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HYDRODYNAMIC LIMIT OF A GINZBURG-LANDAU LATTICE MODEL IN A SYMMETRIC RANDOM MEDIUM

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Abstract: The hydrodynamic behaviour of certain stochastic models is investigated in the presence of random conductivities. It is shown that this randomness of the medium averages out in such a way that the effective, macroscopic resistance of the medium turns out to be the the limiting mean value of microscopic resistances. This means that there in no interplay between nonlinearity of the evolution law and randomness of the medium; the effective conductivity does not depend on the interaction. In the most transparent, one-dimensional case the effective conductivity is just the harmonic mean of the microscopic conductivity. Some extensions including multidimensional systems in a small electric field are also discussed. A complete text is to appear in Commun. Math. Phys.

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1. Formulation of the Problem

Time-dependent Ginzburg-Landau equations describe thermally homogeneous systems near the critical point. In a critical situation the slowly varying, conservative forces, and the rapidly oscillating forces separate in such a radical way that the slowly varying component admits a thermodynamical description, while the oscillating forces are represented by a white noise in space and time. In a thermal equilibrium free energy is the fundamental thermodynamical potential, the simplest Ginzburg-Landau free energy functional reads formally as

$$H(\omega) = \int V(\omega(x)) + \frac{\alpha}{2}|\nabla\omega(x)|^2 \, dx \quad ,$$

where ∇ denotes the gradient of $\omega:\mathbb{R}^d \to \mathbb{R}$. The derivative, $\mathbb{D}H$ of H with respect to the extensive quantity ω plays the role of a chemical potential, its gradient drives the slowly varying, deterministic part of the current of ω ,

$$J_{det}(x,\omega) = -\frac{1}{2}\, c(x)\, \nabla\mathbb{D}H(x,\omega) \ , \ \nabla\mathbb{D}H(x,\omega) = V'(\omega(x)) - \alpha\Delta\omega(x) \ ,$$

where $c > 0$ denotes the conductivity of the (isotropic) medium. The stochastic part of the current is assumed to be of type

$$dJ_{ran}(t,x) = [c(x)]^{1/2}\, dw_t(x) \quad ,$$

where $dw_t(x)$ is a vector valued standard white noise in space and time. Now the evolution law is specified as a single conservation law:

$$d\omega_t(x) + \operatorname{div} J_{det}(x,\omega_t)\, dt = \operatorname{div} dJ_{ran}(t,x) \quad . \tag{1}$$

Stationary states of this reversible evolution are the canonical Gibbs states with energy H at unit temperature.

Our goal is to understand the hydrodynamic behaviour of such models in the case of microscopically random conductivities. Unfortunately, continuum models of this kind are very complicated from a technical point of view, see [Fu]. For example, non-equilibrium solutions to (1) can be constructed only in the one-dimensional case. We are going to discuss in details the following, one-dimensional lattice version of (1). The configurations of our system are then sequences $\omega = (\omega_k)_{k\in\mathbb{Z}}$ indexed by $\mathbb{Z}$, the set of integers, $\omega_k \in \mathbb{R}$, and the evolution is given by

$$d\omega_k = \frac{1}{2} c_k \, [V'(\omega_{k+1}) - V'(\omega_k)] \, dt - \frac{1}{2} c_{k-1} \, [V'(\omega_k) - V'(\omega_{k-1})] \, dt$$

$$+ [c_{k-1}]^{1/2} \, dw_{k-1} - [c_k]^{1/2} \, dw_k \quad , \; \omega_k(0) = \sigma_k \; , \; k\in\mathbb{Z} \; , \quad (2)$$

where $V: \mathbb{R} \to \mathbb{R}$ is a convex potential, w_k, $k\in\mathbb{Z}$ is a family of independent, standard Wiener processes, and c_k, $k\in Z$ is a fixed set of positive numbers. We are interested mainly in such situations when c_k are randomly selected, but in the present, one-dimensional case this structure is not relevant, we only need a law of large numbers for $1/c_k$, which is just the resistance of the bond between sites k and $k+1$. Since (2) is in fact a diffusive system, the rescaled density field, S^ε, should be defined as

$$S_t^\varepsilon(\varphi, \sigma_\varepsilon) = \int \varphi(x) \, \omega_t^\varepsilon(x) \, dx \; , \; \omega_t^\varepsilon(x) = \omega_{[x/\varepsilon]}(t/\varepsilon^2) \; , \; \sigma_\varepsilon = \omega_0^\varepsilon \quad , \; (3)$$

where $\varepsilon > 0$ denotes the scaling parameter, and $[u]$ is the integer part of $u \in \mathbb{R}$. According to the philosophy of hydrodynamic limits, we expect that $S_t^\varepsilon(t,\varphi)$ converges in probability to a deterministic limit $\int \varphi(x) \, \varrho_t(x) \, dx$ as $\varepsilon \to 0$, and ϱ_t, the limiting density, satisfies a nonlinear diffusion equation.

