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Francesco Nicolò*

*Permanent address: Dipartimento di Matematica, II Universita di Roma (Tor Vergata) 00170 - Roma ITALY

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1) Introduction

In recent years the renormalization group has shown itself as a very powerful tool in constructive quantum field theory. Conversely, some important aspects of this theory at the perturbative level, in particular the renormalization problem, have been traditionally studied by different techniques built around the Feynman diagrams formalism and the conceptual framework of the renormalization group has not been widely used.

Nevertheless this is possible and allows, in my opinion, a better understanding of the purely perturbative results and, at the same time a clearer connection between the perturbative and the constructive aspects of quantum field theory.

In this article I want to present some general ideas and results which have been developed in the last few years at the perturbative and at the constructive level of quantum field theory using an approach deeply based on the renormalization group. I will put the emphasis mainly on the "perturbative" results and essentially briefly discuss the following topics:

1) (Perturbative) Renormalization theory with the R.G. approach.
2) The bounds for the coefficients of the perturbative series.
3) Some ideas about the possible definition of the "non renormalizable" theories.

Some remaining interesting topics studied with the approach described here, but which will not be discussed in this article, are the proof of the Borel summability of the Φ_4^4 planar theory (with negative coupling constant) and the search for a non-Gaussian fixed point in the planar Φ^4 theory at dimension $d = 4 + \varepsilon$.

These subjects have been discussed in different works by many people; the approach and the results obtained, which I will discuss in detail, are due to Gallavotti and myself [1], [2], to Gallavotti and Felder [3], and to G. Felder [4]. A similar approach to some of these problems has been developed independently by J. Polchinskii [9].

2) The Euclidean field theory and the definition of the renormalization group

Formally a field theory is specified giving its interaction $V(\phi)$. In the euclidean formalism ϕ is a generalized gaussian random field whose "covariance" $C(x,y)$ has the following Fourier transform

$$\hat{C}(p) = \frac{1}{p^2+1} \tag{1}$$

when 1 is the "mass" of the theory.

$C(x,y)$ diverges at the contact ($|x-y|\to 0$) for d (dimension) greater or equal to 2, therefore we need to regularise it. We define a regularised field as the gaussian field associated with a new covariance: $C^{[\leq N]}(x,y)$ obtained modifying the previous one in such way that

$$\begin{aligned} &\text{i)} \quad \lim_{|x-y|\to 0} C^{[\leq N]}(x,y) < +\infty \\ &\text{ii)} \quad \lim_{N\to\infty} C^{[\leq N]}(x,y) = C(x,y) \end{aligned} \tag{2}$$

in some appropriate sense.

As an example of regularization we can choose the following one (Pauli-Villars)

$$\hat{C}^{[\leq N]}(p) = \frac{1}{p^2+1} - \frac{1}{p^2+\gamma^{2[N+1]}} \tag{3}$$

where N is an integer > 0, $\gamma > 1$, which satisfies i) and ii). The field associated with $C^{[\leq N]}$ will be denoted $\phi^{[\leq N]}$ and called "the regularised field with cutoff γ^N".

There are many possible regularisations, each one with their technical advantages and drawbacks. The general belief is that, apart from technicalities, the conclusions, obtained starting with different regularizations, must coincide when the regularisation is removed (in this case performing the limit $N \to \infty$).

To construct a field theory essentially means to prove the existence of the following limits

$$\lim_{\Lambda\to\infty}\lim_{N\to\infty}\frac{\int\left(\Pi_{1}^{n}\,\phi^{[\leq N]}_{(x_i)}\right)C^{V_\Lambda^{(N)}}\;P\left(d\,\phi^{[\leq N]}\right)}{\int C^{V_\Lambda^{(N)}}\;P\left(d\phi^{[\leq N]}\right)} \equiv \lim_{\Lambda\to\infty} S_\Lambda\left(x_1,\ldots,x_n\right) \to S(x_1,\ldots,x_n) \tag{4}$$

and to prove that the distributions $S(x_1,\ldots,x_n)$ (the Schwinger Functions) satisfy a set of properties which, through the Osterwalder-Schrader reconstruction theorem [5], allow the complete reconstruction of the quantum field theory.

The problem associated with the existence of the first limit is called the ultraviolet (u.v.) problem, that associated with the second one (the infinite volume limit) the infrared problem, and the two problems are essentially independent. The u.v. problem is still unsolved in most of the models.

At the beginning to give a meaning to the expressions we write (see for instance (4)), we introduce an ultraviolet cutoff (γ^N in this formalism). This implies that the theory with cutoff γ^N cannot give any insight on distance scales smaller than γ^{-N}. One of the basic ideas of the renormalization group is that of looking at the Hamiltonian defining the theory on different scales.

Starting from a theory with cutoff γ^N one can, in many different ways, perform a kind of "average" over the fields in the Hamiltonian so that the new resulting fields (often called the "block spin" fields) are essentially constant on the smallest scales and describe the theory only for length distances greater than γ^{-K}, with $K < N$.

This "averaging" procedure transforms the original Hamiltonian into a new one: the effective Hamiltonian on scale γ^{-K}. Iterating this transformation an arbitrary number of times we get a sequence $\{H_n\}$

with H_0 = the original regularized Hamiltonian and H_n the effective Hamiltonian with smallest scale $\gamma^{-(N-n)}$ (or with u.v. cutoff $\gamma^{(N-n)}$). The transformation just introduced is called the renormalization group transformation.

$$TH_n = H_{n+1} \quad n \geq 0$$

$$H_0 = \text{regularized Hamiltonian with cutoff } \gamma^N$$

(Note that H_n depends also on N).

The knowledge of the transformation T gives information about the properties of the sequence $\{H_n\}$ and therefore of the theory under investigation. In particular the knowledge of the fixed points of T and of its linearization DT around them gives the possibility of getting significant insights both in the ultraviolet and in the infrared properties of the theory.

