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Les rencontres physiciens-mathématiciens de Strasbourg - RCP25, 1988, tome 39
« Conférences de P. Degond, R. Gérard, C. Itzykson, F. Wegner et Mlle S. Rousset », ,
exp. n° 2, p. 21-31

<http://www.numdam.org/item?id=RCP25_1988__39__21_0>

Berry's Phase and Quantized Hall Effect

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The connection between Berry's phase and the quantized Hall effect is reviewed. In the first section an introduction to the quantized Hall effect is given, in the second Berry's phase is introduced and determined. In the third section Avron's and Seiler's proof of quantized transport is given in an elementary way and the connection to Berry's phase is made. Remarks are added on the quantization of the Hall conductance in periodic potentials and on a generalization to the fractional quantized Hall effect. Since this seminar was given nearly two years ago I take the liberty to include a few new remarks and references.

1. The Quantized Hall-Effect

In 1980 K. v. Klitzing discovered the quantized Hall effect [1]. He observed that at low temperatures effectively two dimensional systems of electrons (like MOSFETs) subjected to a strong perpendicular magnetic field show a peculiar behaviour as a function of some external parameter like the magnetic field or the gate voltage (a voltage perpendicular to the two dimensional system which controls the density of electrons): Over large regions of this parameter the electronic current flows practically without dissipation and the voltage between two points is given by the current flowing between these points multiplied by the Hall conductance G_H which to very high precision is given by e^2/h times an integer, where e is the charge of an electron and h is Planck's constant.

For free and noninteracting electrons the eigenenergies of the Hamiltonian are quantized, $E_n = \hbar\omega_c(n+\frac{1}{2})$ where $\omega_c = \frac{eB}{m}$ is the cyclotron frequency in the magnetic field B. Each filled Landau band contributes exactly e^2/h to the Hall conductance. If the Fermi energy is between two Landau levels then G_H is indeed an integer multiple of e^2/h.

If one neglects the interaction, but takes into account that the electrons move in a potential which in reality will be disordered, then one can still give arguments for this quantization of the Hall conductance. The arguments to be considered here are gauge or topological arguments: They use that in a system with a hole (which cannot be entered by the electrons) through which there is a magnetic flux Φ a change of this flux by one flux quantum $\Phi_0 = h/e$ yields eigenstates with exactly the same energies. An intuitive argument is the following: Suppose the electrons live on a cylindric ring. Besides the magnetic field perpendicular to this ring there is a flux Φ through the ring. Suppose the potential fluctuations are so weak that there is still an energy gap between the Landau levels and suppose close to the edges there is an ordered region in which the potential increases linearly. Then an adiabatic change of the flux Φ by Φ_0 will move the electrons near the edges to the next eigenstate. Thus for each filled Landau level one electron is carried from one edge to the other. The change of the flux Φ in a time interval Δt induces a voltage $V_H = \Phi_0/\Delta t$ around the ring. The current is $I = en/\Delta t$ for n filled Landau levels. Thus one finds the Hall conductance $G_H = I/V_H = ne^2/h$. Normally the Landau levels will not be completely occupied or empty. One has good reasons, however, to believe, that the states in the tails of the Landau levels are localized so that they do not contribute to the current. Thus independent of whether these states are occupied or not one observes the quantized conductance.

Laughlin who probably first gave a gauge argument [2], used that $\partial E/\partial\Phi$ equals the current around the cylinder. Apparently for one Landau level one has $\Delta E = eV_H$, $\Delta\Phi = \Phi_0$, where V_H is now the potential difference between the edges of the cylinder and thus he obtains the same Hall conductance G_H. In his argument the variation of Φ is a virtual one. It serves only to obtain a relation between the current and the voltage but it does not generate them.

