we take its *coboundary*

$\delta = \{A_j^{-1}A_i\}_{i,j\in \mathbf{I}} \in Z^1(\mathbf{U};GL(n;\mathcal{A}))$. We have $\delta \in Z^1(\mathbf{U};\Lambda_I)$ (A_i and A_j have the *same asymptotic expansion* $\hat{A}$). We denote $A_j^{-1}A_i = A_{i,j}$.

By definition $\mu(\hat{A})$ is the image of δ in $H^1(S^1;\Lambda_I)$. *If* $P \in GL(n;\mathbf{C}\{x\})$, and $\hat{B} = P\hat{A}$, we can choose $A_i = PA_i$; then $\mu(\hat{B}) = \mu(\hat{A})$. In the following we will set

$\mathbf{I} = [1,...,p]$ ("$p+1=1$"), the bijection between $\mathbf{I}$ and $[1,...,p]$ being chosen such that $U_{\iota,\iota+1} = U_\iota \cap U_{\iota+1} \neq \varnothing$ $(\iota=1,...,p)$, and such that the bisecting lines of the arcs $U_{\iota,\iota+1}$ *turn* ***clockwise,*** when ι increases.

If $\Sigma = \{d_1, d_2..., d_p\} \subset S^1$, we will say that the covering $\mathbf{U}$ is *"adapted"* to Σ if

$$U_{\iota,\iota+1} = U_\iota \cap U_{\iota+1} \cap \Sigma = \{d_\iota\} \ (\iota=1,...,p).$$

Let $k_1 > k_2 > ... > k_r > 0$. Let $A \in GL(n;\mathbf{C}\{x\}_{1/k_1,1/k_2,...,1/k_r}[x^{-1}])$.

If[2] $\Sigma = \Sigma(\hat{A}) = \{d_1, d_2..., d_p\}$, we can built a covering $\mathbf{U} = \{U_\iota\}_{\iota\in \mathbf{I}}$, *adapted to* Σ, with $U_\iota \cap U_{\iota+1}$ *bisected* by d, with *opening* $\leq \pi/k_1$ $(\iota=1,...,p)$; such a covering is said k_1*-adapted to* Σ. We can choose

$A_\iota = S_{k_1,k_2,...,k_r;d}\,\hat{A}$ ($d \in U_\iota$, *arbitrary*[3] *between* d_ι *and* $d_{\iota+1}$; $\iota=1,...,p$). Then the 1-cocycle

[1] We will make this hypothesis for *all* the coverings in the following.
[2] More generally we can also take $\Sigma(A) \subset \Sigma$ *finite.*
[3] The values of A_i obtained for the different d glue together by analytic continuation in an analytic matrix always denoted A_i.

$\mathbf{St}(\boldsymbol{U};\hat{A}) = \{A_{\iota+1}{}^{-1}A_\iota\}_{\iota\in\boldsymbol{I}}$ is well defined ; the image of $\mathbf{St}(\boldsymbol{U};\hat{A})$ in $H^1(S^1;\Lambda_I)$ is clearly $\mu(\hat{A})$. We will denote by $\mathbf{St}(\hat{A})$ the 1-cocycle $\mathbf{St}(\boldsymbol{U};\hat{A})$ up to the choice of $\boldsymbol{U}$ (satisfying our hypothesis), and *identify* it to the set of groups $\{(A_{\iota,\iota+1})_{d_\iota}\}_{\iota\in\boldsymbol{I}}$.

If $\boldsymbol{U}$ is an open covering of S^1, and $\mathcal{F}$ a sheaf of groups on S^1,we denote by

$$i_{\boldsymbol{U}}: Z^1(\boldsymbol{U};\mathcal{A}) \longrightarrow H^1(S^1;\mathcal{F})$$ the *natural injection.*

Let $k > 0$. We denote by $\Lambda^{\geq k}$ the subsheaf of Λ_I of germs $I + \Phi$ *with* Φ ***exponentially flat of order*** $\geq k$.

Definition 4.

Let $k > 0$. Let $\Sigma = \{d_1, d_2..., d_p\} \subset S^1$, and an open covering $\boldsymbol{U}$ "adapted" to Σ. A 1-cochain $\delta \in C^1(\boldsymbol{U};\Lambda_I)$ is said "k-summable", if $\delta = \{A_{\iota,\iota+1}\}_{\iota\in\boldsymbol{I}}$, with $A_{\iota,\iota+1} \in \Gamma(U_{\iota,\iota+1};\Lambda^{\geq k})$, and if each $A_{\iota,\iota+1}$ can be (uniquely of course) "analytically" extended in an element of $\Gamma(V_{\iota,\iota+1};\Lambda^{\geq k})$ where $V_{\iota,\iota+1}$ is an open arc of $(\boldsymbol{R},0)$ with ***opening*** *π/k "containing" $U_{\iota,\iota+1}$ ($\iota=1,...,p$).*

We will denote by $H^{1;\geq k}(S^1;\Lambda^{\geq k}) \subset H^1(S^1;\Lambda_I)$ the subset of the images of the *k-summable 1-cocycles.*

Theorem 12.*(Martinet-Ramis* [MR 1], I-*6.*)

Let $k > 0$.

(i) The Malgrange-Sibuya isomorphism

$$GL(n; \mathbf{C}\{x\}[x^{-1}]) \backslash GL(n;\mathbf{C}[[x]][x^{-1}]) \xrightarrow{\mu} H^1(S^1;\Lambda_I).$$

induces an isomorphism

$$GL(n; \mathbf{C}\{x\}[x^{-1}]) \backslash GL(n;\mathbf{C}\{x\}_{1/k}[x^{-1}]) \xrightarrow{\mu} H^{1;\geq k}(S^1;\Lambda^{\geq k}).$$

(ii) If $\delta \in Z^1(\boldsymbol{U};\Lambda^{\geq k})$ is a k-summable 1-cocycle, then $\mathbf{St}(\boldsymbol{U};\mu^{-1}i_{\boldsymbol{U}}(\delta)) = \delta$.

Let now Δ be a differential operator; we denote by $\Lambda(\Delta_0)$ the sheaf (on S^1) of solutions of $End\,\Delta$ and $\Lambda_I(\Delta)$ the subsheaf of solutions of $End\,\Delta$ ***asymptotic to identity;*** $\Lambda_I(\Delta_0)$ is a subsheaf of Λ_I.

Let now Δ_0 be a differential operator with a fundamental solution $F_0 = x^L \boldsymbol{U}\, e^{Q(1/u)}$; we denote by ∇_0 the meromorphic connection defined by Δ_0 , $\boldsymbol{q} = \boldsymbol{q}(Q)$, $N\Sigma(\boldsymbol{q}) = \{k_1, k_2,..., k_r\}$ the set of values taken by $\delta(q_i - q_j)$ $(q_i \neq q_j)$, and n the rank of Δ_0 . Let $End\,\Delta_0 = d/dx - [A_0,.]$.

Let $\boldsymbol{d} \in (\boldsymbol{R},0)$, be a direction and $d \in S^1$ its projection. To the choice of $\boldsymbol{d} \in (\boldsymbol{R},0)$ is associated a *"branch"* of Logarithm and a *"sum"* $F_{0,\boldsymbol{d}}$ of $F_0 = x^L \boldsymbol{U} e^{Q(1/u)}$, analytic on an open sector bisected by d.

The map

$$\lambda_{\boldsymbol{d}} : GL(n;\boldsymbol{C}) \longrightarrow \Lambda(\Delta_0)_d$$
$$\lambda_{\boldsymbol{d}} : \quad C \longrightarrow F_{0,\boldsymbol{d}}\, C\, (F_{0,\boldsymbol{d}})^{-1}$$

is an ***isomorphism*** *of groups.*

Let

$$\Lambda(\Delta_0;\boldsymbol{d};F_0) = \lambda_{\boldsymbol{d}}(\Lambda(Q;d))$$
$$\Lambda^k(\Delta_0;\boldsymbol{d};F_0) = \lambda_{\boldsymbol{d}}(\Lambda^k(Q;d))$$
$$\Lambda^{\geq k}(\Delta_0;\boldsymbol{d};F_0) = \lambda_{\boldsymbol{d}}(\Lambda^{\geq k}(Q;d))$$
$$\Lambda^{<k}(\Delta_0;\boldsymbol{d};F_0) = \lambda_{\boldsymbol{d}}(\Lambda^{<k}(Q;d)).$$

It is easy to see that $\Lambda(\Delta_0;\boldsymbol{d};F_0)$, $\Lambda^k(\Delta_0;\boldsymbol{d};F_0)$, $\Lambda^{\geq k}(\Delta_0;\boldsymbol{d};F_0)$, and $\Lambda^{<k}(\Delta_0;\boldsymbol{d};F_0)$ *does **not** depend on the choice of* F_0 and $\boldsymbol{d}$; moreover $\Lambda(\Delta_0;\boldsymbol{d};F_0) = \Lambda_1(\Delta_0)_d$. We can set:

$\Lambda^k(\Delta_0;\boldsymbol{d};F_0) = \Lambda^k(\Delta_0)_d$, $\Lambda^{\geq k}(\Delta_0;\boldsymbol{d};F_0) = \Lambda^{\geq k}(\Delta_0)_d$, $\Lambda^{<k}(\Delta_0;\boldsymbol{d};F_0) = \Lambda^{<k}(\Delta_0)_d$.

All these *groups*[1] are *subgroups* of $\Lambda(\Delta_0)_d$, and, when the direction d varies, we get *subsheaves* $\Lambda^k(\Delta_0)$, $\Lambda^{\geq k}(\Delta_0)$, and $\Lambda^{<k}(\Delta_0)$ of $\Lambda_1(\Delta_0)$. (When d moves the groups remain "in general" the *"same"*. They can *"jump"* only for a finite set of values of d, the *"Stokes lines"*.)

Let

$$Sto(\Delta_0;\boldsymbol{d};F_0) = \lambda_{\boldsymbol{d}}(\mathrm{Sto}(Q;d))$$
$$Sto^k(\Delta_0;\boldsymbol{d};F_0) = \lambda_{\boldsymbol{d}}(\mathrm{Sto}^k(Q;d))$$
$$Sto^{\geq k}(\Delta_0;\boldsymbol{d};F_0) = \lambda_{\boldsymbol{d}}(\mathrm{Sto}^{\geq k}(Q;d))$$
$$Sto^{<k}(\Delta_0;\boldsymbol{d};F_0) = \lambda_{\boldsymbol{d}}(\mathrm{Sto}^{<k}(Q;d)).$$

It is easy to see that $Sto(\Delta_0;\boldsymbol{d};F_0)$, $Sto^k(\Delta_0;\boldsymbol{d};F_0)$, $Sto^{\geq k}(\Delta_0;\boldsymbol{d};F_0)$, and $Sto^{<k}(\Delta_0;\boldsymbol{d};F_0)$ *does **not** depend on the choice of* F_0 and $\boldsymbol{d}$. We can set:

$Sto(\Delta_0;\boldsymbol{d};F_0) = Sto(\Delta_0)_d$, $Sto^k(\Delta_0;\boldsymbol{d};F_0) = Sto^k(\Delta_0)_d$, $Sto^{\geq k}(\Delta_0;\boldsymbol{d};F_0) = Sto^{\geq k}(\Delta_0)_d$, $Sto^{<k}(\Delta_0;\boldsymbol{d};F_0) = Sto^{<k}(\Delta_0)_d$. If $d \notin \Sigma(\Delta_0)$, $Sto(\Delta_0)_d$ is *reduced to identity.*

From *proposition 10* and *lemma 12*, we get

Proposition 11.