In the approach we are discussing the basic ideas of the renormalization group are implemented in the following way: We start decomposing the field $\phi^{[\leq N]}$ as a sum of independent gaussian fields, each one being approximately associated to a well defined range of momenta (length scales). One of the possible ways of performing this decomposition is the following:

$$\phi_x^{[\leq N]} = \sum_0^N{}' \phi_x^{(J)} \tag{5}$$

$$C_{xy}^{[\leq N]} = \sum_0^N{}' C_{xy}^{(J)} \tag{6}$$

In particular with the Pauli-Villars regularization (6) becomes

$$\hat{C}_{(p)}^{[\leq N]} = \frac{1}{p^2+1} - \frac{1}{p^2+\gamma^{2(N+1)}} = \sum_0^N{}_J \left(\frac{1}{p^2+\gamma^{2J}} - \frac{1}{p^2+\gamma^{2(J+1)}}\right) \equiv \sum_0^N{}' \hat{C}_{(p)}^{(J)} \tag{7}$$

The gaussian measure $P(d\phi^{[\leq N]})$ can be written as the product of N+1 independent gaussian measures associated to the covariances $C^{(J)}$'s

$$P(d\phi^{[\le N]}) = \prod_{0}^{N}{}_{J} P(d\phi^{(J)}) \tag{8}$$

The next step is not to integrate expressions like (4) with respect to the measure $P(d\phi^{[\le N]})$, but to integrate them with respect to the measures $P(d\phi^{(J)})$ of "definite frequency," one after the other, starting from the highest ones and going down.

We introduce the notion of "effective potential" (very similar to the notion of effective Hamiltonian discussed before) which in this approach is the object one essentially studies and is defined in the following iterative way:

$$C^{(TV_\Lambda^{(K+1)})(\phi^{[\le K]};N)} \equiv C^{V_\Lambda^{(K)}(\phi^{[\le K]};N)} \equiv \int P(d\phi^{(k+1)})C^{V_\Lambda^{(K+1)}(\phi^{[\le K+1]};N)} \tag{9}$$

$V^{(N)}(\phi^{[\le N]};N)$ is the original interaction with cutoff γ^N.

The knowledge and the control of the effective potential "on scale K", VK, $V_\Lambda^{(K)}(\phi^{[\le K]};N)$ allows us to build the Schwinger functions of the theory ([1], [2]). We will therefore concentrate on it.

$V^{(K)}(\phi^{[\le K]};N)$ still depends on N through the cutoff γ^N, we are therefore mainly interested in the existence and properties of the limit

$$\lim_{N\to\infty} V_\Lambda^{(K)}(\phi^{[\le K]};N) = V_\Lambda^{(K)}(\phi^{[\le K]}) \tag{10}$$

If one is interested in perturbative results the main goal will be to be able to write $V_\Lambda^{(K)}(\phi^{[\le K]};N)$ as a formal power series in some relevant parameters ("the physical coupling constants") whose coefficients are uniformly bounded in the cutoff N. If, conversely, one looks for more ambitious constructive results one has also to be concerned with the more difficult task of proving or disproving the convergence of the formal series and in the second case of showing that from the divergent series one can, nevertheless, build a function with the expected properties which has this series as an asymptotic one. Sometimes the constructive approach requires less than knowing the existence of the limit (10). For instance, it could be enough to prove that

$$C^{-E_-|\Lambda|} \leq \int P\left(d\phi^{[\leq N]}\right) C^{V_\Lambda^{(N)}} \leq C^{E_+|\Lambda|}$$

where $E\pm$ are functions of the coupling constants of the theory, uniform in the ultraviolet cutoff. This property is called the ultraviolet stability of the theory. While the perturbative results have been obtained for a large class of theories, the so-called renormalizable and super-renormalizable theories, the constructive results just mentioned have been proven true only for a few theories, whose space time dimensions are always less than 4.

3) The perturbative results

The results we want to discuss are valid for a class of interactions, that is for a family of scalar field theories

We define

$$V^{(N)}\left(\phi^{[\leq N]},N\right) = \sum_{1}^{t-2}{}_{\alpha}\, \lambda_b^{(\alpha)}(N) \int :\left(\phi_x^{[\leq N]}\right)^{2\alpha} : d^d x \;+\; \lambda_b^{(t-1)}(N) \int :\left(\underline{\partial\phi}_x^{[\leq N]}\right)^2 : d^d x$$

$$\equiv \sum_{1}^{t-1}{}_{\alpha}\, \lambda_b^{(\alpha)}(N)\; I_N^{(\alpha)}\left(\phi^{[\leq N]}\right) \qquad (11$$

where $\lambda_b^{(\alpha)}(N)$ are the so-called "bare-coupling constants," they depend on the cutoff N and, in principle, as we are now interested in perturbative results they can be thought of as defined by formal power series in some "relevant" parameter. In this case the expression (11) is a formal series; otherwise it is a perfectly defined function of the gaussian field $\phi^{[\leq N]}$. It is well known, by standard arguments, that defining the function

$$\sigma(\alpha) = 2\alpha\left(\frac{d-2}{2}\right) - d \qquad \alpha\varepsilon[1,t-2] \qquad (12$$

the theory is

superrenormalizable	if	$\sigma(\alpha) < 0$	for $\forall\alpha \leq t-2$
renormalizable	if	$\sigma(\alpha) = 0$	for $\alpha = t-2$
non-renormalizable	if	$\sigma(\alpha) > 0$	for some $\alpha \leq t-2$

We are for the moment interested in the first two cases and in particular in the second one, which is harder than the first. We will consider in particular the following example of a renormalizable theory: $t = 4$, $d = 4$, $(\sigma(2) = 0)$ which is usally called the Φ_4^4 theory.

As we said, we have to look for a useful expansion of the effective potential $V^{(J)}(\phi^{[\leq J]};N)$. From the definition (9) of the effective potential the obvious idea is to build the sequence $\{V^{(N)},\ldots, V^{(K)},\ldots\}$

and thus integrating the potential on the previous scale with respect to the gaussian field of definite frequency.

The integration on each scale produces a formal power series (often called a cummulant expansion) namely

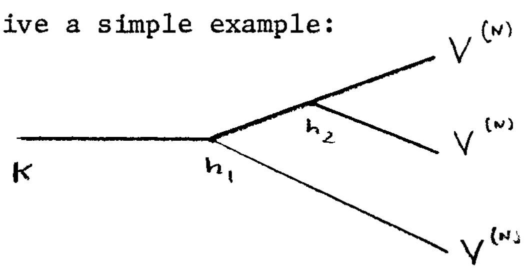
$$V^{(J)}\left(\phi^{[\leq J]};N\right) = \sum_{1}^{\infty} n \frac{1}{n!} E^{T}_{[J+1]} \left(\underbrace{V^{(J+1)},\ldots,V^{(J+1)}}_{n\text{-times}}\right)$$

$$E^{T}_{[J+1]}\left(V^{(J+1)},\ldots,V^{(J+1)}\right) = \frac{\partial^n}{\partial\tau^n} \log \int P(d\phi^{(J+1)}) C^{\tau V(J+1)}\Big|_{\tau=0} \tag{13}$$

is the truncated expectation of order n and the lower index [J+1] indicates that the integration is with respect to the measure $P\left(d\phi^{(J+1)}\right)$. To compute the effective potential on scale K one has to perform this expansion many times from J = N-1 to J = K+1. The final result is that $V^{(K)}\left(\phi^{[\leq K]};N\right)$ can be written as a (formal) multiple series of truncated expectations of truncated expectations of ... and so on and so forth; these expectations being performed with respect to different frequencies, starting from the highest ones.