To expose the problem, let $c_\varepsilon(x) = c_{[x/\varepsilon]}$, $w_t^\varepsilon(x)=\varepsilon w_{[x/\varepsilon]}(t/\varepsilon^2)$, $\nabla_\varepsilon\varphi(x) = \varepsilon^{-1}[\varphi(x+\varepsilon)-\varphi(x)]$, $\nabla_\varepsilon^*\varphi(x) = \varepsilon^{-1}[\varphi(x-\varepsilon)-\varphi(x)]$, then

$$d\omega_t^\varepsilon = -\frac{1}{2}\nabla_\varepsilon^* c_\varepsilon \nabla_\varepsilon V'(\omega_t^\varepsilon)\,dt + \nabla_\varepsilon^*[(c_\varepsilon)^{1/2}\,dw_t^\varepsilon] \quad , \tag{4}$$

$$dS_t^\varepsilon = dM_t^\varepsilon - \frac{1}{2}\int (\nabla_\varepsilon\varphi(x))\, c_\varepsilon(x)\, \nabla_\varepsilon V'(\omega_t^\varepsilon(x))\,dx\,dt \quad , \tag{5}$$

where the martingale part M^ε vanishes as $\varepsilon \to 0$. In view of the principle of local equilibrium, the field V' converges in a weak $\mathbb{L}^2$-sense to a deterministic limit, and so does c_ε by the law of large numbers. This means that we have to evaluate the product of weakly convergent fields; which usually differs from the product of weak limits. Fortunately, there is a trivial particular case, the linear evolution equation corresponding to $V'(x) = x$. Indeed, if $p_c(t,j,k)$ denotes the transition probability of the random walk on $\mathbb{Z}$ with generator $\mathbb{G} = (-1/2)\nabla_1^* c \nabla_1$, and $\rho_k(t)$ is the conditional mean of $\omega_k(t)$ given the medium, then we have an explicit solution $\rho_k(t) = \Sigma_{j\in\mathbb{Z}}\, \rho_j(0)\, p_c(t,j,k)$. Therefore our diffusive scaling (3) results in a limiting equation $\partial\rho/\partial t = (\bar{c}/2)\, \partial^2\rho/\partial t^2$ for the limiting density $\rho = \rho(t,x)$, where $\bar{c}$ denotes the effective diffusion constant of the underlying random walk. For example, if c_k is an ergodic sequence, then $\bar{c}$ is just the harmonic mean of c , i.e. $1/\bar{c} = \langle 1/c_k \rangle$. The linear model can be treated in the same way in all dimensions, the only difference is that no explicit formula of $\bar{c}$ is available if $d > 1$.

The nonlinear problem reduces to the principle of local equilibrium by means of the following trick. Let $\varphi:\mathbb{R}\to\mathbb{R}$ be a smooth test function of compact support, and define φ_ε for $\varepsilon > 0$ by

$$\nabla_\varepsilon^* c_\varepsilon \nabla_\varepsilon \varphi_\varepsilon = \nabla_\varepsilon^* \bar{c} \nabla_\varepsilon \varphi \quad , \; \varphi_\varepsilon(0) = \varphi(0) \quad , \tag{6}$$

where $\bar{c}$ is specified in such a way that $\varphi_\varepsilon \to \varphi$ in $\mathbb{L}^2(\mathbb{R})$ as ε goes to 0. Such a construction is certainly possible if $c_\varepsilon(x) = c_{[x/\varepsilon]}$, where c_k, $k \in \mathbb{Z}$ is an ergodic sequence of harmonic mean $\bar{c}$. The case of several dimensions is more sophisticated. Using φ_ε of (6) as a test function for the weak form of (2) we see that φ_ε cancels the singularity of the evolution law due to the randomness of the medium. Since ∇_ε^* is the formal adjoint of ∇_ε, from (2) by the Ito lemma we obtain