Let $d \in S^1$ and $k > 0$. Let Δ_0 be a given differential operator with a fixed fundamental solution $F_0 = x^L \boldsymbol{U} e^{Q(1/u)}$. We set $\boldsymbol{q} = \boldsymbol{q}(Q)$, and $N\Sigma(\boldsymbol{q}) = \{k_1, k_2 \ldots, k_r\}$ $(k_1 > k_2 > \ldots > k_r)$. Then:

(i) The four sequences

[1] It is possible to give a "direct" definition of these groups, using *Deligne I-filtered structures* (or *Stokes structures*) [Ma 4], [De 3], [De 4].

$$\{id\} \longrightarrow \Lambda^{\geq k}(\Delta_0)_d \longrightarrow \Lambda(\Delta_0)_d \longrightarrow \Lambda^{<k}(\Delta_0)_d \longrightarrow \{id\},$$
$$\{id\} \longrightarrow \Lambda^{k}(\Delta_0)_d \longrightarrow \Lambda^{\geq k}(\Delta_0)_d \longrightarrow \Lambda^{<k}(\Delta_0)_d \longrightarrow \{id\},$$
$$\{id\} \longrightarrow Sto^{\geq k}(\Delta_0)_d \longrightarrow Sto(\Delta_0)_d \longrightarrow Sto^{<k}(\Delta_0)_d \longrightarrow \{id\},$$
$$\{id\} \longrightarrow Sto^{k}(\Delta_0)_d \longrightarrow Sto^{\geq k}(\Delta_0)_d \longrightarrow Sto^{<k}(\Delta_0)_d \longrightarrow \{id\},$$

are ***exact sequences*** *of groups and* ***split.***

(ii) $$\Lambda(\Delta_0)_d = \Lambda^{k_r}(\Delta_0)_d \ltimes \Lambda^{k_{r-1}}(\Delta_0)_d \ltimes \ldots \ltimes \Lambda^{k_1}(\Delta_0)_d\,.$$

Theorem 13.*(Malgrange, Sibuya, Babbitt-Varadarajan* [Ma 3], [Si], [BV].)

Let Δ_1 be a meromorphic differential operator. We denote by ∇_1 the meromorphic connection defined by Δ_1. Let Δ_0 be a differential operator with a fixed fundamental solution $F_0 = x^L \mathcal{U}\, e^{Q(1/u)}$. We denote by ∇_0 the meromorphic connection defined by Δ_0 Then:

(i) There is a natural isomorphism ($\nu = \nu_{\nabla_1}$):

$$\begin{array}{lcl} \text{Marked pairs } (\nabla,\xi)\text{, where } \nabla \text{ is a} & \overset{\nu}{\longrightarrow} & H^1(S^1;\Lambda(\Delta)) \\ \textbf{meromorphic} \text{ connections } \textbf{formally} & & \\ \text{equivalent to } \nabla_1 \text{ .and } \xi \text{ an} & & \\ \textbf{isomorphism} \text{ between } \nabla \text{ and } \nabla_1 . & & \end{array}$$

(ii) If $\nabla_1 = \nabla_0$, the natural isomorphism ν induces an isomorphism:

$$\begin{array}{lcl} \textbf{Meromorphic} \text{ connections } \nabla \textbf{ formally} & \overset{\nu}{\longrightarrow} & \mathcal{U}(\Delta_0)\backslash H^1(S^1;\Lambda(\Delta_0)) \\ \qquad \text{equivalent to } \nabla_0 . & & \end{array}$$

(The group $\mathcal{U}(\Delta_0)$ is acting by ***conjugation*** *on $\Lambda(\Delta_0)$.)*

Definition 5.

Let Δ_0 be a given differential operator with a fixed fundamental solution $F_0 = x^L\mathcal{U}\, e^{Q(1/u)}$. We set $\boldsymbol{q} = \boldsymbol{q}(Q)$, $N\Sigma(\boldsymbol{q}) = \{k_1, k_2,\ldots, k_r\}$, and denote by $\Sigma(\boldsymbol{q}) = \{d_1, d_2\ldots, d_p\}$ the projection of $Fr(\boldsymbol{q})$ on S^1. Let $\boldsymbol{U} = \{U_\iota\}_{\iota\in \boldsymbol{I}}$, be an open covering k_1-adapted to $\Sigma(\boldsymbol{q})$. Then, a 1-cochain

$\delta \in C^1(\boldsymbol{U};\Lambda(\Delta_0)) = Z^1(\boldsymbol{U};\Lambda(\Delta_0))$ is said a ***"Stokes cochain",*** *if*

$\delta = \{A_{\iota,\iota+1}\}_{\iota\in \boldsymbol{I}}$ ($\boldsymbol{I} = \{1,\ldots,p\}$), with $(A_{\iota,\iota+1})_{d_\iota} \in Sto(\Delta_0)_{d_\iota}$ ($\iota = 1,\ldots,p$).

Let $\boldsymbol{d} \in Fr(\boldsymbol{q})$, d its projection on S^1, and let ρ be a representation of $\Pi(\boldsymbol{q})$ in $GL(n;\boldsymbol{C})$. It is easy to check that $\lambda_{\boldsymbol{d}}(\rho(\gamma_{\boldsymbol{d}})) \in \Lambda(\Delta_0)_d$ depends *only* on $d \in S^1$.

Lemma 15.

Let Δ_0 be a fixed differential operator with a fundamental solution $F_0 = x^L\mathcal{U}\, e^{Q(1/u)}$.

We set $q=q(Q)$, $N\Sigma(q)=\{k_1, k_2,\ldots, k_r\}$ $(k_1 > k_2 > \ldots > k_r)$, *and we denote by* $\Sigma(q)$ *the projection of* $Fr(q)$ *on* S^1. *Let* $\boldsymbol{U} = \{U_\iota\}_{\iota\in I}$, *be an open covering* k_1*-adapted to* $\Sigma(q)$.

The natural map

$$\text{Representations of } \Pi(q) \text{ in } GL(n;C) \xrightarrow{z_{\boldsymbol{U}}} \{\textbf{Stokes} \text{ cocycles of } Z^1(\boldsymbol{U};\Lambda(\Delta_0))\}$$

$$\rho \xrightarrow{z_{\boldsymbol{U}}} \text{"}\{\lambda_d(\rho(\gamma_d))\}\text{"} \ (d \in \Sigma(q))$$

is a ***bijection***.

Theorem 14.

Let Δ_0 *be a given differential operator with a fixed fundamental solution* $F_0 = x^L U e^{Q(1/u)}$. *We set* $q=q(Q)$, $N\Sigma(q)= \{k_1, k_2,\ldots, k_r\}$ $(k_1 > k_2 > \ldots > k_r)$, *and denote by* $\Sigma(q) =\{d_1, d_2\ldots, d_p\}$ *the projection of* $Fr(q)$ *on* S^1. *Let* $\boldsymbol{U} = \{U_\iota\}_{\iota\in I}$, *be an open covering* k_1*-adapted to* $\Sigma(q)$. *Then:*

(i) Let[1]*:*

$\hat{H} = \hat{H}_1\hat{H}_2 \ldots\hat{H}_r$, *where* $\hat{H}_i \in GL(n;C[[x]][x^{-1}])$ *is* k_i***-summable*** *for* $i = 1,\ldots,r$. *We suppose that* $\hat{F} = \hat{H}x^L U e^{Q(1/u)}$ *is a* ***formal*** *fundamental solution of a* ***meromorphic*** *differential operator* Δ. *Then the 1-cocycle* $\mathbf{St}(\boldsymbol{U};H)$ *is a* ***Stokes cocycle***.

(ii) Let $\delta \in C^1(\boldsymbol{U};\Lambda(\Delta_0)) = Z^1(\boldsymbol{U};\Lambda(\Delta_0))$ *be a* ***Stokes cocycle***. *Then,* $\Lambda(\Delta_0) \subset \Lambda_I$, $\delta \in Z^1(\boldsymbol{U};\Lambda_I)$, *and if* $\hat{H} = \mu^{-1} i_{\boldsymbol{U}}(\delta)$*:*

(a) $\hat{H} = \hat{H}_1\hat{H}_2 \ldots\hat{H}_r$, *where* $\hat{H}_i \in GL(n;C[[x]][x^{-1}])$ *is* k_i***-summable*** *for* $i = 1,\ldots,r$;

(b) $\hat{F} = \hat{H}x^L U e^{Q(1/u)}$ *is a formal fundamental solution of a* ***meromorphic*** *differential operator* Δ*,* ***formally*** *equivalent to* Δ_0.

Moreover:

$\delta = \mathbf{St}(\boldsymbol{U};H) = \mathbf{St}(\boldsymbol{U};\mu^{-1} i_{\boldsymbol{U}}(\delta))$, *and, if* ∇ *is the meromorphic connection associated to* Δ, $\nu(\nabla) = i_{\boldsymbol{U}}(\delta)$.

(iii) Let $\alpha \in H^1(S^1;\Lambda(\Delta_0))$, *then there exists* ***one*** *and* ***only one Stokes cocycle*** $\delta \in Z^1(\boldsymbol{U};\Lambda(\Delta_0))$ *such that* $\alpha = i_{\boldsymbol{U}}(\delta)$ *(that is representing* α*)*[2].

We will first prove *assertion (i).*

---

[1] It is important to notice that this definition is stated in such a way that it is ***not*** *necessary to know theorem 5 or theorem 7* to apply it (see footnote below). Of course one can also apply it in the situation of *theorem 5 or theorem 7...*

[2] Assertion *(iii)* is due to *M. Loday-Richaud* [LR 1]. Her proof is *completely different:* she gives an *explicit algebraic* ***algorithm*** in order to *compute explicitely* δ, from α. She uses *Malgrange-Sibuya* theory but **not** Gevrey asymptotics and multisummability; so it is possible, using her result and noting that assertions *(i)* and *(ii)* are proved here ***without*** *any use of theorem 5 or theorem 7*, to get a ***new*** *proof of theorem 7* [LR 1]. Cf. also [BV].

Using the construction of *theorem 10,* we can associate to $\hat{F} = \hat{H}F_0$ a *representation* $\rho(H)$ of $\Pi(q)$ in $GL(n;\mathbf{C})$, satisfying *Stokes conditions*. We have

$\mathbf{St}(\mathbf{U};H) = z_{\mathbf{U}}(\rho(\hat{H}))$, *and* $\mathbf{St}(\mathbf{U};\hat{H})$ is a *Stokes cocycle.*

We will ***admit*** *assertion (ii), for a moment.*

Assertion (iii) follows easily from *assertions (ii)* and *(iii):*

Let $\alpha \in H^1(S^1;\Lambda(\Delta_0))$. From *theorem 13,* we get a meromorphic connection $\nabla = \nu^{-1}(\alpha)$, formally equivalent to ∇_0 . We choose a differential operator Δ representing ∇; then there exists a fundamental solution $\hat{F} = \hat{H}F_0$ of Δ, with $\hat{H} \in GL(n;\mathbf{C}[[x]][x^{-1}])$. From *theorem 7* we get a decomposition

$\hat{H} = \hat{H}_1\hat{H}_2 \ldots\hat{H}_r$, *where* $\hat{H}_i \in GL(n;\mathbf{C}[[x]][x^{-1}])$ *is* k_i*-**summable** for* $i = 1,\ldots,r$.