It turns out that it is possible to build a very useful graphical notation for this expansion. In fact a "tree" can be uniquely associated, to each of its terms, together with a set of frequencies $\underline{h}$.

We do not give here a formal definition of a tree which has been given in any detail in [1] and [2]. Intuitively a tree can be drawn starting with an horizontal line on the left and making it bifurcate a finite but arbitrary number of times in an arbitrary number of lines. Let us just give a simple example:



$(\theta,\underline{h})$, $h_2 > h_1$

(fig 1

The final lines (on the right) are associated to $V^{(N)}$. Each bifurcation denotes a truncated expectation and the frequencies (h_1, h_2) associated with the bifurcations tell with respect to which measures $\left(P(d\phi^{(h_1)}), P(d\phi^{(h_2)})\right)$ the truncated expectation is performed.

For instance the tree θ of fig. 1 with the frequency assignment $\underline{h} = (h_1, h_2)$ corresponds to the following term of the expansion of $V^{(K)}$

$$E_{>K}\{E^T_{[h_1]}\{E^T_{[h_2]}(V,V),V)\} \tag{14}$$

when V is the simple expectation of $V^{(N)}$, up to the frequency at which the truncated expectation is performed; in this case the V's correspond to: $E_{>h_2}(V^{(N)})$, $E_{>h_2}(V^{(N)})$, $E_{>h_1}(V^{(N)})$ respectively and

$$E_{>K}(-) \equiv \int \prod_{K+1}^{N}{}_i \, P(d\phi^{(i)})(-).$$

With these notations the tree expansion for the effective potential $V^{(K)}$ is the following one:

$$\begin{aligned} V^{(K)}(\phi^{[\leq K]};N) &= \Sigma_\theta \Sigma_{\underline{h}} V(\theta;\underline{h};K) = \\ &= \sum_{1}^{t-1}{}_\alpha \lambda_b^{(\alpha)}(N) I_K^{(\alpha)} + \sum_{\theta \neq \theta_0} \sum_{\substack{\underline{h} \\ \{h_v \leq N}} V(\theta;\underline{h};K) \end{aligned} \tag{15}$$

when the first term on the right hand side of (15) is just the original interaction (see (11)), written now as a function of the fields $\phi^{[\leq K]}$, $\partial\phi^{[\leq K]}$. This term is associated to the tree θ_0 made by a single line (without bifurcations) which correspond to a simple expectation (remember also that $E_{>K}(:\phi^{[\leq N]})^\alpha:) = :(\phi^{[\leq K]})^\alpha:)$.

The second term in the right hand side of (15) is associated with the remaining "non-local part" of the effective potential on scale K. $\underline{h}$ is the frequency assignment for the family of bifurcations $\{v\}$ of a generic tree and $h_v \leq N$ is the constraint on the sum over the frequencies due to the fact that we start with the regularised field $\phi^{[\leq N]}$.

The relevant features of this expansion are:

i) The generic term $V(\theta,\underline{h},K)$ has the following structure

$$V(\theta;\underline{h};K) = \Sigma_p \int dx_1 \ldots dx_m V(\theta,\underline{h},K;x_1 \ldots, x_m.P)P(\phi^{[\leq K]}, \partial\phi^{[\leq K]}) \tag{16}$$

when P is a Wick polynomial in $\phi^{[\leq K]}$ and $\partial\phi^{[\leq K]}$ and m is the number of final lines of θ.

ii) The tree expansion of $V^{(K)}(\phi^{[\leq K]};N)$ is an expansion in the $\lambda_b^{(\alpha)}(N)$, the order in the bare coupling constants is the number of the final lines of the generic tree θ.

iii) The "dangerous" dependence on the cutoff N is in the constraint $h_v \leq N$ (see (15)). The divergences we expect in the limit $N \to \infty$, if we do not choose appropriately the $\lambda_b^{(\alpha)}(N)$'s, arise because the multiple series $\Sigma_{\underline{h}} V(\theta,\underline{h},k)$ diverge.

This point is crucial in this approach to the Renormalization theory. All the divergences are due to the divergences of the sums over the frequencies associated to the bifurcations. If we just keep the $\lambda_b^{(\alpha)}(N)$ fixed and compute the different terms of (15), it is easy to see that, for instance in the Φ_4^4-theory, already at the second order in the $\lambda_b^{(\alpha)}(N)$ we produce divergences. Namely:

$$\lim_{N\to\infty} \Sigma_{\substack{h \\ \{h_v \leq N\}}} V(\theta = -\!\!\!<;h,k) = \infty \tag{17}$$

The well-known strategy to cure these and the other higher-order divergences is to make a clever choice of the $\lambda_b^{(\alpha)}(N)$ as a final power series in some other parameter in such a way that, in this new expansion, all the divergences cancel for each θ independently at each fixed order in the new parameter. Formally this idea can be stated in that way:

a) Write the bare coupling constants $\lambda_b^{(\alpha)}(N)$ as formal power series in the new "physical" coupling constants $g^{(\alpha)}(\alpha:1,\ldots,t-1)$ with coefficients N-dependent:

$$\lambda_b^{(\alpha)}(\underline{g},N) = \Sigma_{\underline{m}} C^{(\alpha)}(N;\underline{m})\underline{g}^{\underline{m}} \tag{18}$$

when $\underline{m} = (m_1,\ldots,m_{t-1})$, m_i: positive integer.

b) Write the effective potential $V^{(K)}(\phi^{[\leq K]};N)$ as a formal series in the $g^{(\alpha)}$'s and adjust the coefficients $C^{(\alpha)}(N;\underline{m})$ in such a way that all the coefficients of this new expansion of $V^{(K)}(\phi^{[\leq K]};N)$ be finite in the $N \to \infty$ limit.

A field theory is (perturbatively) renormalisable if the program described in (a) and (b) can be accomplished.