A beautiful argument which allows to take the interaction into account and which is mathematically rigorous was given by Avron and Seiler [3]. It is in many respects much more general and more precise than the arguments before, although it has one drawback. It has to be assumed and the authors are aware of this, that the system is in a state which is not degenerate with other eigenstates, an assumption which in practice is not well fulfilled, since the metallic wires allow for a quasicontinuum of states. It is not obvious how to incorporate the idea of localized states into the proof. The proof uses both the idea of a flux Φ_1 which induces the voltage and that of a flux Φ_2 which serves to determine the current. Moreover Φ_1 and Φ_2 have to be averaged over one flux quantum, but this was also true for the arguments given before. Since $\Phi_0 = 4.135 \cdot 10^{-15}$ Vs is very small, Φ_1 has to be varied over a large number of flux quanta anyhow and Φ_2 can hardly be kept constant within one flux quantum in an experiment which has to be performed in strong magnetic fields.

2. Berry's Phase

In 1984 M.V. Berry investigated a problem [4] which at first glance has no relation to the quantized Hall effect. He considered a quantum mechanical system undergoing an adiabatic change of the Hamiltonian. If initially the state is in an eigenstate $|\Omega\rangle$ of the Hamiltonian H, then it will remain close to an eigenstate all the time provided this eigenstate is never degenerate with any other eigenstate. The question Berry posed himself was: What is the change in phase of the state, if finally the Hamiltonian returns to the original one? In leading order of course there is a dynamical phase φ given by $-\hbar^{-1}$ multiplied by the time integral over the energy of the eigenstate. Berry found that there is an additional geometric phase γ in the adiabatic limit which depends only on the contour passed by the Hamiltonian in the course of the time. This contribution is now called Berry's phase.

It turns out that this phase is closely related to the quantized Hall effect. We will see (towards the end of sect. 3) that the Hall conductance is given by Berry's phase which is obtained by going in the space of fluxes Φ_1, Φ_2 around a square of size $\Phi_0{*}\Phi_0$. First, however let us determine Berry's phase: Be t the time. The system is governed by the time-dependent Hamiltonian $H(\varepsilon t)$ where the adiabatic limit is given by $\varepsilon \to 0$. We introduce a new variable $\tau = \varepsilon t$. The state $|\Omega_V(t)\rangle$ of the system be close to the eigenstate $|\Omega(\tau)\rangle$ of $H(\tau)$

$$(H(\tau)-E(\tau))\,|\Omega(\tau)\rangle = 0. \tag{2.1}$$

Inserting

$$|\Omega_V(t)\rangle = e^{i\chi(t)}\,\{|\Omega(\tau)\rangle + \varepsilon\,|\bar{\Omega}(\tau)\rangle + O(\varepsilon^2)\} \tag{2.2}$$

into the time dependent Schrödinger equation

$$H(\varepsilon t)\,|\Omega_V(t)\rangle = i\hbar\,\frac{d}{dt}|\Omega_V(t)\rangle \tag{2.3}$$

yields

$$E(\tau)|\Omega(\tau)\rangle + \varepsilon H(\tau)|\bar{\Omega}(\tau)\rangle = -\,\hbar\dot{\chi}(\tau)|\Omega(\tau)\rangle - \varepsilon\hbar\dot{\chi}(\tau)|\bar{\Omega}(\tau)\rangle + i\varepsilon\,\hbar\,\frac{d}{d\tau}|\Omega(\tau)\rangle + O(\varepsilon^2) \tag{2.4}$$

Multiplication by $\langle\Omega(\tau)|$ gives

$$E(\tau) = -\hbar\dot{\chi}(t) + i\hbar\varepsilon\langle\Omega(\tau)|\frac{d}{d\tau}|\Omega(\tau)\rangle + O(\varepsilon^2) \tag{2.5}$$

so that we obtain

$$\chi(t) = -\frac{1}{\hbar\varepsilon}\int^{\varepsilon t} E(\tau')d\tau' + i\int_C^{\varepsilon t} <\Omega(\tau')|\frac{d}{d\tau'}|\Omega(\tau')>d\tau' + O(\varepsilon) \tag{2.6}$$

The first contribution is the integral over the eigenenergy mentioned initially, the second one taken over a closed contour C is called the Berry phase $\gamma(C)$.