We have $\rho(\hat{H}) = \rho_{irr}(\nabla)$. Let $z_{\mathbf{U}}(\rho(\hat{H})) = \delta \in Z^1(\mathbf{U};\Lambda(\Delta_0))$. We have $i_{\mathbf{U}}(\delta) = \alpha$, and δ is a *Stokes cocycle representing* α.

It remains to prove *unicity*. Let $\delta \in Z^1(\mathbf{U};\Lambda(\Delta_0))$, with $i_{\mathbf{U}}(\delta) = \alpha$. From *assertion (ii)* we get $\delta = \mathbf{St}(\mathbf{U};\mu^{-1}i_{\mathbf{U}}(\delta)) = \mathbf{St}(\mathbf{U};\mu^{-1}(\alpha))$, but $\mathbf{St}(\mathbf{U};\mu^{-1}(\alpha))$ depends *only* on α; *unicity* of δ follows.

Before the proof of *assertion (ii),* we will give some consequences of *theorem 14.*

Proposition 12.

Let Δ_0 *be a given differential operator with a fixed fundamental solution* $F_0 = x^L U e^{Q(1/u)}$. *We set* $q=q(Q)$, $N\Sigma(q) = \{k_1, k_2,\ldots,k_r\}$ $(k_1 > k_2 > \ldots > k_r)$, *and denote by* $\Sigma(q) = \{d_1, d_2\ldots, d_p\}$ *the projection of* $Fr\,(q)$ *on* S^1. *Let* $\mathbf{U} = \{U_\iota\}_{\iota\in \mathbf{I}}$ *be an open covering* k_1*-adapted to* $\Sigma(q)$. *Then the natural map*

Representations of the group $\Pi(q)$ $\longrightarrow$ $H^1(S^1;\Lambda(\Delta_1))$
in $GL(n;\mathbf{C})$, *satisfying the*
Stokes conditions

$$\rho \longrightarrow z_{\mathbf{U}}(\delta)$$

*is a **bijection** commuting with the action of* $(\mathcal{U}(F_0);\mathcal{U}(\Delta_0))$.

Theorem 10 follows from *theorem 13* and *proposition 12.*

It remains now to prove *assertion (ii)* of *theorem 14.*

Let Δ_0 be a *given* differential operator with a fixed fundamental solution $F_0 = x^L U e^{Q(1/u)}$. We set $q=q(Q)$, $N\Sigma(q)=\{k_1, k_2,\ldots, k_r\}$ $(k_1 > k_2 > \ldots > k_r)$, and denote by $\Sigma(q)=\{d_1, d_2\ldots, d_p\}$ the projection of $Fr\,(q)$ on S^1. Let $\mathbf{U} = \{U_\iota\}_{\iota\in \mathbf{I}}$ $(\mathbf{I}=\{1,\ldots,p\})$, be an open covering k_1*-adapted* to $\Sigma(q)$.

Let $\delta \in C^1(\mathbf{U};\Lambda(\Delta_0)) = Z^1(\mathbf{U};\Lambda(\Delta_0))$ *be a* ***Stokes cocycle****. Then,* $\Lambda(\Delta_0) \subset \Lambda_I$, $\delta \in Z^1(\mathbf{U};\Lambda_I)$. Let $\hat{H} = \mu^{-1} i_{\mathbf{U}}(\delta)$. We will prove that δ is a ***Stokes cocycle*** by ***a descending*** *recurrence* on $i = r, r-1, \dots, 1$.

Our *recurrence hypothesis* is:

(Hyp i) Let $\delta^i = \{A_{\iota,\iota+1}\}_{\iota\in I} \in C^1(\mathbf{U};\Lambda(\Delta_0)) = Z^1(\mathbf{U};\Lambda(\Delta_0))$ *be a Stokes cocycle* ***satisfying:***

$(A_{\iota,\iota+1})_{d_\iota} \in Sto^{k_i}(\Delta_0)_{d_\iota}$ $(\iota=1,\dots,p;$ $Sto^{\leq k_i} = Sto^{<k_{i-1}}$, *if* $i > 1$, *and* $Sto^{\leq k_1} = Sto)$.

Then, if $\hat{H}^i = \mu^{-1} i_{\mathbf{U}}(\delta^i)$:

(a_i) $\hat{H}^i = \hat{H}_i \hat{H}_{i+1} \dots \hat{H}_r$, *where* $\hat{H}_j \in GL(n;\mathbf{C}[[x]][x^{-1}])$ *is* k_j-***summable*** *for* $j = i,\dots,r$.

(b_i) $\hat{F}^i = \hat{H}^i x^L U e^{Q(1/u)}$ *is a formal fundamental solution of a* ***meromorphic*** *differential operator* Δ^i, ***formally*** *equivalent to* Δ_0.

Moreover:

$\delta^i = \mathbf{St}(\mathbf{U};\hat{H}^i) = \mathbf{St}(\mathbf{U};\mu^{-1} i_{\mathbf{U}}(\delta^i))$, *and, if* ∇^i *is the meromorphic connection associated to* Δ^i, $\nu(\nabla^i) = i_{\mathbf{U}}(\delta^i)$

Assertion (ii) is (Hyp 1).

We will first prove *(Hyp r).*

Let $\delta^r = \{A_{\iota,\iota+1}\}_{\iota\in I} \in C^1(\mathbf{U};\Lambda(\Delta_0)) = Z^1(\mathbf{U};\Lambda(\Delta_0))$ *be a Stokes cocycle* ***with:***

$(A_{\iota,\iota+1})_{d_\iota} \in Sto^{k_r}(\Delta_0)_{d_\iota}$

We have (for $d_\iota \in (\mathbf{R},0)$, *"above"* d_ι)

$\lambda_{d_\iota}^{-1}(A_{\iota,\iota+1})_{d_\iota} = C_{d_\iota;r}$, or $(A_{\iota,\iota+1})_{d_\iota} = F_{0,d_\iota} C_{d_\iota;r} (F_{0,d_\iota})^{-1}$; if $V_{\iota,\iota+1}$ is the open arc of $(\mathbf{R},0)$ bisected by d_ι, with opening π/k_r, $C_{d_\iota;r} \in \mathrm{Sto}^{k_r}(Q;d))$, *and*

$F_{0,d_\iota} C_{d_\iota;r} (F_{0,d_\iota})^{-1}$ is the germ of a function of $\Gamma(V_{\iota,\iota+1};\Lambda^{\geq k_r})$. Then the 1-cocycle δ^r is k_r*-summable*. It follows from *theorem 12* that $\hat{H}^r = \hat{H}_r$ is k_r*-summable,* and (a_r) is proved; (b_r) follows from *theorem 13*.

We suppose now that *(Hyp j)* is true *for* $r \geq j \geq i > 1$, and will prove *(Hyp i–1)*.

Let $\delta^{i-1} = \{A^{i-1}_{\iota,\iota+1}\}_{\iota\in I} \in C^1(\mathbf{U};\Lambda(\Delta_0)) = Z^1(\mathbf{U};\Lambda(\Delta_0))$ *be a Stokes cocycle* ***with:***

$(A^{i-1}_{\iota,\iota+1})_{d_\iota} \in Sto^{\leq k_{i-1}}(\Delta_0)_{d_\iota}$.

Let $\{C^{i-1}_{d_\iota}\} = z_{\mathbf{U}}^{-1}(\delta^{i-1})$. We have $C^{i-1}_{d_\iota} \in \mathrm{Sto}^{\leq k_{i-1}}(Q;d))$, and, from the *decomposition (Lemma 12):*

$$\Lambda^{\leq k_{i-1}}(Q;d) = \Lambda^{k_r}(Q;d) \ltimes \Lambda^{k_{r-1}}(Q;d) \ltimes \dots \ltimes \Lambda^{k_{i-1}}(Q;d),$$

we get, for $C^{i-1}_{d_\iota} \in \Lambda^{\leq k_{i+1}}(Q;d)$, *a decomposition:*

$$C^{i-1}_{d_\iota} = C_{d_\iota;r} C_{d_\iota;r-1} \dots C_{d_\iota;i-1}, \text{ with } C_{d_\iota;j} \in \Lambda^{k_j}(Q;d)$$

(j=r,...,i–1).

We have $\quad C^{i-1}_{d_\iota} = C^i_{d_\iota} C_{d_\iota;i-1}$, with $C^i_{d_\iota} \in \Lambda^{\leq k_i}(Q;d)$, *and* $C_{d_\iota;i-1} \in \Lambda^{k_{i-1}}(Q;d)$.

Whe have $(A^i_{\iota,\iota+1})_{d_\iota} = \lambda_{d_\iota}(C^i_{d_\iota})$ (it is *independant* of the choice of $d_\iota \in (\boldsymbol{R},0)$, "above" d_ι), and $\delta^i = \{A^i_{\iota,\iota+1}\}_{\iota\in \boldsymbol{I}} \in Z^1(\boldsymbol{U};\Lambda(\Delta_0))$. If $\hat{H}^i = \mu^{-1} i_{\boldsymbol{U}}(\delta^i)$; then $\delta^i = \mathbf{St}(\boldsymbol{U};\hat{H}^i)$.

If we set:

$S^+_{k_i,k_{i+1},\dots,k_r;d_\iota}\hat{H}^i = H^i{}_{d_\iota}{}^+$, and

$S^-_{k_i,k_{i+1},\dots,k_r;d_\iota}\hat{H}^i = H^i{}_{d_\iota}{}^-$, we get:

$(H^i{}_{d_\iota}{}^-)^{-1}H^i{}_{d_\iota}{}^+ = (A^i_{\iota,\iota+1})_{d_\iota} = \lambda_{d_\iota}(C^i_{d_\iota})$, or $H^i{}_{d_\iota}{}^+ F_{0,d_\iota} = H^i{}_{d_\iota}{}^- F_{0,d_\iota} C^i_{d_\iota}$.

We set $\qquad (B_{\iota,\iota+1})_{d_\iota} = H^i{}_{d_\iota}{}^+ \lambda_{d_\iota}(C_{d_\iota;i-1})\,(H^i{}_{d_\iota}{}^+)^{-1}$.

Let $V^i_{\iota,\iota+1}$ and $V^{i-1}_{\iota,\iota+1}$ be the open arcs of $(\boldsymbol{R},0)$ bisected by d_ι with respective openings π/k_i and π/k_{i-1} ($V^{i-1}_{\iota,\iota+1}$ is contained in $V^i_{\iota,\iota+1}$). Then the germ $\lambda_{d_\iota}(C_{d_\iota;i-1})$ is the germ at d_ι of a function $B'_{\iota,\iota+1}$ of $\Gamma(V^{i-1}_{\iota,\iota+1};\Lambda^{\geq k_{i-1}})$ (this follows from $C_{d_\iota;i-1} \in \Lambda^{k_{i-1}}(Q;d)$).The germ $H^i{}_{d_\iota}{}^+$ is the germ at d_ι of a function H^{i+} of $\Gamma(V^i_{\iota,\iota+1};\Lambda)$ *asymptotic to* $\hat{H}^i$ on $V^i_{\iota,\iota+1}$ (and, *a fortiori*, on $V^{i-1}_{\iota,\iota+1}$). We conclude that the germ $(B_{\iota,\iota+1})_{d_\iota}$ is the germ at d_ι of a function $B_{\iota,\iota+1}$ of $\Gamma(V^{i-1}_{\iota,\iota+1};\Lambda^{\geq k_{i-1}})$.