The approach we want to discuss briefly here, based on the Renormalization group ideas, provides a clear and simple proof of (a) and (b) for the renormalizable theories. Let us start sketching the lines of the proof of (a) and (b) given in [1], [2], where the Φ_4^4-theory was discussed in any detail.

We define $V_g^{(N)}(\phi^{[\leq N]};N)$ as the $V^{(N)}(\phi^{[\leq N]};N)$ defined in (11) but with $g^{(\alpha)}$ (some fixed parameters) instead of $\lambda_b^{(\alpha)}(N)$ and consider the tree expansion for $V_g^{(K)}(\phi^{[\leq K]};N)$.

From what was said before (see iii, p. 11) we expect to find some divergences when the limit $N \to \infty$ is performed. To be more precise about the nature of these divergences let us remind ourselves that $V_g^{(K)}(\phi^{[\leq K]};N)$ can be written as a (infinite) sum of terms, each one expressed as the integral with an appropriate coefficient of a Wick monomial P in

$\phi^{[\leq K]}$ and $\partial\phi^{[\leq K]}$

$$\int_{\Lambda^m} dx_1 \; dx_m \; V_N(g,\underline{x},K,P) \; P\left(\phi^{[\leq K]} \; \partial\phi^{[\leq K]}\right) \tag{19}$$

where N is to remind the dependence on the cutoff γ^N. The limit $N \to \infty$ produces the result that for any m there are some P's for which the following integrals diverge:

$$\lim_{N\to\infty} \int_{\Lambda^m} dx_1 \ldots dx_m \left|V_N(g,\underline{x},k,P)\right| = \infty \tag{20}$$

To build the appropriate bare coupling constants and make the theory perturbatively finite one can proceed in the following way. We consider the tree expansion for $V_g^{(K)}\left(\phi^{[\leq K]};N\right)$ up to the second order in g and add to the original interaction a second order term in g: $\Delta V_{g^2}^{(N)}$, again of the form of the terms of equation (11), for what concerns the $\phi^{[\leq N]}$ dependence which exactly cancels the second order divergences of $V_g^{(K)}(\cdot;N)$, in the $N \to \infty$ limit, for any finite K. We start again with the interaction $\left(V_g^{(N)} + \Delta V_{g^2}^{(N)}\right)$, consider the tree expansion up to the third order and the divergences which appear as $N \to \infty$. Again we construct a new term to add to the original interaction $\Delta V_{g^3}^{(N)}$ such that the tree expansion, starting from $\left(V_g^{(N)} + \Delta V_{g^2}^{(N)} + \Delta V_{g^3}^{(N)}\right)$ has not any more divergences up to the order g^3. It is clear that our goal is attained if we are able to build up an iterative procedure for the construction of $\Delta V_{g^m}^{(N)}$ for a generic m.

The recipe for the construction of all the appropriate counterterms is slightly indirect. In fact, we proceed in the following way:

I) We construct a new tree expansion, which we will call "the renormalized tree expansion," starting from $V_g^{(N)}$, not plagued by any infinite when the cutoff is removed.

II) We will prove that this renormalized tree expansion can be obtained performing the usual tree expansion just discussed, hereafter called "the unrenormalized tree expansion" provided we start from a $V^{(N)}$ (see eq. (11)) with the appropriate bare coupling constants.

I) <u>The renormalized tree expansion</u>

We start from the tree expansion defining $V_g^{(K)}\left(\phi^{[\leq K]};N\right)$ and modify it in the following way:

i) To each bifurcation of a generic tree θ we append an index $\tilde{R}$ or $\tilde{L}$; this has to be done in all the possible ways obtaining an expansion in a larger number of terms.

ii) Each bifurcation of a tree is associated to a truncated expectation, the indices $\tilde{R}$ and $\tilde{L}$ tell how it has to be modified:

a) If the index $\tilde{R}$ is appended to a bifurcation v of the tree θ, the truncated expectation $E^T_{[h_v]}(\quad)$ associated to this bifurcation with frequency h_v is a sum of terms of the following kind

$$\int_{\Lambda^m} dx_1 \ldots dx_m W(\theta_v, h_v, \underline{x}, P) P\left(\phi^{[\leq h_v - 1]}, \partial\phi^{[\leq h_v - 1]}\right) \tag{21}$$

assume now that $P(\phi, \partial\phi) = :\phi_{x_1}^{[\leq h_v-1]} \ldots \phi_{x_p}^{[\leq h_v-1]}:$ $p \leq m$, the index $\tilde{R}$ implies to following substitution:

$$:\phi_{x_1} \ldots \phi_{x_p}: \rightarrow R:\phi_{x_1} \ldots \phi_{x_p}: \equiv (1 - L):\phi_{x_1} \ldots \phi_{x_p}: \tag{22}$$

and the operation L is defined by

$$L:\phi_{x_1} \ldots \phi_{x_p}: = \begin{cases} 0 & \text{if } p > 2(t-2) \\ \frac{1}{|\Lambda|} \int_\Lambda d^dx :(\phi_x)^p: & \text{if } 2 < p \leq 2(t-2) \\ \frac{1}{|\Lambda|} \int_\Lambda d^dx :(\phi_x)^2: - \frac{(x_1 - x_2)^2}{2d|\Lambda|} \int_\Lambda d^dx :\left(\partial\phi_x\right)^2:, & \text{if } P=2 \end{cases} \tag{23}$$

Some other similar rules are needed when L operates on monomials P which depend also on $\partial\phi$ fields. They are explicitly given in [1], [2], [3], are finite in number, and we do not report them here.

To complete the description of how the index $\tilde{R}$ modifies the expansion associated with a bifurcation let me remind that in the tree expansion (15) $\Sigma_{\underline{h}}$ is a sum over the frequencies of the various bifurcations of the tree θ, $\underline{h} = \{h_{v_0}, h_{v_1}, \ldots\}$. The set of bifurcations of a generic tree has an obvious partial ordering starting from the initial lines and going down to the root of the tree. Let v be a bifurcation following $\tilde{v}$ and coming immediately before v', then the corresponding frequencies satisfy the following relationship

$$h_{\tilde{v}} > h_v > h_{v'} \tag{24}$$

The presence of the index $\tilde{R}$ at the bifurcation v does not modify this rule.