$$dS_t^\varepsilon(\varphi_\varepsilon,\sigma_\varepsilon) = dM_t^\varepsilon - \frac{1}{2} \int \nabla_\varepsilon^* \bar{c} \nabla_\varepsilon^* \varphi(x) \; V'(\omega_t^\varepsilon(x)) \, dx \, dt \quad , \tag{7}$$

which is much nicer than (5). Now we can apply the principle of local equilibrium to conclude that the field $V'(\omega_t^\varepsilon(\cdot))$ converges weakly to $J'(\rho_t(\cdot))$, where ρ is the limiting density, and

$$J(\rho) = \sup_\lambda \, [\lambda\rho - F(\lambda)] \quad , \; F(\lambda) = \log \int \exp(\lambda x - V(x)) \, dx \; ; \tag{8}$$

notice that $J'(\rho)$ is just the canonical mean value of $V'(\omega_k)$ given the mean spin ρ. On the other hand, $\varphi_\varepsilon \to \varphi$ in $\mathbb{L}^2$, thus $S_t^\varepsilon(\varphi,\sigma_\varepsilon)$ is expected to converge to $\int \varphi(x) \, \rho_t(x) \, dx$, where ρ_t is a weak solution to the limiting equation

$$\frac{\partial \rho}{\partial t} = \frac{1}{2} \frac{\partial}{\partial x}[\bar{c}(x) \, \frac{\partial J'(\rho)}{\partial x}] \quad , \tag{9}$$

this solution is uniquely determined by its initial value. The function $\bar{c}$ is defined by (6) and the requirement $\varphi_\varepsilon \to \varphi$; both should be satisfied for all smooth φ of compact support.

2. Main Result and Discussions

Our statement on the hydrodynamic limit of (2) is basically a law of large numbers, it can be formulated in terms of the following functional spaces. The phase space of the rescaled process ω_t^ε is a locally convex, reflexive space $\mathbb{L}_e^2$ defined by means of some Hilbert norms $|\cdot|_r$, $r > 0$; $|u|_r^2 = \int e^{-r|x|} u^2(x)\, dx$. The dual space of $\mathbb{L}_e^2$ will be denoted by $\mathbb{L}_e^{2*}$, while $\mathbb{H}_e^1$ and $\mathbb{H}_e^{1*}$ are the spaces of absolutely continuous $u:\mathbb{R}\to\mathbb{R}$ such that $u,u'\in\mathbb{L}_e^2$ and $u,u'\in\mathbb{L}_e^{2*}$, respectively. The weak topology of $\mathbb{L}_e^2$ plays a crucial role in the proofs, $\mathbb{C}_w$ denotes the space of weakly continuous and bounded $g:\mathbb{L}_e^2\to\mathbb{R}$. At a given level $\varepsilon > 0$ of scaling the configurations of our system are embedded into $\mathbb{L}_e^2$ as step functions of step size ε . In view of this correspondence we say that μ_ε , a family of Borel probabilities on $\mathbb{L}_e^2$, satifies the law of large numbers with asymptotic mean $\varrho\in\mathbb{L}_e^2$ if $\lim \int g\, d\mu_\varepsilon = g(\varrho)$ for all $g\in\mathbb{C}_w$ as $\varepsilon\to 0$.

The potential V is assumed to have three continuous derivatives, V''' is bounded, and $0 < \alpha_1 \le V''(x) \le \alpha_2$ for all x . The evolution is defined by (4), where $c_\varepsilon = c_\varepsilon(x)$ is a step function of step size ε such that $\alpha_1 \le c_\varepsilon(x) \le \alpha_2$, and we have some continuously differentiable $\bar{c}:\mathbb{R}\to\mathbb{R}$ such that

$$\lim_{\varepsilon\to 0} \int \varphi(x)/c_\varepsilon(x)\, dx = \int \varphi(x)/\bar{c}(x)\, dx \quad \text{for } \varphi\in\mathbb{H}_e^{1*} . \tag{10}$$