We have built a k_{i-1}-*summable cochain* $\beta = \{B_{\iota,\iota+1}\}_{\iota\in \boldsymbol{I}}$. We check easily that

$$\beta \in Z^1(\boldsymbol{U};\Lambda(\Delta^i)).$$

Then it follows from *theorem 12* that $\hat{H}_{i-1} = \mu^{-1} i_{\boldsymbol{U}}(\beta)$ is k_{i-1}-*summable*, and, from *theorem 13 (i)*, that $(\Delta^i)^{\hat{H}_{i-1}} = \Delta^{i-1}$ (***definition*** of Δ^{i-1}) is a ***meromorphic*** differential operator. We set

$$\hat{H}^{i-1} = \hat{H}_{i-1}\hat{H}^i = \hat{H}_{i-1}\hat{H}_i \dots \hat{H}_r .$$

Then $\Delta^{i-1} = (\Delta^i)^{\hat{H}_{i-1}} = (\Delta_0{}^{\hat{H}^i})^{\hat{H}_{i-1}} = \Delta_0{}^{\hat{H}_{i-1}\hat{H}^i} = \Delta_0{}^{\hat{H}^{i-1}}$, and

$\hat{F}^{i-1} = \hat{H}^{i-1}x^{L}\boldsymbol{U}\, e^{Q(1/u)}$ *is a* ***formal*** *fundamental solution of the* ***meromorphic*** *differential operator* Δ^{i-1}, ***formally*** *equivalent to* Δ_0.

Let $H_{d_\iota;i-1}{}^+ = S^+_{k_{i-1},\dots,k_r;d_\iota}\hat{H}_{i-1}$ and $H_{d_\iota;i-1}{}^- = S^-_{k_{i-1},\dots,k_r;d_\iota}\hat{H}_{i-1}$.

We find:

$H_{d_\iota;i-1}{}^+ H^i{}_{d_\iota}{}^+ F_{0,d_\iota} = H_{d_\iota;i-1}{}^- H^i{}_{d_\iota}{}^- F_{0,d_\iota} C^i_{d_\iota} C_{d_\iota;i-1}$

$H_{d_\iota;i-1}{}^+ H^i{}_{d_\iota}{}^+ F_{0,d_\iota} = H_{d_\iota;i-1}{}^- H^i{}_{d_\iota}{}^- F_{0,d_\iota} C^{i-1}{}_{d_\iota;}$

$H^{i-1}{}_{d_\iota}{}^+ F_{0,d_\iota} = H^{i-1}{}_{d_\iota}{}^- F_{0,d_\iota} C^{i-1}{}_{d_\iota;}$.

Then $\delta^{i-1} = \mathbf{St}(\boldsymbol{U};\hat{H}^{i-1}) = \mathbf{St}(\boldsymbol{U};\mu^{-1} i_{\boldsymbol{U}}(\delta^{i-1}))$. We have got *(Hyp i-1)* and *assertion (ii)* of *theorem 14* is proved by recurrence. That ***concludes*** the proof of *theorem 14*.

Examples.

As an illustration of the preceding constructions, it is possible to compute the "wild groups" and their representations for *Airy equation* and *Kummer equations*. This is a simple reformulation of computations of [MR 3], *chapter 3*.

Remark.

For $d \in (\boldsymbol{R},0)$, $\gamma_d \in \Pi(q)$ will later (see *6*, infra) correspond to a loop pointed at "$\boldsymbol{R}^+$"= "$\{0\}$"$\times \boldsymbol{R}^+ \in$"$\{0\}$"$\times$ S^1 ("$\boldsymbol{R}^+$" is a point of the universal covering of the real blow-up of the origin in the analytic halo).

We start from "$\boldsymbol{R}^+$" and go (on "$\{0\}$"$\times(\boldsymbol{R},0)$) to "0"$\times \boldsymbol{d} \in$"$\{0\}$"$\times(\boldsymbol{R},0)$; then we turn clockwise around "$]0,+\infty]$" $\times\{d\}$ in the universal covering of $\boldsymbol{C}^*$ with an analytic halo at zero and go back to "0"$\times d$; after that we return to "$\boldsymbol{R}^+$" (on "$\{0\}$"$\times(\boldsymbol{R},0)$).

So the groups $\Pi(q)$ and $G\Pi(q)$ are "wild fundamental groups *pointed* at "$\boldsymbol{R}^+$"$\in$"$\{0\}$"$\times$ S^1".

Stokes operator $St_d(\Delta_0)$ corresponds to the *"wild monodromy"* along the loop γ_d for the vector space of "germs of solutions of the differential operator Δ (formally equivalent to $\Delta_0 : \Delta = \Delta_0{}^{\hat{H}}$) at "$\boldsymbol{R}^+$"", *modulo the isomorphism* between this vector space and the vector space of *formal* solutions of Δ_0 (given by the *"analyticity"* of H near 0 *in the analytic halo* and the choice of the principal determination of Logarithm).

The *"wild connections"* induced by ∇_0 and ∇ in a *"small"* sector of the *universal covering of the analytic halo,* bisected by $\boldsymbol{R}^+$, are *the same* (H is a *wild analytic* function in such a sector), so, the representation $\rho(\nabla)$, up to conjugation, can be interpreted as a *representation* of the *"wild fundamental group"* $\Pi(q)$ into the group of *linear permutations* of the *germs of horizontal sections of* ∇ "at "$\boldsymbol{R}^+$"$\in$"$\{0\}$"$\times$ S^1" (identified with the formal solutions of Δ_0 like above). Finally we have got a *"wild monodromy"*. This "wild monodromy" express the *"difference"* between ∇ and ∇_0. In fact we want to understand ∇ *independantly* of ∇_0. In order to do that we will first *translate* ∇_0 in terms of *representation.*

Let

$$\boldsymbol{E} = \bigcup_q E(q) = \varinjlim_q E(q).$$

Let $\mathcal{T}(q)$ *be the exponential torus associated to* $q = \{q_1, q_2, \ldots, q_n\} \subset \boldsymbol{E}$ ($\mathcal{T}(q) = Aut_{K_v}\mathbb{L}_v$). To *natural injections*

$$E(q) \longrightarrow \boldsymbol{E}$$

correspond *natural projections*

$$\mathcal{T}(q) \longrightarrow \mathcal{T}.$$

We set

$\mathcal{T} = \varprojlim_q \mathcal{T}(q)$. By definition $\mathcal{T}$ is the ***exponential torus***; it is a ***commutative*** *group*. The algebraic torus $\mathcal{T}(q)$ are endowed with the *Zariski topology,* and $\mathcal{T}$ is endowed with the corresponding *direct limit topology.*

Lemma 16.

(i) Let $\kappa : \mathcal{T} \longrightarrow \mathbf{C}^*$ *be a continuous homomorphism of groups. Then there exists* $q \in \mathbf{E}$*, uniquely determined, such that* κ *is equal to the composition of the natural projection* $\mathcal{T} \longrightarrow \mathcal{T}(\mathbf{q})$ $(\mathbf{q}=\{q\})$ *and of the character* $q: \mathcal{T}(\mathbf{q}) \longrightarrow \mathbf{C}^*$ *. (We will identify* κ *and* q*.)*

(ii) Let F *be a finitely dimensional* **C***-vector space* $(n = \dim_{\mathbf{C}} F)$*, and* $\theta: \mathcal{T} \longrightarrow GL(F)$ *be a continuous homomorphism of groups . Let* $G = \theta(\mathcal{T})$.

Then there exists a basis of F *such that the subgroup* G *of* $GL(F)$*, identified by the choice of this basis to* $GL(n;\mathbf{C})$*, is* ***diagonal.*** *If* $\phi_1, \phi_2,\ldots, \phi_n : G \longrightarrow \mathbf{C}^*$ *are the corresponding homomorphisms of groups (if* $g \in G$*,* $\phi_1(g)$ *is the first entry of* g *on the diagonal...), and if* q_i *is associated to* $\kappa_i = \phi_i\theta$*, like in (i) it is possible to associate to* κ *the set* $\mathbf{q} = \{q_1, q_2,\ldots, q_n\} \subset \mathbf{E}$*,* ***independent*** *of the choice of the basis of* F*, and* θ *is the composition of the natural projection* $\mathcal{T} \longrightarrow \mathcal{T}(\mathbf{q})$ *and of* $(q_1, q_2,\ldots, q_n) \longrightarrow GL(n;\mathbf{C}) = GL(F)$*.*

For $\tau \in \mathcal{T}$*,* $\theta(\tau) = Diag\,(q_1(\tau), q_2(\tau),\ldots, q_n(\tau))$

In the situation of *lemma 16 (ii)*, we will set $\mathbf{q} = \mathbf{q}_\theta$. From a given $\mathbf{q}=\{q_1, q_2,\ldots, q_n\}\subset \mathbf{E}$ we get a representation $\theta: \mathcal{T} \longrightarrow GL(n;\mathbf{C})$*, uniquely determined up to conjugation,* such that $\mathbf{q} = \mathbf{q}_\theta$.

Let ∇ be a ***formal*** *connection.* There exists a representation[1] $\theta: \mathcal{T} \longrightarrow GL(n;\mathbf{C})$*, uniquely determined up to conjugation,* such that $\mathbf{q}(\nabla) = \mathbf{q}_\theta$. More precisely:

Let $F_0(x) = x^L U e^{Q(1/u)}$, with $u^\nu = x$, be a *formal fundamental solution* of the *formal connection* ∇ ($\mathbf{q}(\nabla)$ is the set of the diagonal entries of $Q = Diag\,(q_1, q_2,\ldots, q_n)$).

Let (γ_0) be *the* free group generated by γ_0 . We define *an action of the group* (γ_0) *on the lattice* $\mathbf{E}$ by

$q\gamma_0\,(u) = q(e^{-2i\pi/\nu}u)$, and *an action of the group* (γ_0) *on the exponential torus* $\mathcal{T}$ by $\gamma_0\,\tau(q) = \tau(q\gamma_0)$, for $\tau \in \mathcal{T}$ *and* $q \in \mathbf{E}$ arbitrary.

By definition the *wild formal fundamental group* $\pi_{1,sf}((\mathbf{C}^*,0);"\mathbf{R}^+")$ of $(\mathbf{C},0)$ pointed at "$\mathbf{R}^+$" is the *semi-direct product*

$(\gamma_0)\ltimes \mathcal{T}$ built from the action of (γ_0) on $\mathcal{T}$.