To see that the Berry phase depends only on the contour C but not on the parametrization by τ we introduce a surface in the space of Hamiltonians which contains the contour parametrized by Φ_1 and Φ_2. Then we may write $H(\tau) = H(\Phi_1(\tau),\Phi_2(\tau))$. With the abbreviation $\partial_i = \partial/\partial\Phi_i$ the phase can be written as contour integral in the Φ_1-Φ_2 space

$$\gamma(C) = i\int_C <\Omega|\partial_i\Omega>d\Phi_i(\tau) \tag{2.7}$$

which is independent of the parametrization on τ. By means of Stokes' theorem it can be written as a surface integral

$$\gamma(C) = i\int_S(\partial_1<\Omega|\partial_2\Omega>-\partial_2<\Omega|\partial_1\Omega>)d\Phi_1 d\Phi_2 = i\int_S(<\partial_1\Omega|\partial_2\Omega>-<\partial_2\Omega|\partial_1\Omega>)d\Phi_1 d\Phi_2 \tag{2.8}$$

It can easily be seen that the integrand in the last line is invariant under phase changes $|\Omega> \to \exp(i\lambda(\Phi_1,\Phi_2))|\Omega>$. In order to evaluate (2.8) one has to determine $|\partial_i\Omega>$. From $(H-E)|\Omega> = 0$ one obtains $(\partial_i H-\partial_i E)|\Omega> + (H-E)|\partial_i\Omega> = 0$ from which one deduces

$$|\partial_i\Omega> = - G\partial_i H|\Omega> + |\Omega><\Omega|\partial_i\Omega>. \tag{2.9}$$

with the resolvent $G = (1- |\Omega><\Omega|)\,(H-E)^{-1}\,(1- |\Omega><\Omega|)$.

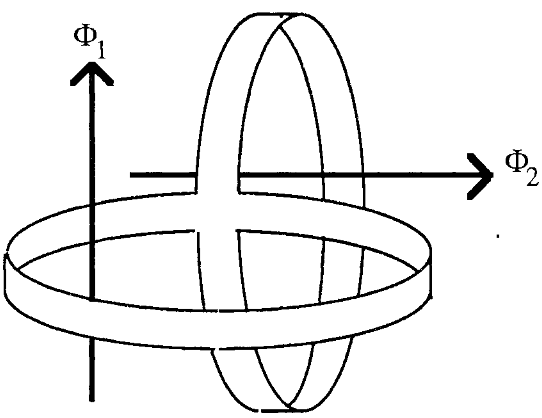
The appearence of the resolvent makes clear that one obtains large contributions to $\gamma(C)$ close to a degeneracy of eigenstates. The generic case is the Hamiltonian

$$H(\vartheta) = \vartheta_0(\tau) + \underline{\vartheta}(\tau)\underline{\sigma} = \begin{pmatrix} \vartheta_0+\vartheta_3 & \vartheta_1-i\vartheta_2 \\ \vartheta_1+i\vartheta_2 & \vartheta_0-\vartheta_3 \end{pmatrix} \tag{2.10}$$

with eigenyalues $E_\pm = \vartheta_0 \pm |\underline{\vartheta}|$. The phase $\gamma(C)$ equals $\pm\,\Omega(C)/2$, where $\Omega(C)$ is the solid angle enclosed by the vector $\underline{\vartheta}(\tau)$. This implies that a spin in a magnetic field picks up the phase $\gamma(C) = -\,m_B\Omega(C)$ where m_B is the spin component in **B** direction. Such a phase can only be seen in interference experiments. Moreover to change the Hamiltonian as a function of time does not seem to be the appropriate way to observe this phase. However one can change the surrounding while a particle moves along some path. Such experiments have been performed and confirm the predictions by Berry. Berry's phase has been observed for example for polarized light in optical fibers by Tomita and Chiao [5] and for polarized neutrons in a helical magnetic field by Bitter and Dubbers [6]. In the Aharonov-Bohm-effect [7] electrons move around a tube pinched by a magnetic flux which yields interferences, although the electrons do not directly feel the magnetic field inside the tube, an effect which can also be understood in terms of Berry's phase.