b) If the index $\tilde{L}$ is appended to the bifurcation V (in [1], [2] instead of putting an $\tilde{L}$ at V the whole subtree which has V as the lowest bifurcation is encircled by a frame) the truncated expectation $E^T_{[hv]}(\ )$ is modified substituting for the term $:\phi_{x_1}\ldots\phi_{x_m}:$ of (21) the term $-L:\phi_{x_1}\ldots\phi_{x_m}:$ defined in eq. (23). Moreover in the "$\Sigma_{\underline{h}}$" of the tree expansion the sum over h_v does not run anymore on the values prescribed by the inequality (24). Conversely, one has to make the following substitution

$$\sum_{h_v = h_{v'}-1}^{h_{\tilde{v}}-1} \longrightarrow \sum_{h_v = 0}^{h_{v'}} \tag{25}$$

Given a tree θ with a definite choice of $\tilde{R}$ and $\tilde{L}$ at its bifurcations, applying prescriptions i) and ii) we obtain a well defined expression for the term of the tree expansion associated to θ and to this choice of the $\tilde{R}$ and $\tilde{L}$ indices. Iterating this procedure for all the terms of the tree expansion we get a modified expansion: "the renormalized tree expansion". This is interesting as we are able to prove:

α) The coefficients of the "renormalized tree expansion" are finite as $N \to \infty$ for any $V^{(K)}_{r,g}(\phi^{[\le K]};N)$. (here r means "renormalized")

β) This new expansion is also obtained if, instead of the interaction $V^{(N)}_g(\phi^{[\le N]};N)$ we start with an interaction

$$V^{(N)}_g + \Delta V^{(N)}_{g^2} + \Delta V^{(N)}_{g^3} + \dots$$

and we perform the original, hereafter called the "unrenormalized," tree expansion.

This fact, remembering that all the added counterterms $\Delta V^{(N)}_{g^k}$ must have the form

$$\Delta V^{(N)}_{g^K} = \Sigma_\alpha O(g^K) I^{(\alpha)}_N \left(\phi^{[\le N]}\right) \tag{26}$$

see (11), is what is commonly expressed saying that one looks for a "Lagrangian renormalization."

Therefore from β) one can finally obtain the expression (as a formal power series in g) for the bare coupling constants of the theory, the result we were looking for. In fact

$$V^{(N)}\left(\phi^{[\le]};N\right) = V^{(N)}_g + \Sigma_2^\infty{}_K\, \Delta V^{(N)}_{g^K} = \Sigma_\alpha \lambda^{(\alpha)}_b(g;N) I^{(a)}_N \tag{27}$$

The proof of statements α) and β) is in [1], [2]. The proof of α) is stated in great detail; that of β) is also complete, but it is given in a more implicit way. Here I want to sketch a proof of β) suggested to me in that form by J. Feldman.

As said before, to prove (β) means to prove that the "renormalized tree expansion" (proved to be finite, in (2)) is a "Lagrangian renormalization." This means that it is possible to construct an interaction of the form (11) such that its "unrenormalized" tree expansion for the effective potential, $V^{(K)}\left(\phi^{[\leq K]};N\right)$, for a generic K, is equal to the "renormalized" tree expansion of $V^{(K)}_{r,g}\left(\phi^{[\leq K]};N\right)$. Of course the unrenormalized expansion is made in terms of the bare coupling constants while the renormalized one, in terms of the physical ones $g^{(\alpha)}$, g if we are interested in the Φ^4_4-theory. We proceed by induction.

We assume that the effective potential on scale K+1, $V^{(K+1)}(\cdot,N)$ can be written via the renormalized tree expansion. Then we construct the effective potential on scale K starting from that one scale K+1 and prove that again $V^{(K)}(\cdot,N)$ can be expressed via the renormalized tree expansion. The proof is complete if we prove that $V^{(N)}$ also can be written via the renormalized tree expansion (we have to specify what it does really mean in this case) and that this is compatible with having the expression, required at the beginning, given in (11) or (27).

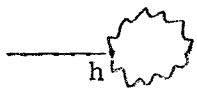
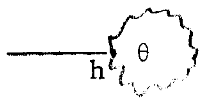
Let us prove the inductive step from the scale K+1 to the scale K. The assumption on $V^{(K+1)}(;N)$ means that we can write it in the following way:

$$V^{(K+1)}(;N) = \underset{K+1}{\bullet}\!\!\overline{} + \sum_{\substack{\theta \\ \theta\neq\theta_0}} \sum_{h=\theta}^{K+1} \underset{K+1}{\bullet}\!\!\overline{}\!\!\underset{h}{\tilde{L}}\{\theta\} + \sum_{\substack{\theta \\ \theta\neq\theta_0}} \sum_{K+2}^{N} \underset{K+1}{\bullet}\!\!\overline{}\!\!\underset{h}{\tilde{R}}\{\theta\} \qquad (28$$

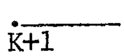
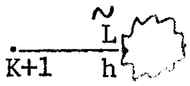
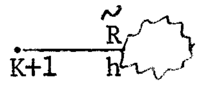
The first term of the right hand side of (28) is the trivial ($\equiv$ without bifurcations) tree and corresponds to $\Sigma_{\alpha}\, g^{(\alpha)} I^{(\alpha)}_{K+1}$ (remember that, by assumption), the right hand side of (28) is the renormalized tree expansion).

$\bullet\!\!\overline{}_{h}\{\theta\}$ corresponds to the contribution of a generic tree θ when we sum over all the frequencies except the last one which has the value h;

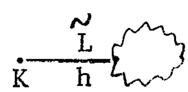
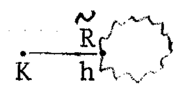
θ has to be thought of with an index $\tilde{L}$ or $\tilde{R}$ appended to each bifurcation. We also denote

 $\equiv \sum_{\substack{\theta \\ \theta \neq \theta_0}}$  (29

and rewrite eq. (28) as

$V^{(K+1)}(:N) =$  $+ \sum_{h=0}^{K+1}$  $+ \sum_{h=K+2}^{N}$  (30

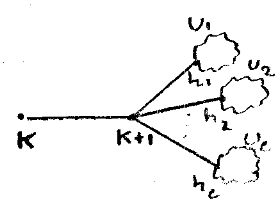
To get $V^{(K)}(;N)$ from $V^{(K+1)}(;N)$, which, one has to remember, is obtained with an unrenormalized tree expansion from $V^{(N)}(;N)$, one has only to perform a cumulant expansion with respect to the measure $P\left(d\phi^{(K+1)}\right)$. We obtain

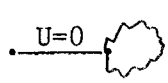
$V^{(K)}\left(\phi^{[\leq K]};N\right) =$  $+ \sum_{h=0}^{K+1}$  $+ \sum_{h=K+2}^{N}$  $+$

$$+ \sum_{\ell=2}^{\infty} \frac{1}{\ell!} E^{T}_{[K+1]} \underbrace{\left(V^{(K+1)},\ldots,V^{(K+1)}\right)}_{\ell\text{-times}} \qquad (31$$

and we have to prove that this expression can be put in the same form as (30), with, obviously, K instead of K + 1.