Let $\hat{M} = U^{-1}e^{2i\pi L}U$ be the formal monodromy matrix associated to F_0. *We set*

$$\rho(\nabla)(\gamma_0) = \hat{M}, \text{ and, for } \tau \in \mathcal{T},$$

$$\rho(\nabla)(\tau) = Diag\,(q_1(\tau), q_2(\tau),\ldots, q_n(\tau)).$$

We have
$$\hat{M}^{-1}Q(1/u)\,\hat{M} = Q(e^{-2i\pi/\nu}u)$$
$$\hat{M}^{-1}Diag\,(q_1, q_2,\ldots, q_n)\,\hat{M} = (q_1\gamma_0\,, q_2\gamma_0\,,\ldots, q_n\gamma_0\,)$$
$$\hat{M}^{-1}Diag\,(q_1(\tau), q_2(\tau),\ldots, q_n(\tau))\,\hat{M} = (q_1\gamma_0\,(\tau), q_2\gamma_0\,(\tau),\ldots, q_n\gamma_0\,(\tau))$$

[1] In the following *all the representations are supposed continuous.*

$$\rho(\nabla)(\gamma_0)^{-1}\rho(\nabla)(\tau)\,\rho(\nabla)(\gamma_0) = \hat{\mathrm{M}}^{-1}\rho(\nabla)(\tau)\,\hat{\mathrm{M}} = \rho(\nabla)(\gamma_0\,\tau).$$

So we have defined a ***linear representation***

$$\rho(\nabla)\colon \pi_{1,sf}((\boldsymbol{C}^*,0);\text{"}\boldsymbol{R}^+\text{"}) = (\gamma_0) \ltimes \boldsymbol{T} \longrightarrow GL(n;\boldsymbol{C}),$$

*associated to the **formal connection*** ∇. (This representation is, up to conjugation, independant of the order of the Jordan blocks of L on the diagonal.)

We will see now that, given a linear representation

$$\rho_1\colon \pi_{1,sf}((\boldsymbol{C}^*,0);\text{"}\boldsymbol{R}^+\text{"}) \longrightarrow GL(n;\boldsymbol{C}),$$

*there exists a **unique formal connection*** ∇, such that $\rho_1 = \rho(\nabla)$. Moreover, the formal connection ∇ *depends only of the class of* ρ_1 *up to equivalenceby the adjoint action of* $GL(n;\boldsymbol{C})$

We set $\rho_1(\gamma_0) = \hat{\mathrm{M}}$ and $\rho_1(\boldsymbol{T}) = \mathrm{T}_1$. We set $\boldsymbol{q} = \boldsymbol{q}_\theta$; θ being the restriction of ρ_1 to $\boldsymbol{T}$, $\boldsymbol{q}$ is Galois invariant (it is invariant by the action of M). We can choose a basis of $GL(n;\boldsymbol{C})$ in such a way that T_1 is a diagonal group: $\mathrm{T}_1 = \{Q(\tau) = Diag\,(q_1(\tau),q_2(\tau),\ldots,q_n(\tau)) / \tau \in \boldsymbol{T}\}$ ($\boldsymbol{q} = \{q_1, q_2,\ldots, q_n\}$, and $Q = Diag\,(q_1, q_2,\ldots, q_n)$). Using a method of [BJL], [J], we can suppose moreover that we have chosen our basis such that $U\hat{\mathrm{M}}U^{-1}$ is in Jordan form. Then let L be such that $e^{2i\pi L} = U\hat{\mathrm{M}}U^{-1}$ (L is defined up to multiplication on the right by a diagonal matrix $Diag\,(x^{m_1}, x^{m_2},\ldots,x^{m_n})$, $m_i \in \boldsymbol{Z}$). Then $F_0 = x^L\,U\,e^Q$ is a fundamental solution of a rational differential operator Δ_0 and the corresponding connection ∇_0 is *independant of the choice* of the *basis* and of the *integers* m_i, and *invariant by conjugation on* ρ_1. We have clearly $\rho(\nabla) = \rho_1$.

So we get

Theorem 15.

The natural map

$$\begin{array}{ccl} \textit{\textbf{Formal meromorphic} connections} & \overset{\rho}{\longrightarrow} & \textit{Finite dimensional linear representations} \\ & & \textit{of the group } \pi_{1,sf}((\boldsymbol{C}^*,0);\text{"}\boldsymbol{R}^+\text{"}), \\ & & \textit{up to conjugation.} \\ \nabla & \longrightarrow & \rho(\nabla) \end{array}$$

*is an **isomorphism**.*

This isomorphism is compatible with *sums, duality, tensor products,...* It is an *isomorphism of Tannakian categories.*

If now ∇ is a *germ of **meromorphic** connection,* we get from ∇ ***two linear*** *representatons:*

$$\rho(\nabla)\colon \pi_{1,sf}((\boldsymbol{C}^*,0);\text{"}\boldsymbol{R}^+\text{"}) \longrightarrow GL(n;\boldsymbol{C}), \textit{ and}$$
$$\rho(\nabla)_{irr}\colon \quad G\Pi(\boldsymbol{q}) \longrightarrow GL(n;\boldsymbol{C}).$$

The respective restrictions of these representations $\rho(\nabla)$ and $\rho(\nabla)_{irr}$ to the respective subgroups (γ_0) of $\pi_{1,sf}((C^*,0);"R^+")$ and $G\Pi(q)$ are clearly equal. Conversely, two linear representations

$$\rho_1 : \pi_{1,sf}((C^*,0);"R^+") \longrightarrow GL(n;C), \text{ and}$$
$$\rho_2 : G\Pi(q) \longrightarrow GL(n;C),,$$

admitting *equal restrictions* to the subgroups

$(\gamma_0) \subset \pi_{1,sf}((C^*,0);"R^+")$ and $(\gamma_0) \subset G\Pi(q)$,

being given, it is in general *impossible* to find a germ of *meromorphic connection* ∇ such that $\rho(\nabla):= \rho_1$ and $\rho(\nabla)_{irr} = \rho_2$: ρ_1 and ρ_2 must satisfy a *"Stokes condition"* (cf. *theorem 10*).

Proposition 13.

The natural map

$$\begin{array}{ccc} \textit{Germs of } \textbf{\textit{meromorphic}} \textit{ connections } \nabla & \xrightarrow{\rho} & \textit{Pairs of representations of the groups} \\ \textbf{\textit{formally}} \textit{ equivalent to } \nabla_0 . & & \pi_{1,sf}((C^*,0);"R^+") \textit{ and } G\Pi(q) \textit{ in} \\ & & GL(n;C) \textit{ coincident on the two subgroups} \\ & & \textit{corresponding to } (\gamma_0), \textit{ and satisfying} \\ & & \textit{the } \textbf{\textit{Stokes conditions}}. \\ \nabla & \longrightarrow & (\rho(\nabla),\rho_{irr}(\nabla)), \end{array}$$

*is a **bijection**.*

The next step now is to build a new group $\pi_{1,s}((C^*,0);"R^+")$, the *wild fundamental group* of $(C,0)$, pointed at "R^+", satisfying the following properties:

(i) The wild fundamental group is a semi-direct product

$$\pi_{1,s}((C^*,0);"R^+") = \pi_{1,sf}((C^*,0);"R^+") \ltimes \mathcal{R}$$
$$\pi_{1,s}((C^*,0);"R^+") = ((\gamma_0) \ltimes T) \ltimes \mathcal{R},$$

where $\mathcal{R}$ (the *resurgent group*) is the *"exponential"* of a *free Lie algebra Lie* $\mathcal{R}$ (the *resurgent Lie algebra*), with infinitely many generators.

(ii) To each germ ∇ of rank n *meromorphic connection*, we can associate a linear representation`

$$\rho(\nabla) : \pi_{1,s}((C^*,0);"R^+") \longrightarrow GL(n;C),$$

such that the restriction of $\rho(\nabla)$ to $\pi_{1,sf}((C^*,0);"R^+")$ is $\rho(\nabla)$, and such that, $\rho(\nabla)$ *being known*, the knowledge of the restriction of $\rho(\nabla)$ to the resurgent group $\mathcal{R}$ is *equivalent* to the knowledge of the representation

$$\rho(\nabla)_{irr}: G\Pi(q) \longrightarrow GL(n;C) \; (q = q(\nabla)).$$

(iii) If a finite dimensional representation[1] of the wild fundamental group

$\rho_0 : \pi_{1,s}((C^*,0);"R^+") \longrightarrow GL(n;C)$, is given

[1] The restriction to T of such a representation will be *allway* supposed *continuous* in the following.

we denote by ρ_1 the restriction of ρ_0 to $\pi_{1,sf}((C^*,0);"R^+")$, and

$\rho_2 : G\Pi(q) \longrightarrow GL(n;C)$ the representation corresponding to the restriction of ρ_0 to the *resurgent group* $\mathcal{R}$ (and the knowledge of ρ_1 ...), with $q = q_{\rho_0}$. Then the pair (ρ_1,ρ_2) satisfies the *"Stokes conditions"*, there exists *(Proposition 13)* an uniquely determined germ of meromorphic connection ∇ such that

$(\rho(\nabla),\rho(\nabla)_{irr}) = (\rho_1,\rho_2)$, and $(\rho(\nabla),\rho(\nabla)_{irr})$ comes from the representation

$$\rho(\nabla) : \pi_{1,s}((C^*,0);"R^+") \longrightarrow GL(n;C)$$

defined by ∇ by the construction of *(ii)*.

Let $q = \{q_1, q_2,..., q_n\} \subset E$, and, after ordering, let Q be the diagonal matrix $Q = Diag\{q_1, q_2,..., q_n\}$. Let $\mathcal{T}(q)$ be the exponential torus associated to q, and let $\mathrm{T}(Q)$ be its representation in $GL(n;C)$ given by Q.

Let $\tau \in \mathcal{T}(Q)$. It is *represented* by the matrix

$$Q(\tau) = Diag\,(q_1(\tau),q_2(\tau),...,q_n(\tau)) \in \mathrm{T}(Q) \subset GL(n;C).$$

Lemma 17.

Let $q = \{q_1, q_2,..., q_n) \subset E$, and, after ordering, let Q be the diagonal matrix $Q = Diag(q_1, q_2,..., q_n\}$. Let $C \in End(n;C)$, $C = (c_{i,j})$ $(q_{i,j} = q_i - q_j)$. Then:

(i) $$\tau C \tau^{-1} = Q(\tau)\, C\, Q(\tau)^{-1} = (c_{i,j}\, q_{i,j}(\tau)).$$

(ii) Let $q \in E$, $q \neq 0$ and

$$C_q = (a_{i,j}), \text{ with } a_{i,j} = 0 \text{ if } q_i - q_j \neq q, \text{ and } a_{i,j} = c_{i,j} \text{ if } q_{i,j} = q.$$

Then:

$$\tau C_q \tau^{-1} = Q(\tau)\, C_q\, Q(\tau)^{-1} = q(\tau)\, C_q\,.$$

(iii) Let $Dia(C)$ be the diagonal matrix with the same diagonal entries than C:

$\tau C \tau^{-1} = Dia(C) + \sum_{i,j} q_{i,j}(\tau)\, C_{q_{i,j}}$ *(with $C_q = 0$ if $q = 0$), and such a decomposition is **uniquely** determined: if*

$$\tau C \tau^{-1} = Dia(C) + \sum_{i,j} q_{i,j}(\tau)\, A_{q_{i,j}}, \text{ then } A_{q_{i,j}} = C_{q_{i,j}}.$$

(iv) Let $d \in (R,0)$. If $C \in \mathrm{Sto}(Q;d)$, then:

$\tau C \tau^{-1} = I + \sum q(\tau)\, C_q$, *the sum being extended to $q = q_{i,j}$, with $q_i <<_d q_j$,*

$$\tau C \tau^{-1} = I + \sum_{q \in E_d(q)} q(\tau)\, C_q\,.$$

(v) Let $d \in (R,0)$. If $C \in$ Lie $\mathrm{Sto}(Q;d)$ (Lie algebra of $\mathrm{Sto}(Q;d)$), then:

$\tau C \tau^{-1} = \sum q(\tau)\, C_q$, *the sum being extended to $q = q_{i,j}$, with*

$q_i <<_d q_j$,

$$\tau \mathrm{C} \tau^{-1} = \sum_{q \in E_d(q)} q(\tau)\, \mathrm{C}_q .$$

The only non trivial point is *unicity* in *(iii)*.