Our basic result is the following

THEOREM. Suppose that the initial configuration, σ_ε converges weakly in $\mathbb{L}_e^2$ to some $\sigma\in\mathbb{H}_e^1$ as $\varepsilon\to 0$. Then $S_t^\varepsilon(\varphi,\sigma_\varepsilon)\to$ $\int \varphi(x)\,\varrho_t(x)\,dx$ in probability for each $\varphi\in\mathbb{H}_e^{1*}$, where ϱ is the weak solution to (9) with initial value $\varrho_0=\sigma$. ▮▮▮

A complete proof of this result is to be published in [Fr3], the resolvent approach of [Fr1],[Fr2] is used. Following the perturbative argument of [FM], the following, weakly asymmetric problem can also be treated. Let $e_\varepsilon:\mathbb{R}\to\mathbb{R}$ be uniformly bounded, and consider

$$d\omega_t^\varepsilon = \nabla_\varepsilon^*[e_\varepsilon c_\varepsilon B(\omega_t)]\,dt - \frac{1}{2}\nabla_\varepsilon^* c_\varepsilon \nabla_\varepsilon V'(\omega_t^\varepsilon(\cdot))\,dt + \nabla_\varepsilon^*\sqrt{c_\varepsilon}\,dw_t^\varepsilon \qquad (11)$$

where c_ε is a random conductivity as before, e_ε is interpreted as an external, electric field. We expect that the electric field averages out in a direct way; if

$$\lim_{\varepsilon\to 0}\int \varphi(x)e_\varepsilon(x)\,dx = \int \varphi(x)\bar{e}(x)\,dx \quad \text{for } \varphi\in\mathbb{H}_e^{1*}, \qquad (12)$$

then the limiting equation reads as

$$\frac{\partial\varrho}{\partial t} = \frac{\partial}{\partial x}[\bar{e}(x)\bar{c}(x)\bar{B}(\varrho)] + \frac{1}{2}\frac{\partial}{\partial x}[\bar{c}(x)\frac{\partial\varrho}{\partial x}] \quad , \qquad (13)$$

where $\bar{B}$ is the canonical equilibrium expectation of B.

In two or more space dimensions there is no explicit way to describe the medium and the effective conductivities. Microscopic conductivities are associated to the bonds of $\mathbb{Z}^d$, they can also be interpreted as the transition rates of a symmetric random walk on $\mathbb{Z}^d$. Suppose that this random walk admits a diffusion limit with diffusion constant $\bar{c}>0$; using the methods of [Fr2] and the ideas outlined below, it seems to be possible to pass to the hydrodynamic limit, the effective conductivity will be just $\bar{c}$.

A direct extension of the methods of [Fr2] allows us to consider models containing random, ferromagnetic interaction terms in the Ginzburg-Landau free energy. More exactly, let

$$H(\omega) = \sum_{k\in\mathbb{Z}^d} [\ V(\omega_k) + \sum_{|j-k|=1} b_{kj}\ U(\omega_k-\omega_j)\ ] \quad , \tag{14}$$

where V is the same as before, $b_{kj}=b_{jk} > 0$, $U(x)=U(-x)$, $U''(x) \geq 0$, and both U'' and b_{kj} are bounded. In this general case the Wiener process w^ε of (4) is vector valued, ∇_ε and ∇_ε^* are the discrete gradient of step size ε , and its adjoint, respectively, and V' should be replaced by the gradient of H . If b_{kj} is an ergodic system of bond variables, then the subbaditive ergodic theorem shows that the limiting equation does not depend on the randomness of b , but explicit expressions are not available.

3. On the Idea of the Proof

Like in all references enclosed below, our basic information on the dynamics concerns the smoothness of time averages of type

$$X_z^\varepsilon(\sigma,g) = \int_0^\infty e^{-zt}\ g(\omega_t^\varepsilon)\ dt \quad , \ z > 0 \ , \ g\in\mathbb{C}_w \ , \ \sigma=\omega_0^\varepsilon \quad . \tag{15}$$