The last term of (31) can be expressed in terms of trees:

$\sum_{\ell=2}^{\infty} \frac{1}{\ell!} E^{T}_{[K+1]}(\qquad) = \sum_{\ell=2}^{\infty} \frac{1}{\ell!} \sum_{\{U\}} \sum_{\{h_1,\ldots,h_\ell\}}$  (32

when the indices $\{U\} = \{U_1,\ldots,U_\ell\}$ which are applied to the ℓ bifurcations immediately before the lowest one, run over the set $\{\tilde{R},\tilde{L},0\}$, U = 0 just means that the term  which merges in the lowest bifurcation is the trivial tree.

Remembering that in eq. (29) for all the trees θ the sums over the inner frequencies are performed, we can rewrite (32) as

$$\sum_{2}^{\infty}{}_{\ell}\ \frac{1}{\ell!}\,E^{T}_{[K+1]}\big(\underbrace{V^{(K+1)},\ldots,V^{(K+1)}}_{\ell}\big) = \underset{K}{\cdot}\!\!\frac{\quad}{}\!\!\underset{K+1}{} \;=\; (L + (1-L))\ \underset{K}{\cdot}\!\!\frac{\quad}{}\!\!\underset{K+1}{}$$

$$= -\tilde{L}\Big(\underset{K}{\cdot}\!\!\frac{\quad}{}\!\!\underset{K+1}{}\Big) + \underset{K}{\cdot}\!\!\frac{\quad}{}\!\!\underset{K+1}{\overset{\tilde{R}}{}} \tag{33}$$

(Note that the index $\tilde{L}$ appended implies that the operation L is performed and also that the lowest bifurcation frequencies run from 0 to the root frequency K; here, anyway there is not a summation on the lowest frequency.)

Inserting (33) into (31) we see that the first term of the right hand side of (33) cancels the last term of the $\sum_{0}^{K+1}$ of (31) while the second term of the right hand side of (33) just adds to the corresponding sum, obtaining

$$V^{(K)}\big(\phi^{[\leq K]};N\big) = \underset{K}{\cdot\!\!-\!\!\cdot} + \sum_{0}^{K}{}_{h}\ \underset{K}{\cdot}\!\!\frac{\quad}{}\!\!\underset{h}{\overset{\tilde{L}}{}} + \sum_{K+1}^{N}{}_{h}\ \underset{K}{\cdot}\!\!\frac{\quad}{}\!\!\underset{h}{\overset{\tilde{R}}{}} \tag{34}$$

which is just eq. (30) with K instead of K+1.

To complete the proof of (β) we have to check the first step of the instructive procedure. If we put in eq. (34) K = N we get:

$$V^{(N)}\big(\phi^{[\leq N]};N\big) = \Sigma_{\alpha}\, g^{(\alpha)} I_{N}^{(\alpha)} + \sum_{0}^{N}{}_{h}\ \underset{N}{\cdot}\!\!\frac{\quad}{}\!\!\underset{h}{\overset{\tilde{L}}{}} \tag{35}$$

and from the meaning of the index L it follows that the right hand side of (35) can be written as

$$\Sigma_{\alpha}\, \lambda_{b}^{(\alpha)}(\underline{g};N)\, I_{N}^{(\alpha)}\big(\phi^{[\leq N]}\big)$$

defining the bare coupling constants. Therefore (β) is proven and proving the renormalizability of the theory amounts only to proving the statement (α).

We do not do it here and refer to [1] and [2] for a proof. We remark that this result is, in some sense, evident as the renormalized expansion has been built in such a way as to remove the divergences arising when different coordinates coincide, just performing subtractions which produce some extra zeroes in the integrands (see prescriptions in i), ii)). Although the guiding idea is clear, nevertheless the proof is a bit involved due to the fact that one has to prove that the introduction of the $\tilde{R}$ and $\tilde{L}$ at the high bifurcations does not destroy the effect of the same operations at the lower frequency bifurcations.

Before discussing the second and third point of this paper, I want to present a slightly different way of looking at the same results which has the advantage of making more evident the connection between the perturbative and the non-perturbative approach and, moreover, sheds some light on a possible way of defining the non renormalizable theories. We start again from eqs. (30) and (34) and observe that in the expression (30) for $V^{(K+1)}(\,;N)$ the local part of the effective potential is totally contained in the first two terms. We can regroup them together and write:

$$V^{(K+1)}(\,;N) = \sum_\alpha r^{(\alpha)}(K+1;\,N)\,I^{(\alpha)}_{K+1} + \sum_{h=K+2}^{N} \underset{K+1\qquad h}{\bullet\!\!\xrightarrow{\quad\tilde{R}\quad}} \tag{36}$$

where in $r^{(\alpha)}(K+1,\,N)$, we call "dimensional running coupling constants," we have omitted the dependence on the $g^{(\alpha)}$'s. Looking at the expression of $V^{(K)}$ obtained from $V^{(K+1)}$ we can rewrite, following (31) and (34)

$$V^{(K)}(\,;N) = \sum_\alpha r^{(\alpha)}(K+1,\,N)\,I^{(\alpha)}_{K} + \sum_{h=K+2}^{N} \underset{K\qquad h}{\bullet\!\!\xrightarrow{\quad\tilde{R}\quad}} +$$

$$+ \sum_{\ell=2}^{\infty} \frac{1}{\ell!}\, E^T_{[K+1]}\big(\underbrace{V^{(K+1)},\ldots,V^{(K+1)}}_{\ell\text{-times}}\big) =$$

$$= \sum_\alpha r^{(\alpha)}(K;\,N)\,I^{(\alpha)}_{K} + \sum_{h=K+1}^{N} \underset{K\qquad h}{\bullet\!\!\xrightarrow{\quad\tilde{R}\quad}} \tag{37}$$

which immediately implies, remembering eq. (33):

$$r^{(\alpha)}(K + 1;\ N) + L\Big(\cdot\underset{K\qquad K+1}{\overset{L}{\longrightarrow}}\Big) = r^{(\alpha)}(K;\ N) \tag{38}$$

and as

$$L\ (\qquad) = \sum_{2}^{\infty}{}_{\ell}\ \frac{1}{\ell!}\ L\ E^{T}_{[K+1]}\ \big(\underbrace{V^{(K+1)},\dots,V^{(K+1)}}_{\ell}\big)$$

it is clear that $r^{(\alpha)}(K{:}\ N)$ depends on $r^{(\alpha)}(K + 1;\ N)$ and on $V^{(K+1)}$; $V^{(K+1)}$ in turn depends on $r^{(\alpha)}(K + 1;\ N)$ and also through $\sum_{K+2}^{N}{}_{h}\ \cdot\underset{K\qquad h}{\overset{\tilde{R}}{\longrightarrow}}$ on $V^{(K+2)}$ which depends on $r^{(\alpha)}(K^{+2};N)$ and so on and so forth, to conclude that $r^{(\alpha)}(K;\ N)$ will depend on all the $r^{(\alpha)}(h;\ N)$ with $h \geq K + 1$.