Let $(p_1, p_2, \dots, p_\nu)$ be a ***Z**-basis* of the lattice $\boldsymbol{E(q)}$

We have an *isomorphism*

$$(p_1, p_2, \dots, p_\nu) : \boldsymbol{T(q)} \longrightarrow (\boldsymbol{C}^*)^\nu$$

$$(p_1, p_2, \dots, p_\nu) : \quad \tau \longrightarrow (p_1(\tau), p_2(\tau), \dots, p_\nu(\tau)).$$

We set $p_k(\tau) = \tau_k$ $(k=1,\dots,\nu)$. Then each $q_{i,j}(\tau)$ is a *monomial* in the variables $\tau_k \in \boldsymbol{C}^*$ and the *distincts* $q_{i,j}(\tau)$ are *independant on* $\boldsymbol{C}$.

The decomposition *(iii)* appears as a ***"Fourier decomposition"*** of the *"unfolding"* $\tau \mathrm{C} \tau^{-1}$ of the matrix C by the *adjoint action of the exponential torus* $\boldsymbol{T(q)}$.

Let $\Delta = d/dx - A$, where $A \in End(n; \boldsymbol{C}\{x\}[x^{-1}])$, be a germ of meromorphic differential operator at the origin of the complex plane $\boldsymbol{C}$.

Let $\hat{F}(x) = \hat{H}(x)\, x^L\, U\, e^{Q(1/u)}$ be a formal fundamental solution of Δ as above. We set

$$F_0(x) = x^L\, U\, e^Q,\ \boldsymbol{q} = \boldsymbol{q}(Q), \text{ and denote } n \text{ the rank of } \Delta.$$

Let $\boldsymbol{d} \in Fr\,(\boldsymbol{q})$ and $\mathrm{St}_d(\Delta)$ the corresponding Stokes matrix. For ***every*** $\tau \in \boldsymbol{T}$, the matrix $\tau\, \mathrm{St}_d(\Delta) \tau^{-1}$ belongs to the representation of $Gal_K(\Delta)$ in $GL(n;\boldsymbol{C})$. associated to $\hat{F}$, the matrix $\mathrm{St}_d(\Delta)$ is *unipotent* and $\tau\,(Log\ \mathrm{St}_d(\Delta))\tau^{-1}$ belongs to the representation of *Lie* $Gal_K(\Delta)$ in $End(n;\boldsymbol{C})$ associated to $\hat{F}$, that is corresponds to a *Galois derivation* of the field $K<\hat{F}>$. Then it follows from *Lemma 17* ($\mathrm{St}_d(\Delta) \in \mathrm{Sto}(Q;\boldsymbol{d})$) that we have a uniquely determined decomposition

$$\tau\,(Log\ \mathrm{St}_d(\Delta))\tau^{-1} = \sum q(\tau)\, Log\ \mathrm{St}_d(\Delta)_q ,$$ *the sum being extended to* $q = q_{i,j}$, *with* $q_i <<_d q_j$, or

$$\tau\,(Log\ \mathrm{St}_d(\Delta))\tau^{-1} = \sum_{q \in E_d(q)} q(\tau)\, Log\ \mathrm{St}_d(\Delta)_q ,$$

with each $Log\ \mathrm{St}_d(\Delta)_q$ belonging to the representation of *Lie* $Gal_K(\Delta)$ in $End(n;\boldsymbol{C})$. associated to $\hat{F}$, that is corresponding to a ***Galois derivation*** of the field $K<\hat{F}>$. *We have performed a **"Fourier analysis of the infinitysimal Stokes phenomena"**.*

Theorem 16.

Let $\Delta = d/dx - A$, where $A \in End(n;\boldsymbol{C}\{x\}[x^{-1}])$, be a germ of meromorphic differential operator at the origin of the complex plane $\boldsymbol{C}$. We set $\boldsymbol{q} = \boldsymbol{q}(\Delta)$, and denote by *$n$ the rank of Δ. Then, for each $\boldsymbol{d} \in Fr\,(\boldsymbol{q})$, $\tau\,(Log\ St_d(\Delta))\tau^{-1}$ belongs to Lie $Gal_K(\Delta)$, and we have an uniquely determined decomposition*

$$\tau\,(Log\ St_d(\Delta))\tau^{-1} = \sum q(\tau)\ Log\ St_d(\Delta)_q$$ *, the sum being extended to* $q = q_{i,j}$, *with* $q_i <<_d q_j$, *or*

$$\tau\,(Log\ St_d(\Delta))\tau^{-1} = \sum_{q \in E_d(q)} q(\tau)\ Log\ St_d(\Delta)_q\ ,$$

with each $Log\ \mathrm{St}_d(\Delta)_q$ *belonging to Lie* $Gal_K(\Delta)$.

Moreover $\tau\,(Log\ St_d(\Delta)_q)\tau^{-1} = q(\tau)\ Log\ St_d(\Delta)_q$ *and*
$\hat{M}\mathrm{St}_d(\Delta)_q\hat{M}^{-1} = \mathrm{St}_{exp(-2i\pi)d}(\Delta)_q$ *, for every* $q \in E$.

It is now natural to introduce the ***free complex Lie algebra*** *Lie* $\mathcal{R}$ generated by all the *"letters"* $\dot{\Delta}_{q,d}$ where *(q,d) is chosen such that* $q \in E$ and $d \in Fr\ q$ (i.e. such that e^q is *"maximally decaying"* on ***d***). We will name it *resurgent Lie algebra*[1].

In the situation of *theorem 16* we get a *linear representation*

$$Lie\ \rho_{res}(\Delta): \quad Lie\ \mathcal{R} \longrightarrow End\,(n;C)$$

$$Lie\ \rho_{res}(\Delta): \quad \dot{\Delta}_{q,d} \longrightarrow St_d(\Delta)_q \ \text{ if } d \in Fr\,(q), \text{ and}$$

$$Lie\ \rho_{res}(\Delta): \quad \dot{\Delta}_{q,d} \longrightarrow 0, \quad \text{if } d \notin Fr\,(q).$$

We define an action of the *wild formal fundamental group*

$\pi_{1,sf}((C^*,0);"R^+") = (\gamma_0) \ltimes T$ on the *resurgent Lie algebra Lie* $\mathcal{R}$ by

$$\gamma_0\ \dot{\Delta}_{q,d}\ \gamma_0^{-1} = \dot{\Delta}_{q,exp(-2i\pi)d} \quad \text{, and}$$

$$\tau\ \dot{\Delta}_{q,d}\ \tau^{-1} = q(\tau)\ \dot{\Delta}_{q,d} \quad .$$

If we denote by $\rho(\Delta)$ the representation

$\rho(\Delta): \pi_{1,sf}((C^*,0);"R^+") \longrightarrow GL\,(n;C)$ associated to the *formal connection* defined by the differential operator Δ, the above action is *"compatible" with the pair of representations* $(\rho(\Delta)\,, Lie\ \rho_{res})$ *(theorem 16).*

Proposition 14.

The natural map

Pairs of representations $(\rho_1, L\rho)$ *of the group* $\pi_{1,sf}((C^*,0);"R^+")$ *in* $GL(n;C)$ *and of the Lie algebra Lie* $\mathcal{R}$ *in* $End(n;C)$ *"compatible" with the action of* $\pi_{1,sf}((C,0);"R^+")$. *on Lie* $\mathcal{R}$.	$\longrightarrow$	*Pairs of representations of the groups* $\pi_{1,sf}((C^*,0);"R^+")$ *and* $G\Pi(q_{\rho_1})$ *in* $GL(n;C)$*, coincident on the two subgroups corresponding to* (γ_0)*, and satisfying the* ***Stokes conditions,*** *up to conjugation.*

$$(\rho_1, L\rho) \longrightarrow (\rho_1, \rho_2)$$

[1] Because it contains *all Ecalle's resurgent algebras.*

with $\qquad Log\ \rho_2\ (\gamma_d) = \sum q(\tau)\ L\rho(\dot{\Delta}_{q,d}), \qquad$ *for every* $d \in Fr(q_{\rho_1})$.

is a ***bijection***.

From *Propositions 13 and 14,* we get a *first version* of the ***"wild Riemann Hilbert correspondence":***

Theorem 17.

The natural map

Germs of ***meromorphic*** *connections at the origin .*	$\longrightarrow$	*Pairs of* ***representations*** *of the* ***group*** $\pi_{1,sf}((C^*,0);"R^+")$ *in* $GL(n;C)$ *and of the* ***Lie algebra*** *Lie* $\mathcal{R}$ *in* $End(n;C)$ ***"compatible"*** *with the action of* $\pi_{1,sf}((C^*,0);"R^+")$ *on Lie* $\mathcal{R}$*, up to conjugation..*

$$\nabla \longrightarrow (\rho(\nabla), Lie\ \rho_{res}(\Delta))$$

is a ***bijection***.

In order to get the searched result, that is the *classification of germs of meromorphic connections in terms of representations of* ***group***, it only remains to replace the *resurgent* ***Lie algebra*** *Lie* $\mathcal{R}$ by a ***group***, the *resurgent group* $\mathcal{R}$ (the *"exponential"* of *Lie* $\mathcal{R}$), and the *action* of the *wild formal fundamental group* $\pi_{1,sf}((C^*,0);"R^+")$ on the *Lie algebra Lie* $\mathcal{R}$ by an *action* of the same group on the *group* $\mathcal{R}$. Then we will get a pair of representations $(\rho(\nabla), \rho_{res}(\Delta))$, respectively of the groups $\pi_{1,sf}((C^*,0);"R^+")$ and $\mathcal{R}$ in $GL(n;C)$, *compatible* with the *action* of the first group in the second, that is a *representation of the* ***semidirect*** *product* (defined by the *same action*)

$$\pi_{1,sf}((C^*,0);"R^+") \ltimes \mathcal{R} \text{ in } GL(n;C).$$

Let X be a *set*. We denote [S](LA 4.10) by L_X the *free complex Lie algebra* on X, by $\hat{L}_X$ its *completion*, by Ass_X the *complex associative algebra* on X, by $\hat{Ass}_X$ its *completion*, by $\hat{\mathcal{M}}_X$ the ideal generated in $\hat{Ass}_X$ by X, by $\Delta : \hat{Ass}_X \longrightarrow \hat{Ass}_X \otimes \hat{Ass}_X$ the *diagonal map*, and by $\hat{G}_X$ the set of $\beta \in I + \hat{\mathcal{M}}_X$ with $\Delta\,\beta = \beta \otimes \beta$.

There is a natural *isomorphism*

$$exp:\ \hat{\mathcal{M}}_X \longrightarrow I + \hat{\mathcal{M}}_X.$$

We can *identify* $\hat{L}_X$ with the set of *primitive elements* of $\hat{Ass}_X$. Then we get by restriction of the exponential an *isomorphism*

$$exp:\ \hat{L}_X \longrightarrow \hat{G}_X\,.$$

By the *Campbell-Hausdorff formula* we get a *group structure* on $\hat{G}_X$.