3. Quantized Conductance

Avron and Seiler [3] (see also [8,9]) considered a system with two holes (regions from which the electrons are excluded) punched by fluxes Φ_1 and Φ_2. Provided the system is in a state which for arbitrary Φ_1 and Φ_2 is not degenerate then the adiabatic change of Φ_1 which induces a voltage $\dot{\Phi}_1$ around the hole #1 yields after average over Φ_1 and Φ_2 over a flux quantum $\Phi_0 = h/e$ a vanishing conductance around the loop #1 and a Hall conductance around loop #2 which is an integer multiple of e^2/h. The authors need not make any assumptions concerning the dimensionality of the system. They need not assume a strong magnetic field. Moreover their argument holds for a system with many particle interaction.



The argument proceeds as follows: The magnetic field of the system can be described by a vector potential $A(r) = A_0(r) + \Phi_1\Gamma_1(r) + \Phi_2\Gamma_2(r)$. Outside the flux tubes one has curl $\Gamma_i(r) = 0$. Thus inside the region where the electrons move one has $\Gamma_i(r) = \text{grad}\,\Lambda_i(r)$. The potential $\Lambda_i(r)$ is only defined modulo 1. It grows by unity by circling the hole #i, since the integral over A along a closed path yields the enclosed flux. The Hamiltonian reads

$$H = \sum_n \frac{1}{2m}[\frac{\hbar}{i}\nabla_n - eA(r_n)]^2 + V(r). \tag{3.1}$$

The current can be obtained from

$$<\frac{\partial H}{\partial \Phi_i}> = -\frac{e}{2}<\sum_n\{v_n,\Gamma_i\}> = -\int d^3r\ \Gamma_i(r)\ <j(r)> = -\int d^3r\ \mathrm{grad}\Lambda_i(r)\ <j(r)>$$

$$= -\int_0^1 d\lambda_i \int df\ <j(r)> = -I_i \tag{3.2}$$

where the df integral has to be taken over the surface $\Lambda_i(r) = \lambda_i$. Thus the right hand side is the λ_i-average over the currents through these surfaces. From this expression for the current one obtains

$$<\Omega_V(t)|I_i|\Omega_V(t)> = -<\Omega|\partial_iH|\Omega> - \varepsilon<\bar{\Omega}|\partial_iH|\Omega> - \varepsilon<\Omega|\partial_iH|\bar{\Omega}> \tag{3.3}$$

The first term on the right handside yields $<\Omega|\partial_iH|\Omega> = \partial_i<\Omega|H|\Omega>$. To evaluate the other terms we first realize that inserting eq. (2.5) into (2.4) yields $(H-E)|\bar{\Omega}(\tau)> = i\hbar\frac{d}{d\tau}|\Omega(\tau)> - i\hbar|\Omega><\Omega|\frac{d}{d\tau}|\Omega>$ and thus

$$|\bar{\Omega}> = i\hbar G\frac{\partial}{\partial\tau}|\Omega(\tau)> = i\hbar\dot{\Phi}_1\frac{1}{\varepsilon}G|\partial_1\Omega> \tag{3.4}$$

with the resolvent G introduced after eq. (2.9). With this eq. one obtains

$$\varepsilon<\bar{\Omega}|\partial_iH|\Omega> = -i\hbar\dot{\Phi}_1<\partial_1\Omega|G\partial_iH|\Omega> = i\hbar\dot{\Phi}_1<\partial_1\Omega|\partial_i\Omega> - i\hbar\dot{\Phi}_1<\partial_1\Omega|\Omega><\Omega|\partial_i\Omega>, \tag{3.5}$$

from which we obtain

$$<\Omega_V(t)|I_i|\Omega_V(t)> = -\partial_i<\Omega|H|\Omega> + i\hbar\dot{\Phi}_1(<\partial_1\Omega|\partial_i\Omega> - <\partial_i\Omega|\partial_1\Omega> - <\partial_1\Omega|\Omega><\Omega|\partial_i\Omega>$$

$$+ <\partial_i\Omega|\Omega><\Omega|\partial_1\Omega>) \tag{3.6}$$

The last two terms cancel since $<\Omega|\partial_i\Omega>$ is purely imaginary. The average over Φ_i of the first term vanishes, since $<\Omega|H|\Omega>$ is periodic in Φ_i with period Φ_0. Thus one obtains for the average over Φ_1 and Φ_2

$$\bar{I}_i = \frac{ie^2\dot{\Phi}_1}{2\pi h}\int_0^{\Phi_0} d\Phi_1 \int_0^{\Phi_0} d\Phi_2(<\partial_1\Omega|\partial_i\Omega> - <\partial_i\Omega|\partial_1\Omega>). \tag{3.7}$$