Equation (38) is therefore a recursive relation for the running coupling constants.

Let us point out now another feature connected to the introduction of the running coupling constants.

Looking back at eq. (38) we have

$$\sum_{K+1}^{N}{}_{h}\ \cdot\underset{K\qquad h}{\overset{\tilde{R}}{\longrightarrow}} = \sum_{2}^{\infty}{}_{\ell}\ \frac{1}{\ell!}\ \sum_{K+1}^{N}{}_{h}\ E_{>K}\ RE^{T}_{[h]}\ \big(\underbrace{V^{(h)},\dots,V^{(h)}}_{\ell\text{-times}}\big) \tag{39}$$

We can rewrite, using this relation, the effective potential for a generic $K \leq N$ as

$$V^{(K)}(\ ;N) = \Sigma_{\alpha}\ r^{(\alpha)}(K;\ N) I^{(\alpha)}_{K} + \sum_{2}^{\infty}{}_{\ell}\ \frac{1}{\ell!}\ \sum_{K+1}^{N}{}_{h} RE^{T}_{[h]}\big(V^{(h)},\dots,V^{(h)}\big) \tag{40}$$

Starting from K = N and going down we see that we produce a tree expansion of a third type different from both a unrenormalized and the renormalized one. We write it

$$V^{(K)}(\ ;N) = V_{r}(\theta_{0},\ K) + \sum_{\theta\neq\theta_{0}}{}_{\theta}\ \sum_{(h_{v}\leq N)}{}_{\underline{h}}\ V_{r}(\theta,\ \underline{h},\ K) \tag{41}$$

when

$$V_r(\theta_0, K) = \Sigma_\alpha r^{(\alpha)}(K; N) I_K^{(\alpha)}$$

$$V_r(\theta, \underline{h}, K) = \frac{1}{S!} E_{>K} RE^T_{[hv_0]}\left(V_r(\theta_1; \underline{h}^{(1)}; h_{v_0}), \ldots, V_r(\theta_s; \underline{h}^{(s)}; h_{v_0})\right)$$

with θ a generic tree whose lowest frequency bifurcation is h_{v_0}, such that in V_0 s-subtrees merges with frequencies respectively $\underline{h}^{(1)}, \ldots, \underline{h}^{(s)}$ and roots h_{v_0}.

All that means that this new expansion differs from the unrenormalized one because each bifurcation has appended an index $\tilde{R}$ and because the final lines do not bring anymore the bare coupling constants $\lambda_b^{(\alpha)}(\underline{g}, N)$ nor the "physical" ones $g^{(\alpha)}$, but, conversely, each final line brings a running coupling constant $r^{(\alpha)}(h; N)$ where h is the frequency of the first bifurcation where the final line emerges.

This "running" tree expansion proves itself very useful when it is associated to the recursion relation for the running coupling constants (38) which we rewrite as

$$r^{(\alpha)}(K; N) = r^{(\alpha)}(K + 1, N) + \sum_{\theta \neq \theta_0} \sum_{\substack{\underline{h} \\ h_{v_0} = K+1 \\ h_v \leq N}} r^{(\alpha)}(\theta, \underline{h}, N) \tag{43}$$

and

$$\Sigma_\alpha r^{(\alpha)}(\theta, \underline{h}, N) I_K^{(\alpha)} = \frac{1}{S!} L\, E^T_{[K+1]}\left(V_r(\theta_1, \underline{h}^{(1)}, K+1), \ldots, V_r(\theta_s, \underline{h}^{(s)}, K+1)\right) \tag{44}$$

Equations (43) and (44) provide a recursion relation for the $\{r^{(\alpha)}(h; N)\}$. Let us assume one can find a "solution" to it bounded in an appropriate way (see later to give a precise meaning to it). Then it is intuitively clear that to prove the (perturbative) finiteness of the theory one is left to check that the "running" tree expansion is finite. This proof is the particular case of that for the renormalised expansion, where all the indices appended to the bifurcations are $\tilde{R}$-indices and where the $g^{(\alpha)}$'s are substituted by the appropriate bounds for the $r^{(\alpha)}$'s solutions of (43), (44).

Therefore apart from some technical aspects which suggest it is better to deal with "adimensional running coupling constants," we are going to define in a moment, the proof of the renormalizability of the theory can be rephrased in the following way:

α') Prove that the recursion relation for the running coupling constants admit a solution as a formal power series in some parameters that we fix on a definite scale and call "physical coupling constants" (for instance we can define $r^{(\alpha)}(0; N) \equiv g^{(\alpha)}$) and that the coefficients of this formal series are uniformly bounded in N.

β') Prove that the running tree expansion has all its coefficients uniformly bounded in N when the adimensional coupling constants are bounded by a constant.

This approach, developed by Felder and Gallavotti in [3] has some advantages with respect to that proposed in [1], [2] and previously described. Its content for the renormalizable theories is exactly the same as that of the previous one, but equations (41),...(44) are meaningful in the $N \to \infty$, in a sense we are going to make precise, also for the non-renormalizable theories, the crucial difference being now that in this case it is not possible to satisfy (α').