More exactly, both X^ε and its variational derivative $\mathbb{D}X_z^\varepsilon(x,\sigma,g)$ are equicontinuous functions of the initial configuration σ with respect to the weak topology of $\mathbb{L}_e^2$, see [Fr1],[Fr2]. This delicate property is due to the parabolic structure of the evolution equation, cf. $V'' > 0$. Let $f_z^\varepsilon(\sigma)$ denote the conditional expectation of X^ε given the initial value σ , then $g(\sigma) = z\ f_z^\varepsilon(\sigma) - \mathbb{G}_\varepsilon f_z^\varepsilon(\sigma)$, where $\mathbb{G}_\varepsilon$ denotes the generator of the rescaled process ω_t^ε . In view of the resolvent approach, we want to pass along subsequences to the resolvent equation of the limiting semigroup defined by (9), it reads as

$$g(\varrho) = z\, f_z(\varrho) + \frac{1}{2} \iint [\partial_x J'(\varrho(x))]\bar{c}(x)\, \partial_x \mathbb{D} f_z(x,\varrho) \ , \ \varrho \in \mathbb{H}_e^1 \ . \quad (16)$$

The original resolvent equation $g = zf - \mathbb{G}_\varepsilon f$ is an elliptic equation in the functional space $\mathbb{C}_w$, so it is quite natural to look for its weak form. Elements of the dual space of $\mathbb{C}_w$ are measures on $\mathbb{L}_e^2$, therefore we have to integrate with respect to a clever measure. Let $\mu_{\lambda,\varepsilon}$ denote the Gibbs state with energy $H(\omega) = \Sigma\, V(\omega_k) - \Sigma\, \lambda_k^\varepsilon \omega_k$, where λ^ε is a real sequence indexed by $\mathbb{Z}$; the projection of this measure on $\mathbb{L}_e^2$ will be denoted by the same symbol, thus $\mu_{\lambda,\varepsilon}$ is concentrated on step functions of step size ε . Integrating by parts we obtain that if $\lambda^\varepsilon(x) = \lambda^\varepsilon_{[x/\varepsilon]}$, then

$$\int g(\sigma)\, \mu_{\lambda,\varepsilon}(d\sigma) = z \int f_z^\varepsilon(\sigma)\, \mu_{\lambda,\varepsilon}(d\sigma)$$

$$+ \frac{1}{2} \iint (\nabla_\varepsilon \lambda^\varepsilon(x))\, c_\varepsilon(x)\, \nabla_\varepsilon \mathbb{D} f_z(x,\sigma)\, dx\, \mu_{\lambda,\varepsilon}(d\sigma) \ , \quad (17)$$

and now we are in a position to use the trick of (6). Indeed, in the present, one-dimensional case we can define $\lambda_k^\varepsilon = \lambda^\varepsilon(\varepsilon k)$ by

$$c_\varepsilon(x)\, \nabla_\varepsilon \lambda^\varepsilon(x) = (I_\varepsilon \bar{c}(x))\, \nabla_\varepsilon J'(I_\varepsilon \varrho(x)) \quad , \quad \lambda^\varepsilon(0) = J'(I_\varepsilon \varrho(0)) \ , \quad (18)$$

where $I_\varepsilon \varphi(x)$ is the integral mean of φ over $[\varepsilon k, \varepsilon k+\varepsilon)$ with $k = [x/\varepsilon]$. This transformation removes the singularity of (17) due to the randomness of c_ε , and a direct calculation shows that $\mu_{\lambda,\varepsilon}$ satisfies the law of large numbers with asymptotic mean ϱ . Therefore, a compactness argument based on the continuity properties of

X_z^ε and $\mathbb{D}X_z^\varepsilon$ allows us to pass from (17) to (16). Since (16) is uniquely solved, each subsequence converges to the very same limit, and the proof can be completed in the same way as in [Fr2].

In the multidimensional case (18) can not be solved, then we define λ^ε by

$$\delta_\varepsilon \lambda^\varepsilon + \nabla_\varepsilon^*(c_\varepsilon \nabla_\varepsilon \lambda^\varepsilon) = \nabla_\varepsilon^*(I_\varepsilon \bar{c}\, \nabla_\varepsilon J'(I_\varepsilon \varrho)) \quad , \tag{19}$$

where $\delta_\varepsilon > 0$ goes to zero as $\varepsilon \to 0$. This resolvent equation is uniquely solved if $\varrho \epsilon L^2(\mathbb{R}^d)$, and $\lambda \to J'(\varrho)$ in $\mathbb{L}^2(\mathbb{R}^d)$ whenever $\bar{c}$ and ϱ are smooth enough. This means that the corresponding family $\mu_{\lambda,\varepsilon}$ satisfies the law of large numbers with asymptotic mean ϱ , thus we can proceed as before, the general case of $\varrho \epsilon \mathbb{H}_e^1$ reduces to this one by an easy approximation procedure.

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