The average current around hole #1 vanishes since the integrand vanishes. The Hall current around hole #2 can be written

$$\bar{I}_2 = \frac{e^2\dot{\Phi}_1}{2\pi h}\gamma(C), \tag{3.8}$$

where $\gamma(C)$ is the Berry phase for the system where one starts with $\Phi_1 = \Phi_2 = 0$, increases Φ_1 by Φ_0, then Φ_2 by Φ_0, then one decreases Φ_1 by Φ_0 and finally Φ_2 by Φ_0. In the following it will be shown that $\gamma(C)$ is quantized. It assumes only values which are integer multiples of 2π. For this purpose we express $\gamma(C)$ as contour integral

$$\gamma(C) = i\int_0^{\Phi_0} d\Phi_2[<\Omega|\partial_2\Omega>(\Phi_0,\Phi_2) - <\Omega|\partial_2\Omega>(0,\Phi_2)] - (1\leftrightarrow 2). \tag{3.9}$$

Now $|\Omega(\Phi_0,\Phi_2)>$ and $|\Omega(0,\Phi_2)>$ are connected by

$$|\Omega(\Phi_0,\Phi_2)> = \exp[i\theta_1(\Phi_2)+2\pi i\sum_n \Lambda_1(r_n)]\ |\Omega(0,\Phi_2)>. \tag{3.10}$$

Then the integrand in (3.9) yields $id\theta_1(\Phi_2)/d\Phi_2$. With a similar expression for $|\Omega(\Phi_1,\Phi_0)>$

$$|\Omega(\Phi_1,\Phi_0)> = \exp[i\theta_2(\Phi_1)+2\pi i\sum_n \Lambda_2(r_n)]\ |\Omega(\Phi_1,0)>. \tag{3.11}$$

one obtains

$$\gamma(C) = -\theta_1(\Phi_0) + \theta_1(0) + \theta_2(\Phi_0) - \theta_2(0). \tag{3.12}$$

Since from (3.10) and (3.11) one has

$$|\Omega(\Phi_0,\Phi_0)> = \exp[i\theta_2(\Phi_0)+i\theta_1(0)+2\pi i\sum_n(\Lambda_1(r_n)+\Lambda_2(r_n))]\ |\Omega(0,0)>$$

$$= \exp[i\theta_1(\Phi_0)+i\theta_2(0)+2\pi i\sum_n(\Lambda_1(r_n)+\Lambda_2(r_n))]\ |\Omega(0,0)>, \tag{3.13}$$

it turns out that the right hand side of (3.12) is indeed 2π times an integer and thus the Hall conductance is quantized.

The argument given for the 'integer' quantized Hall effect can be extended to the fractional quantized Hall effect discovered in 1982 by Tsui, Störmer, and Gossard [10], where the Hall conductance is a fraction (with a small numerator q) times e^2/h. In the topological framework it can be understood, if q eigenstates are nearly degenerate and equally populated [8,11]. The argument will be outlined in the appendix. It has been shown that interacting electrons in a periodic potential on a torus yield such degenerate eigenstates [12-14] and that a perturbation of this system by a random potential yields a lifting of this degeneracy which as a function of the size of the system becomes exponentially small [11,15], so that the assumption of the degeneracy is quite reasonable.

Systems of two dimensional electrons in a magnetic field and a periodic potential had been considered much earlier. If in a system with periodic potential a magnetic field is added so that the flux per elementary cell is $(p/q)\Phi_0$ where p and q are prime, then each band will break into q subbands, and on the other hand if to a system in a magnetic field a periodic potential is added, then each Landau level will break into p subbands. This was nicely demonstrated by Hofstadter [16], who plotted the energy bands for all values of q up to 50. Thouless et al [17] have shown in 1982 that each subband contributes an integer times e^2/h to the Hall conductance. They used already arguments similar to that of Avron and Seiler. Starting from the Kubo-formula they had to integrate over the magnetic Brillouin zone, so that the two components of the momentum held the place of Φ_1 and Φ_2 in Avron's and Seiler's calculation. Moreover they did not only show the quantization of G_H, but gave the contributions of the subbands in terms of the solution of a Diophantine equation. For a survey see the article by Thouless [18].