To make this argument more precise and to complete the discussion about the perturbative approach showing the results on the bounds of the coefficients of the effective potential (from which it is not hard to derive those for the Schwinger functions) we start defining the adimensional running coupling constants

$$r^{(\alpha)}(K;\ N) = \gamma^{-\sigma(\alpha)K}\lambda^{(\alpha)}(K;\ N)$$

$$\sigma(\alpha) = \begin{matrix} 2\alpha\left(\frac{d-2}{2}\right) - d & \alpha \leq t-2 \\ 0 & \alpha = t-1 \end{matrix} \qquad (45$$

(In the Φ^4_4 case, α: 1,2,3, $\sigma(1) = -2$, $\sigma(2) = \sigma(3) = 0$) and we rewrite the recursion relation (43) as:

$$\lambda^{(\alpha)}(K;\ N) = \gamma^{-\sigma(\alpha)}\lambda^{(\alpha)}(K+1;\ N) + \sum_{\theta \neq \theta_0} \sum_{\substack{\underline{h} \\ h_{v_0}=K+1 \\ h_v \leq N}} \sum_{\underline{\alpha}} \beta^{(\alpha)}(\theta, \underline{h}, \underline{\alpha}) \prod_{1}^{n}{}_i \lambda^{(\alpha_i)}(h_i;\ N) \qquad (46$$

when

$\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$

$\{h_i\}$: frequencies of the bifurcations where final lines merge

n = number of final lines of θ

and

$$\gamma^{\sigma(\alpha)K} r^{(\alpha)}(\theta, \underline{h}, N) \equiv \sum_{\underline{\alpha}} \beta^{(\alpha)}(\theta, \underline{h}, \underline{\alpha}) \Pi_i \lambda^{(\alpha_i)}(h_i;\ N) \qquad (47$$

We observe that the dependence on N of $\lambda^{(\alpha)}(K;\ N)$ is due to the fact that if they are solutions of the non-linear equation (46) they will depend on N as the kernel of the equation does. Its N-dependence is due to the condition $\sum_{h_v < N}$ of (46). This suggests that one could consider as the "tree equation" for the running coupling constant the analogous one to (46) but with the constraint $h_v \leq N$ removed. Namely

$$\lambda^{(\alpha)}(K) = \gamma^{-\sigma(\alpha)}\lambda^{(\alpha)}(K+1) + \sum_{\theta | \theta_0} \sum_{\substack{\underline{h} \\ \{h_{v_0}=K+1\}}} \sum_{\underline{\alpha}} \beta^{(\alpha)}(\theta, \underline{h}, \underline{\alpha}) \prod_{1}^{n(\theta)}{}_i \lambda^{(\alpha_i)}(h_i) \qquad (48$$

It would be very interesting to investigate if this equation has a solution and if this solution is connected in the obvious way with the solution of (46)

$$\lambda^{(\alpha)}(K) = \lim_{N\to\infty} \lambda^{(\alpha)}(K;\ N) \qquad (49$$

Nothing is known about the existence and the nature of these solutions so there is not very much to add to this point. One has, of course, to realise that in searching for solutions of the functional equation (48) one is looking for more strong results than the purely perturbative ones.

Going back to the perturbative problem we can summarize the description of this last approach with the following results (proved in [1], [2] for the renormalizable case, explicitly for the Φ_4^4 theory, and in [3] for the non-renormalizable situation as well).

Theorem 1

Let $V^{(K)}(\ :N)$ be the effective potential on scale K and consider its "running" tree expansion (41); the order n in the running coupling constants is made by a finite sum of terms of the following kind:

$$\int_{\Lambda^n} dx_1 \ldots dx_n\ V^{(K)}(x_1, \ldots, x_n, P)\ P\ (\phi^{[\leq K]},\ \partial\phi^{[\leq K]}) \qquad (50$$

where P is a Wick monomial of degree p in the ϕ's and the $\partial\phi$'s on scale K; then the following estimates hold uniformly in N:

$$\int_{\Delta_1 x \ldots x \Delta_n} |V^{(K)}(x_1, \ldots x_n, P)|\, dx_1 \ldots dx_n \leq N(p) C^n \frac{(n(t-2))!}{n!} ||\underline{\lambda}||^n e^{-K\gamma^K d(\Delta_1, \ldots, \Delta_n)} \qquad (51$$

where $\Delta_1, \ldots, \Delta_n\ Q_k$ (family of cubes paving R^d of linear size γ^{-K}), C, K > 0, N(p) is a function > 0 of the only variable p and $||\lambda|| \equiv \sup_{(\alpha,K,N)} |\lambda^{(\alpha)}(K,\ N)|$.

This result, in the Φ_4^4 case, essentially tells that as $t = 4$, the coefficients of the running tree expansion have a n! dependence (n is the order in $||\lambda||$) from which we expect that its Borel series has a finite convergent radius due to an instanton singularity on the negative axis[8]. This is present as well as in the $d = 3$ case for the Φ^4 theory. The next result shows, implicitly that nevertheless in $d = 4$ the situation is more complicated when we look for an expansion in the physical coupling constants instead of the running coupling constants.

Theorem 2

The coefficients $\beta^{(\alpha)}(\theta,\underline{h},\underline{\alpha})$ of the "β-functional" defined in the right hand side of eq. (46) satisfy, uniformly in N, the following estimates

$$|\beta^{(\alpha)}(\theta,\underline{h},\underline{\alpha})| \leq \frac{(n(t-2))!}{n!} C^n \prod_{v>v_0} \gamma^{-\rho(h_v-h_{v^1})} \tag{52}$$

where $C, \bar{\rho} > 0$ and n is the number of final lines of θ. In the Φ_4^4 case we have a n! in the bound (52). (In Φ_3^4, as the theory is superrenormalisable, essentially one has not to introduce the running coupling constants). Due to this n! one can expect that troubles can appear in the expansion of the running coupling constants in terms of the physical ones. The problem is in fact that the coefficients of this expansion bring themselves some n! dependence (where n is now the order of the physical coupling constants). Therefore one could suspect that the final n-dependence of the coefficients of the renormalized tree expansion is worst than the predicted n!-one. This is not the case, but its proof is a complicated task (obtained before with different methods by T. 't Hooft and De Calan, Rivasseau [6], [7]) thoroughly discussed in [1], [2].

To come back about the possibility of using this formalism to attack the problem of a definition and a possible construction of a "non-renormalizable" field theory, we remark that Theorems 1 and 2 are still valid in the non-renormalizable case and the well known impossibility of obtaining an expansion, uniform in the cutoff, in terms of a finite number of physical coupling constants is translated here in the impossibility of satisfying the statement (α') (above, p. 23). Of course a possible way out of this problem is to look for a solution of (48) with appropriate bounds in the frequency. In this case one could give a meaning to a non-renormalisable field theory without ever introducing the physical coupling constants as parameters in which to perform a power series expansion. Some results in this direction have been obtained by G. Felder in the planar case of the Φ_4^4-theory [4].

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