If X is the set of "letters" $\dot{\Delta}_{q,d}$, with (q,d) such that $q \in \boldsymbol{E}$ and $d \in Fr\, q$, we denote
$Lie\, \mathcal{R} = L_X$, $\mathcal{UR} = Ass_X$, $\widehat{\mathcal{UR}} = \widehat{Ass}_X$, $\widehat{\mathcal{MR}} = \widehat{\mathcal{M}}_X$, $\widehat{\mathcal{R}} = \widehat{G}_X$. We get *isomorphisms*

$$exp:\ \widehat{\mathcal{MR}} \longrightarrow I + \widehat{\mathcal{MR}}$$
$$exp:\ Lie\,\widehat{\mathcal{R}} \longrightarrow \widehat{\mathcal{R}}.$$

We denote by $\mathcal{R}$ the subgroup of $\widehat{\mathcal{R}}$ generated by the image of $Lie\,\mathcal{R}$ by *exp;* by *definition* $\mathcal{R}$ is the ***resurgent group***.

Lemma 16 .
We consider the action of the wild formal fundamental group $\pi_{1,sf}((\boldsymbol{C}^*,0)\ "\boldsymbol{R}^+")$ *on the free Lie algebra* $Lie\,\mathcal{R}$ *defined by*

$$\gamma_0\ \dot{\Delta}_{q,d}\ \gamma_0^{-1} = \dot{\Delta}_{q,exp(-2i\pi)d}$$
$$\tau\ \dot{\Delta}_{q,d}\ \tau^{-1} = q(\tau)\ \dot{\Delta}_{q,d}\quad .$$

This action can be extended naturally to $\mathcal{UR}$ *and we get (by restriction) an action on* $\mathcal{R}$, *leaving* $\mathcal{R}$ *invariant, such that*

$$\gamma_0\ exp(\dot{\Delta}_{q,d})\gamma_0^{-1} = exp(\dot{\Delta}_{q,exp(-2i\pi)d})$$
$$\tau\ exp(\dot{\Delta}_{q,d})\tau^{-1} = exp(q(\tau)\ \dot{\Delta}_{q,d}).$$

The ***wild fundamental group*** of the germ of $\boldsymbol{C}^*$ at the origin, *pointed at* "$\boldsymbol{R}^+$", is by *definition* the ***semi-direct*** *product*

$$\pi_{1,s}((\boldsymbol{C}^*,0);"\boldsymbol{R}^+") = \pi_{1,sf}((\boldsymbol{C}^*,0);"\boldsymbol{R}^+") \ltimes \mathcal{R}$$
$$\pi_{1,s}((\boldsymbol{C}^*,0);"\boldsymbol{R}^+") = ((\gamma_0) \ltimes \boldsymbol{T}) \ltimes \mathcal{R}$$

defined by the ***action*** *of* $\pi_{1,sf}((\boldsymbol{C}^*,0);"\boldsymbol{R}^+")$ *on* $\mathcal{R}$ introduced in *lemma 16.*

Let $\alpha_1, \alpha_2,..., \alpha_m \in Lie\,\mathcal{R}$ *independant on* $\boldsymbol{Z}$. Then the subgroup of $\widehat{\mathcal{R}}$ generated by *exp* α_1, *exp* α_2,..., and *exp* α_m is *isomorphic to the free group* generated by the m *"letters" exp* α_1, *exp* α_2,..., *exp* α_m . We get:

Lemma 17 .
If $(\rho_1, L\rho_2)$ *is a pair of representations of the group* $\pi_{1,sf}((\boldsymbol{C}^*,0);"\boldsymbol{R}^+")$ *in* $GL(n;\boldsymbol{C})$ *and of the Lie algebra* $Lie\,\mathcal{R}$ *in* $End(n;\boldsymbol{C}^*)$ *"compatible" with the action of* $\pi_{1,sf}((\boldsymbol{C},0);"\boldsymbol{R}^+")$ *on* $Lie\,\mathcal{R}$, *then there exists a unique representation*
$\rho_2 : \mathcal{R} \longrightarrow GL(n;\boldsymbol{C})$ *such that*
$\rho_2(exp\ \alpha) = exp\ L\rho_2(\alpha)$ *for every* $\alpha \in Lie\,\mathcal{R}$. *This representation is compatible with the action of* $\pi_{1,sf}((\boldsymbol{C},0);"\boldsymbol{R}^+")$ *on* $\mathcal{R}$ *defined in lemma 16.*

We get the "***wild Riemann-Hilbert correspondence***"*:*

Theorem 18.

The natural map

$$\begin{array}{ccc} \textit{Germs of } \textbf{\textit{meromorphic}} \textit{ connections} & \nabla \xrightarrow{\rho_s} & \textit{Finite dimensional } \textbf{\textit{linear}} \\ \textit{at the origin.} & & \textbf{\textit{representations}}^{1} \textit{ of the} \\ & & \textbf{\textit{wild fundamental group}} \\ & & \pi_{1,s}((C^*,0);\text{"}\mathbf{R}^+\text{"}), \textit{ up to} \\ & & \textit{conjugation.} \end{array}$$

$$\nabla \longrightarrow \rho_s(\nabla)$$

*is a **bijection**.*

The *wild Riemann-Hilbert correspondence* is an *equivalence of Tannakian categories.*

Remarks.

1. There are extensions of the *wild Riemann-Hilbert correspondence* to ***non-linear situations*** in relation with problems of ***analytic classification*** (germs of *non linear analytic differential equations,* germs of *analytic diffeomorphisms,* germs of *analytic vector fields...*) [MR 1], [E]. In these generalisations one gets statements similar to *theorem 17*. In the case of *differential equations,* C^n is replaced by an analytic *manifold, End* $(n;\mathbf{C})$ by an analytic *vector field,* and $GL(n;\mathbf{C})$ by the analytic *pseudogroup* of automorphisms of the manifold. *Theorem 18* takes a quite technical form...

2. In such situations *Ecalle* introduces *"hidden variables"* ("variables cachées"). We can easily describe (and extends[2]) his viewpoint using our technics:

Let ∇ be a germ of meromorphic connection and let $\rho_s(\nabla)$ be the corresponding representation got from the wild Riemann-Hilbert correspondence. Let $X(\nabla)$ be the set of "letters" defined by

$X(\nabla) = \{\rho_s(\nabla)(\dot{\Delta}_{q,d})/\ q \in \mathbf{E}$ and $\mathbf{d} \in Fr\ q\}$. There are at most a finite number of values of (q,d) such that the matrix $\rho_s(\nabla)(\dot{\Delta}_{q,d})$ is not zero. If this matrix is *zero,* we *suppress* the coresponding letter. It remains a finite subset. $X'(\nabla)$. We set $Ass_{X'(\nabla)} = \mathcal{UR}(\nabla)$.

If f is a horizontal section of ∇, we consider

$$X(\nabla;f) = \{\rho_s(\nabla)(\dot{\Delta}_{q,d})(f)/\ q \in \mathbf{E} \text{ and } \mathbf{d} \in Fr\ q\}$$

and the set of "letters" $X'(\nabla;f)$ corresponding to $X'(\nabla)$. We set $\widehat{Ass}_{X'(\nabla;f)} = \mathcal{UR}(\nabla;f)$.

The idea is to *interpret* $\widehat{\mathcal{UR}}(\nabla;f)$ as a *"formal function"* on $\widehat{\mathcal{UR}}$ *"extending" f.* This "function" depends on *new (non commutative) variables,* the *"coordinates"* of the

[1] We recall that we suppose all the representations *continuous* on $\mathcal{T}$.

[2] Ecalle uses only particular "one-levelled" lattices.

elements of $\hat{\mathcal{UR}}$. These *"hidden variables"* belongs to the *dual* of $\hat{\mathcal{UR}}$. We will be more precise in *part 6* below, and interpret $\hat{\mathcal{UR}}(\nabla;f)$ as giving birth to a *"formal function"* on a ***principal bundle** with structure group* $\hat{\mathcal{R}}$, corresponding to an ***actual** function extending* f defined on a ***principal bundle** with structure group* $\mathcal{R}$. Moreover there are *natural actions* of $\pi_{1,s}((C^*,0);"R^+")$ on all these objects.

3. The *"Lie-algebra"* $Lie\ \pi_{1,s}((C,0);"R^+")$ of the *wild fundamental group* is the ***semi-direct** product of Lie-algebras* ($Lie\ \pi_{1,s}((C,0);"R^+") = \mathcal{T}$)

$$Lie\ \mathcal{T} \ltimes Lie\ \mathcal{R},$$

associated to the *action* of the *commutative algebra ("Cartan algebra")* $Lie\ \mathcal{T}$ on the *resurgent algebra* $Lie\ \mathcal{R}$ defined by

$$[H,\ \dot{\Delta}_{q,d}\ ] = q(H)\ \dot{\Delta}_{q,d}\ ,$$

$H \in Lie\ \mathcal{T}$, *where*

$$q:\ Lie\ \mathcal{T} \longrightarrow C$$

is the *infinitesimal map* associated to

$$q:\quad \mathcal{T} \longrightarrow C^*.$$

From the wild monodromy representation ρ_s we get a representation

$$Lie\ \rho_s : Lie\ \pi_{1,s}((C,0);"R^+") \longrightarrow End\ (n;C).$$

The restriction of this representation to $Lie\ \mathcal{R}$ is the map $Lie\ \rho_{res}$ of *theorem 17*. It corresponds to *Ecalle's **"bridge equation"** ("équation du pont")*.

We will explain now how to *change* the *"base point"* $"R^+"$ of the wild fundamental group $\pi_{1,s}((C^*,0);"R^+")$.

We will replace $"R^+"$ by $"d" \in \{"0"\} \times S^1$ $(("0",d) = "d")$ or $d \in \{"+\infty"\} \times S^1$ (that we can identify with S^1, the real analytic blow up of the origin in C).

We fix $"d" \in \{"0"\} \times S^1$. Let $"c"$ be an homotopy class of continuous paths on $\{"0"\} \times S^1$ with origin $"d"$ and extremity $"R^+"$ (corresponding to an homotopy class of paths c on S^1). We set

$\pi_{1,s}((C^*,0);"d") = \{"c"\, b "c"^{-1} /\ b \in \pi_{1,s}((C^*,0);"R^+")\}$, and put on this set the evident structure of group; $\pi_{1,s}((C^*,0);"d")$ is *independant* of the choice of c in a sense that we leave to the reader to explicit.

Let now $d \in \{"+\infty"\} \times S^1$, we set

$\pi_{1,s}((C^*,0);d) = \{(\gamma_d^-)^{-1}\, b\, \gamma_d^- /\ b \in \pi_{1,s}((C^*,0);"d")\}$, where the symbol γ_d^- corresponds to the *multisummation operator* S_d^- in "the" direction d^- (S_d^- is interpreted as an analytic continuation along γ_d^-). We put on $\pi_{1,s}((C^*,0);d)$ the evident structure of group.

We can also set

$$\pi_{1,s}((C^*,0);d) = \{(\gamma_d^+)^{-1}\, b\, \gamma_d^+ /\ b \in \pi_{1,s}((C^*,0);"d")\}:$$

there is a natural isomorphism between the two groups on the right side of our equalities.