Acknowledgement. I am indebted to J. Hajdu and R. Seiler for useful discussions.

Appendix

Here I outline the proof for the fractional Hall effect. I consider q eigenstates which are nearly degenerate. I require that they are linear combinations of q orthogonal states $|\Omega_\alpha(\tau)>$

$$H(\tau)\,|\Omega_\alpha(\tau)> = \sum_{\beta=1}^{q} E_{\alpha\beta}|\Omega_\beta(\tau)>. \tag{A.1}$$

Then the state $|\Omega_\gamma(t)>$ at time t can be written

$$|\Omega_\gamma(t)> = \sum_{\alpha\beta} a_\alpha U_{\alpha\beta}(t)\{|\Omega_\beta(\tau)>+\varepsilon|\bar{\Omega}_\beta(\tau)>+O(\varepsilon^2)\}, \tag{A.2}$$

where a_α depends on the initial condition only and the $|\bar{\Omega}_\beta(\tau)>$ are orthogonal to the $|\Omega_\alpha(\tau)>$'s. The matrix is unitary. Insertion into the timedependent Schrödinger equation yields

$$(U^{-1}\frac{dU}{dt})_{\beta\gamma} = \frac{E_{\beta\gamma}}{i\hbar} - \varepsilon<\Omega_\gamma|\frac{d}{d\tau}|\Omega_\beta>. \tag{A.3}$$

If we define the phase $\chi(t)$ by

$$i\chi(t) := \frac{1}{q}\ln \det U(t), \tag{A.4}$$

then we obtain from $i\dot{\chi} = \frac{1}{q}\,\mathrm{tr}(U^{-1}\frac{dU}{dt})$

$$\chi(t) = -\frac{1}{q\hbar}\int dt \sum_{\gamma} E_{\gamma\gamma}(\varepsilon t) + \frac{i}{q}\int_C d\tau \sum_{\gamma} <\Omega_\gamma|\frac{d}{d\tau}|\Omega_\gamma>. \tag{A.5}$$

The second term in this expression is a generalized Berry phase $\gamma(C)$. In analogy to (3.4) and (2.9) one obtains

$$|\bar{\Omega}_\beta(\tau)> = i\hbar \sum_{\gamma} G_{\beta\gamma}\frac{d}{d\tau}|\Omega_\gamma(\tau)> = \frac{i\hbar\dot{\Phi}_1}{\varepsilon}\sum_{\gamma} G_{\beta\gamma}|\partial_1\Omega_\gamma(\tau)>, \tag{A.6}$$

$$|\partial_i\Omega_\beta> = \sum_{\gamma}(-G_{\beta\gamma}\partial_i H|\Omega_\gamma> + |\Omega_\gamma><\Omega_\gamma|\partial_i\Omega_\beta>) \tag{A.7}$$

with the resolvent

$$G_{\beta\gamma} = (1-\sum_{\alpha}|\Omega_\alpha><\Omega_\alpha|)\,(H-E)^{-1}{}_{\beta\gamma}\,(1-\sum_{\alpha}|\Omega_\alpha><\Omega_\alpha|). \tag{A.8}$$

The indices $\beta\gamma$ on the right hand side of (A.8) refer to the matrix E.

The expectation value for the current reads

$$<\Omega_V(t)|I_i|\Omega_V(t)> = -<\Omega_V|\partial_i H|\Omega_V> = -\sum_{\alpha\beta\kappa\lambda}\rho_{\kappa\lambda}U_{\lambda\beta}U^+{}_{\alpha\kappa}<\Omega_\alpha+\varepsilon\bar{\Omega}_\alpha|\partial_i H|\Omega_\beta+\varepsilon\bar{\Omega}_\beta>, \tag{A.9}$$

where ρ is the statistical matrix. If the q states are equally populated, then $\rho_{\kappa\lambda} = \frac{1}{q}\delta_{\kappa\lambda}$ and the expression simplifies to

$$<I_i> = -\frac{1}{q}\sum_{\alpha}<\Omega_\alpha+\varepsilon\bar{\Omega}_\alpha|\partial_i H|\Omega_\alpha+\varepsilon\bar{\Omega}_\alpha>. \tag{A.10}$$

Now again one has $\sum_\alpha \langle\Omega_\alpha|\partial_i H|\Omega_\alpha\rangle = \partial_i \sum_\alpha \langle\Omega_\alpha|H|\Omega_\alpha\rangle$. Further one obtains with (A.6) and (A.7)

$$\langle I_i\rangle = -\frac{1}{q}\partial_i \sum_\alpha \langle\Omega_\alpha|H|\Omega_\alpha\rangle + \frac{i\hbar\dot{\Phi}_1}{q}\sum_\gamma (\langle\partial_1\Omega_\gamma|\partial_i\Omega_\gamma\rangle - \langle\partial_i\Omega_\gamma|\partial_1\Omega_\gamma\rangle)$$

$$+\sum_{\beta\gamma}(-\langle\partial_1\Omega_\gamma|\Omega_\beta\rangle\langle\Omega_\beta|\partial_i\Omega_\gamma\rangle + \langle\partial_i\Omega_\gamma|\Omega_\beta\rangle\langle\Omega_\beta|\partial_1\Omega_\gamma\rangle). \tag{A.11}$$

The first sum vanishes by averaging over Φ_i. The last sum vanishes since the contributions cancel identically. For I_1 also the second sum vanishes. Thus $\langle\bar{I}_1\rangle = 0$. Again I_2 can be expressed by Berry's phase

$$\langle\bar{I}_2\rangle = \frac{e^2\dot{\Phi}_1}{2\pi h}\gamma(C) \tag{A.11}$$

$$\gamma(C) = \frac{i}{q}\int_0^{\Phi_0} d\Phi_2 \sum_\gamma (\langle\Omega_\gamma|\partial_2\Omega_\gamma\rangle(\Phi_0,\Phi_2) - \langle\Omega_\gamma|\partial_2\Omega_\gamma\rangle(0,\Phi_2)) - (1\leftrightarrow 2) \tag{A.12}$$

Similarly to (3.10) and (3.11) one has

$$|\Omega_\gamma(\Phi_0,\Phi_2)\rangle = \sum_\beta \Theta_{1\gamma\beta}(\Phi_2)\exp[2\pi i\sum_n \Lambda_1(r_n)]\,|\Omega_\beta(0,\Phi_2)\rangle, \tag{A.13}$$

$$|\Omega_\gamma(\Phi_1,\Phi_0)\rangle = \sum_\beta \Theta_{2\gamma\beta}(\Phi_1)\exp[2\pi i\sum_n \Lambda_2(r_n)]\,|\Omega_\beta(\Phi_1,0)\rangle. \tag{A.14}$$

where Θ_1 and Θ_2 are unitary matrices. With these expressions (A.12) reduces to

$$\gamma(C) = \frac{i}{q}\int_0^{\Phi_0} d\Phi_2\, \mathrm{tr}(\Theta_1{}^{-1}\partial_2\Theta_1)(\Phi_2) - \frac{i}{q}\int_0^{\Phi_0} d\Phi_1\, \mathrm{tr}(\Theta_2{}^{-1}\partial_1\Theta_2)(\Phi_1) \tag{A.15}$$

$$= \frac{i}{q}\ln\frac{\det\Theta_1(\Phi_0)\,\det\Theta_2(0)}{\det\Theta_1(0)\,\det\Theta_2(\Phi_0)}. \tag{A.16}$$

From (A.13) and (A.14) it follows that $\Theta_1(\Phi_0)\,\Theta_2(0) = \Theta_2(\Phi_0)\,\Theta_1(0)$, thus the argument of the logarithm in (A.16) is unity and $\gamma(C)$ is $2\pi/q$ times an integer.

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