We can now replace $\pi_{1,s}((C^*,0);"R^+")$ by $\pi_{1,s}((C^*,0);"d")$ or $\pi_{1,s}((C^*,0);d)$ in *theorem 18* (by definition $\rho_s(\nabla)("c")$ is the analytic isomorphism of solutions spaces given by the analytic continuation of a fundamental solution F_0 of "the" *formal normal form* corresponding to ∇ along c, $\rho_s(\nabla)(\gamma_d^-)$ is the isomorphism of spaces of solutions given by S_d^-). Elements of $\pi_{1,s}((C^*,0);d)$ are represented by linear permutations of *actual solutions* in a germ of sector bisected by d.

It is possible now to give a ***global*** *version* of our *wild fundamental group*.

Let X be a connected Riemann surface. Let $S = \{a_1,a_2,...,a_m\}$ be a finite subset of X, let x_0 be a base point in $X - S$, *and, for each* $i=1,...,m$, let d_i be a fixed direction "starting from a_i". We choose homotopy classes of paths c_i ("in" $X - S$) with *origin* x_0 and *extremity* a_i, "arriving at a_i *along the direction* d_i"$(i=1,...,m)$. We built, like above, groups

$G_i = \{c_i b c_i^{-1} / b \in \pi_{1,s}((C^*,0);d_i)\}$, $i=1,...,m$ (these groups are *independant* of the choice of c_i in a sense that we leave to the reader to explicit).

By *definition*[1] the *wild fundamental group* "of" $X - S$, *pointed at* x_0, is

$$\pi_{1,s}(X - S,S;x_0) = G_1 * ... * G_m \quad \textit{(free product of groups)},$$

and the *wild fundamental group of* X *is*

$$\pi_{1,s}(X - ...;.) = \varprojlim_{S} \pi_{1,s}(X - S;S;.)$$

(There are some trouble with base points in the limit: we get rid of it as in the ordinary case...)

It is easy to prove the following results (we define $\rho_s(\nabla)(c_i)$ as the analytic isomorphism of solutions spaces given by the analytic continuation along c_i):

We have a ***wild global Riemann-Hilbert correspondence:***

Theorem 19.

Let X be a connected Riemann surface.

The natural map

$$\begin{array}{ccc} \textbf{\textit{Meromorphic}}\ \textit{connections} & \xrightarrow{\rho_s} & \textit{Finite dimensional } \textbf{\textit{linear}} \\ \textit{on } X. & & \textbf{\textit{representations}}^2 \textit{ of the} \\ & & \textbf{\textit{wild fundamental group}} \\ & & \pi_{1,s}(X;.), \\ & & \textit{up to conjugation.} \\ & \nabla \longrightarrow \rho_s(\nabla) & \end{array}$$

is a ***bijection***.

---

[1] Be careful: the group *depends on* X *and* S, not only on $X - S$.

[2] We recall that we suppose all the representations *continuous* on T.

The *wild global Riemann-Hilbert correspondence* is an *equivalence of Tannakian categories*.

We will call the map $\rho_s(\nabla)$ ***wild monodromy representation*** of the *connection* ∇.

Let $\rho_m(\nabla)$ be the (classical) ***monodromy representation*** of the connnection ∇ (local or global case). It is possible to get[1] the *actual monodromy representation* $\rho_m(\nabla)$ from the *wild monodromy representation* $\rho_s(\nabla)$. If X is a connected Riemann surface, we will denote

$$\pi_1(X-...;.) = \underset{\overleftarrow{S}}{Lim}\ \pi_1(X-S;.)\ (S\ \textit{finite}\ \text{subset of}\ X).$$

Proposition 14.

(i) Let $d \in S^1$ be a fixed direction. There exists a "natural" functor $\mathfrak{D}$ from the tensor category of finite dimensional linear representations of $\pi_{1,s}((C^*,0);d)$ to the tensor category of finite dimensional linear representations of $\pi_1((C^*,0);d)$ such that

$$\mathfrak{D}(\rho_s(\nabla)) = \rho_m(\nabla)$$

for every germ of meromorphic connection ∇ at the origin.
This functor is defined by

$$\mathfrak{D}(\rho) = \rho_1(\gamma_{d_1}) \ldots \rho_1(\gamma_{d_p})\rho_1\ ,$$ *where (ρ_1, ρ_2) is the pair of representations in $GL(n;C)$ respectively from $\pi_{1,sf}((C^*,0);d)$ and $G\Pi(q_{\rho_1})$ (pointed at d) associated to ρ ($\boldsymbol{q} = \boldsymbol{q}_{\rho_1}$, and $d_1, ..., d_p$ are the directions of $Fr\,(\boldsymbol{q})$ contained in the interval $[0, 2\pi[\subset (\boldsymbol{R},0)$, **ordered** with the ordering relation induced by $\boldsymbol{R}$).*

(ii) Let X be a connected Riemann surface. There exists a "natural" functor $\mathfrak{D}$ from the tensor category of finite dimensional linear representations of $\pi_{1,s}(X-...;.)$ to the tensor category of finite dimensional linear representations of $\pi_1(X-...;.)$, such that

$$\mathfrak{D}(\rho_s(\nabla)) = \rho_m(\nabla),$$

for every meromorphic connection ∇.

We can reformulate *theorem 6* in a more *"geometric form"* (and extend it to the *global* case), *replacing* the *actual monodromy representation* by the *wild monodromy representation* in *Schlesinger's theorem::*

Theorem 20.

Let $K = \boldsymbol{C}\{x\}[x^{-1}]$. Let ∇ be a germ of meromorphic connection at the origin. We fix a $\boldsymbol{C}$-basis of the space of horizontal sections on a germ of sector bisected by a given

[1] In some sense π_1 is contained in a "completion" of $\pi_{1,s}$ and ρ_s can be extended to this completion "by continuity". Then ρ_m is the restriction to π_1 of this extension.

direction d *and identify the Galois differential group* $Gal_K(\nabla)$ *with its corresponding representation in* $GL(n;\mathbf{C})$.

Then $Gal_K(\nabla)$ *is the Zariski closure of the image in* $GL(n;\mathbf{C})$ *of the wild monodromy representation*

$$\rho_s(\nabla):\ \pi_{1,s}((\mathbf{C}^*,0);d) \longrightarrow GL(n;\mathbf{C}).$$

Theorem 21.

Let X *be a connected Riemann surface. Let* K_X *be the differential vector field of meromorphic functions on* X. *Let* ∇ *be a meromorphic connection on* X, *and* x_0 *a point of* X *regular for* ∇. *We fix a* $\mathbf{C}$*-basis of the space of horizontal sections of* ∇ *on a germ of small "disc" centered at* x_0 *and identify the Galois differential group* $Gal_{K_X}(\nabla)$ *with its corresponding representation in* $GL(n;\mathbf{C})$.

Then $Gal_{K_X}(\nabla)$ *is the Zariski closure of the image in* $GL(n;\mathbf{C})$ *of the wild monodromy representation*

$$\rho_s(\nabla):\ \pi_{1,s}(X;.) \longrightarrow GL(n;\mathbf{C}).$$

Examples and applications.

It is possible to ***compute explicitely the wild monodromy representations*** for the ***generalized confluent hypergeometric differential equations*** (using results of [DM]). These computations use *elementary functions* and *Γ-function.* It is possible to compute the Galois differential groups of the generalized confluent hypergeometric differential equations from these representations. This program is partially achieved [DM], [M1], [M2]. *C. Mitschi* has studied in particular *order* ***seven*** case and got, after *N. Katz* [K3], *generalized confluent hypergeometric differential equations of order seven admitting the* ***exceptional group*** G_2 ***as Galois differential group*** [M2].

It is possible to get an interesting result for the *"inverse problem"* in differential Galois theory from *theorem 18* (or *theorem 17*) [Ra 8]:

Theorem 22.

Let $\mathbf{L}$ *be a complex semi-simple Lie algebra. Let* ρ *be a finite dimensional representation of* $\mathbf{L}$. *Then:*

(i) There exists a rational differential equation $\mathbf{D}$ *on* $P^1(C)$, *with singularities contained in* $\{0,+\infty\}$, 0 *being regular singular and* $+\infty$ *irregular, such that* $Gal_{C(z)}(\mathbf{D})$ *is Zariski connected and such that*

$$Lie\ Gal_{C(z)}(\mathbf{D}) \approx \rho(L)\ \textit{(isomorphism of complex Lie-algebras)}.$$

(ii) There exists a germ of meromorphic differential equation D *at the origin such that* $Gal_K(D)$ *is Zariski connected and such that*

$$Lie\ Gal_K(D) \approx \rho(L).$$

We will end this paragraph by a comparison between *N. Katz's* viewpoint and ours.

Let X^{an} be a compact connected Riemann surface. Let S be a fixed *finite* subset of X^{an}. We denote by $D.E.(X^{an};S))$ the tensor category of *meromorphic connections* on X^{an} *with singularities contained in* S.

To each point z_0 *of* $X^{an} - S$ we can associate a *fibre functor* $\omega(z_0)$ of the tensor category $D.E.(X^{an};S))$:

$\omega(z_0)(\nabla) = \{$*horizontal sections of* ∇ on a germ of neighbourhood of $\nabla\}$.

We will denote by $\pi_1{}^{diff}(X^{an}-S;S;z_0)$ the group $Aut^{an}(\omega(z_0))$ (automorphisms of the fibre functor $\omega(z_0)$).

There is a natural map

$$\pi_{1,s}(X^{an}-S;S;z_0) \longrightarrow \pi_1{}^{diff}(X^{an}-S;S;z_0):$$

each element of $\pi_{1,s}(X^{an}-S;S;z_0)$ defines clearly an automorphism of the fibre functor $\omega(z_0)$.

Let Y be a smooth connected $\boldsymbol{C}$*-scheme* such that the corresponding *analytic variety* is the connected *Riemann surface* $X^{an} - S = Y^{an}$. We denote by $D.E.(Y/\boldsymbol{C})$ the tensor category of *algebraic connections* on Y. The natural map $\nabla \longrightarrow \nabla^{an}$ *gives* an *equivalence of tensor categories* between $D.E.(Y/\boldsymbol{C})$ and $D.E.(X^{an};S)$.

We denote by $\pi_1{}^{diff}(Y/\boldsymbol{C};z_0)$ the group $Aut(\omega(z_0))$ (automorphisms of the fibre functor $\omega(z_0)$).

There is a natural isomorphism between $\pi_1{}^{diff}(X^{an}-S;S;z_0)$ and $\pi_1{}^{diff}(Y/\boldsymbol{C};z_0)$. We get:

Proposition 15.

Let Y *be a smooth connected* $\boldsymbol{C}$*-scheme such that the corresponding analytic variety is the connected Riemann surface* $X^{an} - S = Y^{an}$*, where* X^{an} *is a compact Riemann surface and* S *a finite subset of* X^{an}*. Then* $\pi_1{}^{diff}(Y/\boldsymbol{C};z_0)$ *is an affine pro-algebraic* $\boldsymbol{C}$*-group-scheme and there exists a natural homomorphism of groups*

$$\pi_{1,s}(X^{an}-S;S;z_0) \longrightarrow \pi_1{}^{diff}(Y/\boldsymbol{C};z_0).$$

This map is *not onto*. We ignore if it is *injective*. Anyway $\pi_1{}^{diff}$ appears as an *"algebraic hull"* of $\pi_{1,s}$, just like $\pi_1{}^{diff}$ appears as an *algebraic hull* of $\pi_{1,m}$ in the *fuchsian case*.

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